Prof. Dr. U. Görtz
Dr. F. Fité

Sommersemester 2016

ALGEBRAIC NUMBER THEORY II

Problem Set 5

Due date: 24/5/2016

Exercise 1. Let
$$\cdots \longrightarrow C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \to \dots$$
be a sequence of abelian groups such that for every $q \in \mathbb{Z}$ there exists a group homomorphism $h_q \colon C_q \to C_{q+1}$ such that $h_{q-1} \circ d_q + d_{q+1} \circ h_q = \mathrm{id}_{C_q}$. Show that $C_\bullet$ is exact if one of the following conditions is satisfied:

1. The sequence $C_\bullet$ is a complex, i.e., $d_q \circ d_{q+1} = 0$ for all q.
2. We have $C_q = 0$ for all $q < -1$, $d_0 \circ d_1 = 0$ and that h_q is surjective for all q.

Exercise 2. Let G be a group and A a G-module. A map $\varphi \colon G \to A$ is called a 1-*cocycle* if for every $\sigma, \tau \in G$ one has
$$\varphi(\sigma\tau) = \varphi(\sigma) + \sigma\varphi(\tau).$$
Let $Z^1(G, A)$ denote the group of 1-cocycles (with addition induced by the addition on A). A map $\varphi \colon G \to A$ of the form $\varphi(\sigma) = \sigma(b) - b$ (for some fixed $b \in A$) is called a 1-*coboundary*. Let $B^1(G, A)$ denote the group of 1-coboundaries; check that this is a subgroup of $Z^1(G, A)$. The first cohomology group of A is then defined as the quotient
$$H^1(G, A) = Z^1(G, A)/B^1(G, A)\,.$$

i) Let $G = \mathbb{Z}/2Z = \{1, -1\}$ act on $A = \mathbb{Z}$ in the following way: the nontrivial element -1 satisfies $-1 \cdot a = -a$ for every $a \in \mathbb{Z}$. Compute $H^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$.

ii) Let $G = (\mathbb{Z}/4\mathbb{Z})^\times$ act on $A = \mathbb{Z}/4\mathbb{Z}$ by multiplication. Determine the group $H^1((\mathbb{Z}/4\mathbb{Z})^\times, \mathbb{Z}/4\mathbb{Z})$.

iii) If p is a prime and the action on $\mathbb{Z}/p\mathbb{Z}$ is the natural one by multiplication, what is $H^1((\mathbb{Z}/p\mathbb{Z})^\times, \mathbb{Z}/p\mathbb{Z})$?

Exercise 3. Let L be a finite Galois extension of the field K, and let $G = \mathrm{Gal}(L/K)$.

i) Prove that $H^1(G, L^\times) = 1$.

Hint: Given a 1*-cocycle* $\varphi \colon G \to L^\times$*, construct a* 1*-coboundary from the element*
$$b = \sum_{\sigma \in G} \varphi(\sigma) \cdot \sigma(a)$$
for $a \in L^\times$ *chosen such that* $b \neq 0$*. To show that there exists* $a \in L^\times$ *such that* $b \neq 0$*, use Dedekind's theorem on the independence of characters.*

ii) Suppose that G is cyclic and that σ is a generator of G. Show that if $a \in L^\times$ is such that $N_{L/K}(a) = 1$, then there exists $b \in L^\times$ such that $a = b/\sigma(b)$.

Hint: Show that if $a \in L^\times$ is such that $N_{L/K}(a) = 1$, then there is a 1-cocycle $\varphi\colon G \to L^\times$ uniquely characterized by the condition $\varphi(\sigma) = a$.