Algebraic Number Theory 4 – Formal Groups 2 SS 2017

Prof. Dr. Ulrich Görtz
Dr. Heer Zhao

Problem Sheet 11

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Problem 41

Let $\mathcal{C}$ be an abelian category. An object J of $\mathcal{C}$ is called injective, if the functor $\mathrm{Hom}_{\mathcal{C}}(-, J) : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}b$ is exact, where $\mathcal{A}b$ denotes the category of abelian groups.

For any two objects A and B of $\mathcal{C}$, the abelian group $\mathrm{Hom}_{\mathcal{C}}(A, B)$ carries a natural structure of left $\mathrm{End}_{\mathcal{C}}(B)$-module. We denote by $\mathrm{End}_{\mathcal{C}}(B)-\mathrm{LMod}$ the category of left $\mathrm{End}_{\mathcal{C}}(B)$-Modules.

Now let I be an injective object such that, for any $A \in \mathcal{C}$, there exists a set S and a monomorphism $A \to \prod_S I$. Show that the functor

$$\mathrm{Hom}_{\mathcal{C}}(-, I) : \mathcal{C}^{\mathrm{op}} \to \mathrm{End}_{\mathcal{C}}(I)-\mathrm{LMod},\ A \mapsto \mathrm{Hom}_{\mathcal{C}}(A, I)$$

is a faithful exact functor.

Problem 42

Let k be a field of positive characteristic p.

(1) Express α_{p^r} as a W_n^m.

(2) Prove that the Cartier dual $\alpha_{p^2}^*$ of α_{p^2} is isomorphic to W_2^1.

Problem 43

Let $\mu(n)$ be the Möbius function defined by

$$\mu(n) = \begin{cases} (-1)^{(\text{number of prime divisors of } n)}, & \text{if } n \text{ is square-free;} \\ 0, & \text{otherwise.} \end{cases}$$

Let p be a fixed prime number, we define

$$F(t) := \prod_{p \nmid n} (1 - t^n)^{\frac{\mu(n)}{n}},$$

where

$$(1-t^n)^a := \sum_{k\geqslant 0}\binom{a}{k}(-t^n)^k, \text{ with } \binom{a}{k} := \frac{a\cdot(a-1)\cdots(a-k+1)}{k!}.$$

(1) Show that $\exp(-t) = \prod_{n\geqslant 1}(1-t^n)^{\frac{\mu(n)}{n}}$ in $1+t\cdot\mathbb{Q}[[t]]$.

(2) Show that $F(t)\in 1+t\cdot\mathbb{Z}_{(p)}[[t]]$.

(3) Show that

$$F(t) = \exp(-\sum_{m\geqslant 0}\frac{t^{p^m}}{p^m}).$$

Hint: (1) Use that the Möbius function satisfies

$$\sum_{d|n}\mu(d) = \begin{cases} 1, & \text{if } n=1; \\ 0, & \text{otherwise.}\end{cases}$$

(2) First show that $\binom{a}{k}\in\mathbb{Z}_{(p)}$ for $a\in\mathbb{Z}_{(p)}$.
(3) Show $\log F(t) = -t + \frac{1}{p}\log F(t^p)$ first, and iterate.

Problem 44

Let R be a ring, and $\Lambda_R := \prod_{d\geqslant 1}\mathbb{A}^1_R = \operatorname{Spec} R[U_1, U_2,\cdots]$. We identify sequences $(u_1, u_2,\cdots)$ with power series $1+u_1t+u_2t^2+\cdots$, and use the product of formal power series to make Λ_R into a group scheme over R. Let W_R be the additive group scheme associated to the Witt ring scheme W over R.

Show that the morphism

$$W_{\mathbb{Z}_{(p)}}\to\Lambda_{\mathbb{Z}_{(p)}}, \underline{x} = (x_0, x_1,\cdots)\mapsto E(\underline{x}, t) := \prod_{n\geqslant 0}F(x_n\cdot t^{p^n})$$

is a homomorphism of group schemes over $\mathbb{Z}_{(p)}$.

Hint: First show that $E(\underline{x},t) = \exp\big(-\sum_{l\geqslant 0}w_l(x_0,\cdots,x_l)\frac{t^{p^l}}{p^l}\big)$, where w_l is the l-th Witt polynomial.