Algebraic Number Theory 4 – Formal Groups 2 SS 2017

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Problem Sheet 1

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Throughout this problem sheet, R will be a (commutative and unitary) ring. Let H, H' be two commutative affine group schemes over $\mathrm{Spec} R$, and we denote by $\mathrm{Hom}_R(H, H')$ (resp. $\mathrm{End}_R(H)$) the group of homomorphisms from H to H' (resp. endomorphisms of H).

Problem 1

Suppose that R is integral. Show that

$$\mathrm{Hom}_R(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_a) = \mathrm{Hom}_R(\mathbb{G}_a, \mathbb{G}_{\mathrm{m}}) = 0.$$

Problem 2

Suppose that R is integral. Compute $\mathrm{End}_R(\mathbb{G}_{\mathrm{m}})$.

Problem 3

Let Γ be an abstract finite group. Choose one of the axioms defining the notion of group scheme and verify it for the constant group scheme $\underline{\Gamma}$ over R.

Problem 4

Let $G = \mathrm{Spec} A$ be a commutative affine group scheme over R such that A is a locally free R-module of finite rank. Let A^* be $\mathrm{Hom}_{R-\mathrm{mod}}(A, R)$ endowed with the R-algebra structure induced from the group scheme structure of G, and $G^* := \mathrm{Spec} A^*$, the Cartier dual of G.

(1) Choose one of the axioms defining the notion of group scheme, and verify it for G^*.

(2) Let $n \geqslant 1$. Prove that $\underline{\mathbb{Z}/n\mathbb{Z}}^* \cong \mu_n$.

(3) Let now R be an $\mathbb{F}_p$-algebra, i.e., $pR = 0$ for some prime p. Show that $\underline{\mathbb{Z}/p\mathbb{Z}}$, μ_p, and α_p are pairwise non-isomorphic.