Prof. Dr. U. Görtz
Dr. F. Fité

Sommersemester 2016

ALGEBRAIC NUMBER THEORY II

Problem Set 9

Due date: 21/6/2016

Exercise 1. Let M be an abelian group and f, g endomorphisms of M such that $f \circ g = g \circ f = 0$. The Herbrand quotient is defined as

$$q_{f,g}(M) = \frac{[\mathrm{Ker}(f) : \mathrm{Im}(g)]}{[\mathrm{Ker}(g) : \mathrm{Im}(f)]} \in \mathbb{Q}\,,$$

provided that $[\mathrm{Ker}(f) : \mathrm{Im}(g)]$ and $[\mathrm{Ker}(g) : \mathrm{Im}(f)]$ are finite.

i) Show that if M is finite, then $q_{f,g}(M) = 1$.

ii) Suppose that G is cyclic and that M is a G-module such that the cohomology groups $H^1(G, M), H^2(G, M)$ are finite. If σ is a generator of G, we write $h(M) = q_{\sigma-1,N_G}(M)$. Check that

$$h(M) = \frac{\#H^2(G, M)}{\#H^1(G, M)}\,.$$

iii) Compute $h(\mathbb{Z})$, where G acts trivially on $\mathbb{Z}$.

Exercise 2. Let G be a finite cyclic group. Show that if

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of G-modules, then $h(M) = h(M')h(M'')$, in the sense that whenever two of the three Herbrand quotients are defined, then so is the third one, and that in this case equality holds.

Exercise 3. Let L/K be a finite unramified extension of local fields with Galois group G, and let $U_L = \mathcal{O}_L^\times$. Show that $H_T^r(G, U_L) = 0$ for all $r \in \mathbb{Z}$.

Hint: Show that that there is an isomorphism $L^\times \simeq U_L \times \mathbb{Z}$ of G-modules, where we let G act trivially on $\mathbb{Z}$. Then use Ex. 3 of PS5 and Ex. 2 of PS4.

Exercise 4. Let G be a topological group and let M be a G-module. Show that the following are equivalent:

i) The map $G \times M \to M$ defined by $(g, m) \mapsto {}^g m$ is continuous, where M carries the discrete topology and $G \times M$ is endowed with the product topology.

ii) The stabilizer in G of any element $m \in M$ is open.

iii) $M = \bigcup_H M^H$, where H runs through all the open subgroups of G.