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Sommersemester 2016

ALGEBRAIC NUMBER THEORY II

Problem Set 7

Due date: 7/6/2016

Exercise 1. Let G be a finite group and A a G-module.

i) Show that $\mathrm{Ord}(G) \cdot H_T^q(G, A) = 0$ for all $q \in \mathbb{Z}$.

   Hint: Use dimension shift.

ii) Show that if the multiplication map

$$A \to A\,, \qquad a \mapsto \mathrm{Ord}(G)a$$

   is an isomorphism, then $H_T^q(G, A) = 0$.

Exercise 2. Prove that $H^2(G, \mathbb{Z}) = G^\vee$.

Hint: Consider the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ *and apply the previous exercise.*

Exercise 3. Let G be a finite group and A a finite abelian group of order coprime to the order of G. Prove that any extension

$$1 \to A \to E \to G \to 1$$

is split.

Hint: Use Exercise 1 and Exercise 2 of PS6.
Remark: More generally, the Theorem of Schur and Zassenhaus says that the condition that A be abelian can be dropped.

Exercise 4. Let $m \geq 1$ be an integer, and let k be a field whose characteristic is coprime to m. Let $\overline{k}$ be a separable closure of k, and let $G = \mathrm{Gal}(\overline{k}/k)$. Denote by $\mu_m \subseteq \overline{k}^\times$ the subgroup of $m-$th roots of unity.

i) Show that $k^\times/(k^\times)^m \simeq H^1(G, \mu_m)$.

ii) Assume that $\mu_m \subseteq k^\times$, so that the first part yields an isomorphism

$$\psi \colon k^\times/(k^\times)^m \xrightarrow{\sim} \mathrm{Hom}(G, \mu_m).$$

   Prove that the maps

$$k' \mapsto \psi^{-1}(\mathrm{Hom}(\mathrm{Gal}(k'/k), \mu_m)),$$
$$B \mapsto k(\sqrt[m]{b};\, b \in B)$$

   define a bijection between the finite abelian extensions k'/k inside $\overline{k}$ whose Galois group is annihilated by m and finite subgroups of $k^\times/(k^\times)^m$.