**Algebraic Number Theory 4 – Formal Groups 2** SS 2017

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## Problem Sheet 10

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Throughout this problem sheet, let $p$ be a prime number, and $k$ a (commutative) ring. We denote by $W(k)$ the ring of Witt vectors over $k$. We denote by $V_n(k)$ the ideal $V^n(W(k))$ of $W(k)$.

If $k$ is of positive characteristic $l$, then $k$ is called perfect if the map $x \mapsto x^l$, $x \in k$ is bijective.

### Problem 37

(1) Show that $FV = [p]$ and $V^m(a) \cdot b = V^m(a \cdot F^m(b))$ for any $a, b \in W(k)$.

(2) If $k$ is of characteristic $p$, then $VF = [p]$ and $V^m(a) \cdot V^n(b) = V^{m+n}(F^n(a) \cdot F^m(b))$ for any $a, b \in W(k)$.

(3) Give an example of $k$ such that $VF \neq [p]$.

### Problem 38

Let $R$ be a complete local noetherian ring with perfect residue field $k$ of characteristic $p$. Let $s : k \to R$ be any set-theoretic section of the canonical projection $\pi : R \to k$.

(1) For any $x \in k$, show that the limit $\varinjlim_{n\to\infty} s(x^{p^{-n}})^{p^n}$ exists in $R$.

(2) Let $i(x) := \varinjlim_{n\to\infty} s(x^{p^{-n}})^{p^n}$. Show that $i : k \to R$, $x \mapsto i(x)$ is the unique multiplicative section of $\pi$.

*Hint:* Let $\mathfrak{m}$ be the maximal ideal of $R$. Note that for $a, b \in R$ with $a \equiv b \mod \mathfrak{m}$, we have $a^p \equiv b^p \mod \mathfrak{m}^2$.

**Problem 39**
Let $k$ be of characteristic $p$.

(1) Show that $k$ is integral (resp. reduced) if and only if $W(k)$ is integral (resp. reduced).

(2) Show that $k$ is perfect if and only if $W(k)/pW(k)$ is reduced.

*Hint:* Make use of part (2) of Problem 37.

**Problem 40**
Suppose that $k$ is a field of characteristic $p$. Show that the following are equivalent.

(1) $W(k)$ is noetherian.

(2) $W(k)/pW(k)$ is reduced.

(3) $pW(k) = V(W(k))$.

(4) $k$ is perfect.

*Hint:* Use problem 39, and consider the dimension of the $k$-vector space $V_1(k)/V_1(k)^2$.