Algebraic Number Theory 4 – Formal Groups 2 SS 2017

Prof. Dr. Ulrich Görtz
Dr. Heer Zhao

Problem Sheet 12

Due date: July. 19, 2017

Throughout this problem sheet, k will be a perfect field of positive characteristic p. We will denote the Witt ring scheme over k simply by W. We will denote by E the Dieudonné ring over $W(k)$.

Problem 45

For all positive integers m and n, consider the morphisms

$$\tau_n^m : W_n^m \to W, (x_0, \cdots, x_{n-1}) \mapsto (x_0, \cdots, x_{n-1}, 0, 0, \cdots).$$

Their images form a system of infinitesimal neighborhoods of 0 inside W, and let $\widehat{W} := \varinjlim_{n,m} \tau_n^m(W_n^m)$ be the formal scheme associated to this system. Show that:

(1) For any finite length artinian k-algebra R, $\widehat{W}(R)$ consists of elements $(x_i)_i \in W(R)$ such that all components x_i are nilpotent and almost all are zero.

(2) The addition in W induces a group structure on $\widehat{W}$.

(3) The multiplication in W induces a morphism $W \times \widehat{W} \to \widehat{W}$ which is clearly compatible with the ring structure of W and the group structure of $\widehat{W}$, in other words $\widehat{W}(R)$ is an ideal of $W(R)$ for any finite length artinian k-algebra R.

Hint: For part (2) observe that setting $\deg(X_i) = \deg(Y_i) = p^i$, the Witt polynomials w_ℓ, and hence the polynomials S_ℓ giving the sum of Witt vectors, are homogeneous of degree p^ℓ. For part (3), a suitable variant of this argument is useful.

Problem 46

Let $E(\underline{x}, t)$ be as in Problem 44. Show that the morphism

$$\widehat{W} \to \hat{\mathbb{G}}_{\mathrm{m},k}, \ \underline{x} \mapsto E(\underline{x}, 1)$$

is well-defined, and actually a homomorphism of formal groups, where $\hat{\mathbb{G}}_{\mathrm{m},k}$ denotes the formal completion of the group scheme $\mathbb{G}_{\mathrm{m},k}$.

Problem 47

Let G be a finite group scheme over k of local-local type and of order p^2. Prove that $[p]_G = 0$.

Problem 48

Assume that k is algebraically closed. Determine all isomorphism classes of local-local finite group schemes G satisfying $\mathrm{ord}(G) = p^2$.

Hint: Let M be the Dieudonné module associated to G. Consider the problem in two cases $F \cdot M = 0$ and $F \cdot M \neq 0$. The case $F \cdot M = 0$ is easy.

Suppose that $F \cdot M \neq 0$. Since $[p]_G = 0$ by Problem 47, M is naturally a module over $W(k)/pW(k) = k$. Then $F \cdot M = k \cdot e$ and $e = F(f)$ for some $e, f \in M$. With respect to the basis $\{e, f\}$, F can be expressed as

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \sigma.$$

Describe V with respect to this basis, and study what happens under change of basis.