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Sommersemester 2016

ALGEBRAIC NUMBER THEORY II

Problem Set 8

Due date: 14/6/2016

Exercise 1. Let G be a finite group and let A be a G-module. Denote by $H^q_T(G,A)_p$ the *p-primary part* of $H^q_T(G,A)$, that is, the group of all elements whose order is a power of p. Let G_p denote a p-Sylow subgroup of G. Prove that:

i) $\mathrm{res}_q\colon H^q_T(G,A)_p \to H^q_T(G_p,A)$ is injective.

ii) $\mathrm{cor}_q\colon H^q_T(G_p,A) \to H^q_T(G,A)_p$ is surjective.

Exercise 2. Let G be a finite group and $H \subseteq G$ a subgroup. For each coset $\xi \in H\backslash G$, choose a representative $\sigma(\xi)$, i.e.,

$$G = \bigcup_{\xi \in H\backslash G} H\sigma(\xi) \quad \text{(disjoint union).}$$

For a G-module C and $c \in {}_{N_G}C$, define

$$N'_{G/H}(c) = \sum_{\xi} \sigma(\xi)\cdot c \in {}_{N_H}C\,.$$

i) Let

$$0 \to A \to B \to C \to 0$$

be an exact sequence of G-modules. Show that the diagram

$$\begin{array}{ccc} H_T^{-1}(G,C) & \xrightarrow{\ \delta\ } & H_T^0(G,A) \\ \downarrow{\scriptstyle N'} & & \downarrow{\scriptstyle \mathrm{res}_0} \\ H_T^{-1}(H,C) & \xrightarrow{\ \delta\ } & H_T^0(H,A) \end{array}$$

is commutative. Here, we define $N'(c + I_GC) = N'_{G/H}(c) + I_HC$. Recall that we know from the lecture that $\mathrm{res}_0(a + N_GA) = a + N_HA$.

ii) Deduce that $N' = \mathrm{res}_{-1}$.

iii) Let $\tau \in G$. Show that

$$\sigma(\xi)\cdot\tau\cdot\sigma(\xi\tau)^{-1} \in H \quad \text{for all } \xi \in H\backslash G.$$

Show that the transfer map $\mathrm{ver}\colon G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ is given by the formula

$$\mathrm{ver}(\tau G') = \left(\prod_{\xi\in H\backslash G} \sigma(\xi)\tau\sigma(\xi\tau)^{-1}\right) H'\,.$$

Hint: Use the identification $G^{\mathrm{ab}} = I_G/I_G^2$ *(and similarly for* H*), and that by part ii) the restriction map* $I_G/I_G^2 = H_T^{-1}(G, I_G) \to H_T^{-1}(H, I_G) = I_G/I_H I_G$ *is given by* N'*.*

Exercise 3. Let $G = \mathbb{Z}/6\mathbb{Z}$ act on $A = \mathbb{Z}/3\mathbb{Z}$ in the following way: the action of a generator of G is given by the formula $a \mapsto -a$. Show that:

i) $H_T^q(G, A) = 0$ for $q = 0, -1$. (We will see soon that this implies $H_T^q(G, A) = 0$ for every $q \in \mathbb{Z}$ since G is cyclic.)

ii) The G-module A is, however, not *cohomologically trivial*, that is, there exists $H \subseteq G$ and $q \in \mathbb{Z}$ such that $H_T^q(H, A) \neq 0$.

Exercise 4. Let $G = \mathbb{Z}/2\mathbb{Z}$ act on $A = \mathbb{Z}/8\mathbb{Z}$ in the following way: the action of the non-trivial element of G is given by the formula $a \mapsto 3a$.

i) Show that $H_T^q(G, A) = 0$ for $q = 0, -1$. (By the remark in Ex. 3 i) this implies that the G-module A is cohomologically trivial.)

ii) Let $B = \mathbb{Z}/2$ with trivial G-action. Prove that the G-module $A \otimes B$ is not cohomologically trivial.

Hint: Recall from PS6 Exercise 2 part ii) that $H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ *when* $\mathbb{Z}/2\mathbb{Z}$ *acts trivially on* $\mathbb{Z}/2\mathbb{Z}$*.*