Algebraic Number Theory 4 – Formal Groups 2 SS 2017

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Problem Sheet 2

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Throughout this problem sheet, all group schemes are over a field k and S_r denotes the abstract symmetric group on a finite set of r elements.

Problem 5

Let n be a positive integer. Give explicit (closed) embeddings of $\underline{\mathbb{Z}/n\mathbb{Z}}$ and μ_n into GL_m for some m. In the case that k is of positive characteristic p, do the same for α_p.

Hint: For α_p, try to embed $\mathbb{G}_a$ first.

Problem 6

Let G be a finite group scheme over k. Show that G can be embedded as a closed subgroup scheme of GL_n for some n.

Hint: Look for a finite-dimensional k-vector space R with a linear G-action such that the corresponding morphism $G \to GL(R)$ is a closed embedding. (Here we denote by $GL(R)$ the group scheme over k with S-valued points $GL(R)(S) = GL(R \otimes_k S)$, the group of S-module automorphisms of $R \otimes S$; an isomorphism $R \cong k^n$ induces an isomorphism $GL(R) \cong GL_n$.)

Remark: The same is true if G is any affine group scheme of finite type over k.

Problem 7

Let $\underline{S_n}$ act on the affine space $\mathbb{A}^n$ via $\sigma \cdot (a_1, \cdots, a_n) := (a_{\sigma(1)}, \cdots, a_{\sigma(n)})$ for $\sigma \in S_n$ and $(a_1, \cdots, a_n) \in \mathbb{A}^n(R)$ for R any k-algebra such that $\mathrm{Spec} R$ is irreducible. Show that the quotient for this action is isomorphic to $\mathbb{A}^n$.

Hint: Use the fundamental theorem on symmetric polynomials.

Problem 8
Let k be the field $\mathbb{C}$ of complex numbers. Let $\mu_{n,\mathbb{C}}$ act on $\mathbb{A}^2_{\mathbb{C}}$ via $\zeta \cdot (a, b) := (\zeta a, \zeta^{-1} b)$, where ζ denotes an n-th root of unity, and $(a, b) \in \mathbb{C}^2$. Show that the quotient of this action is isomorphic to $\mathbb{C}[U, V, W]/(UW - V^n)$.

Remark: Problem 7 is an example of smooth quotient, while this problem offers an example of singular quotient.