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Sommersemester 2016

ALGEBRAIC NUMBER THEORY II

Problem Set 6

Due date: 31/5/2016

Exercise 1. Let G be a group and A an abelian group. (In this exercise and Exercise 2 we write all groups multiplicatively.) A *group extension of* G *by* A is a short exact sequence of groups

$$0 \to A \to E \xrightarrow{\pi} G \to 1\,.$$

An extension is said to split if there exists a group homomorphism $\sigma\colon G \to E$ such that $\pi \circ \sigma = \mathrm{id}$.

i) Show that in an extension of G by A, the abelian group A has a G-module structure with action defined by ${}^g a = g' a g'^{-1}$ for $a \in A$ and $g \in G$, and $g' \in E$ any lift of g.

ii) The semidirect product $A \rtimes G$ of a group G and a G-module A is a group with underlying set $A \times G$ and multiplication given by the formula

$$(a, g) \cdot (b, h) = (a \cdot {}^g b, gh)\,.$$

Show that $A \rtimes G$ is indeed a group.

iii) Prove that the group extension $0 \to A \to E \to G \to 1$ splits if and only if E is isomorphic to $A \rtimes G$, where G acts on A as in i).

Exercise 2. Let G be a group and A a G-module. We will be interested in extensions ξ of G by A such that the given G-module structure on A coincides with that induced by ξ as in Exercise 1 ii). Say that two such group extensions $\xi_i\colon 0 \to A \to E_i \to G \to 1$ of G by A with $i = 1, 2$ are equivalent if there exists a group isomorphism $\varphi\colon E_1 \simeq E_2$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & E_1 & \longrightarrow & G & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle\varphi} & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & E_2 & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

is commutative. Let $\mathrm{EXT}(G, A)$ denote the set of equivalence classes of extensions of G by A inducing the given G-module structure on A. Given a group extension $\xi\colon 0 \to A \to E \xrightarrow{\pi} G \to 1$, define

$$c_\xi(g, h) = \sigma(g)\sigma(h)\sigma(gh)^{-1} \in A\,,$$

where $\sigma\colon G \to E$ is a map such that $\pi \circ \sigma = \mathrm{id}$.

i) Prove that c_ξ lies in $Z^2(G, A)$, that its cohomology class γ_ξ only depends on the equivalence class $[\xi]$ of ξ, and that the association $[\xi] \mapsto \gamma_\xi$ gives a bijection between $\mathrm{EXT}(G, A)$ and $H^2(G, A)$.

ii) Let $m, n \in \mathbb{Z}_{>1}$, and let $\mathbb{Z}/m\mathbb{Z}$ act trivially on $\mathbb{Z}/n\mathbb{Z}$. We will see later in the course that

$$H^2(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/(n, m)\mathbb{Z}.$$

Prove this for $(m, n) = (2, 2), (2, 3), (3, 3)$.

Hint: Prove that every extension E of a cyclic group G by an abelian group A contained in the center of E is again an abelian group, and then use i).

Exercise 3. For a cyclic field extension L/K, show that $H^1(\mathrm{Gal}(L/K), L) = 0$.

Exercise 4. For a finite group G, we define the *character group* $G^\vee := \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$ of G. Prove that the natural map $G \to G^{\vee\vee}$ induces an isomorphism $G^{\vee\vee} = G^{\mathrm{ab}}$.