Algebraic Number Theory 4 – Formal Groups 2 SS 2017

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Problem Sheet 3

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Convention

For two (affine) group schemes G and H over a ring R, we denote by $\mathrm{Hom}_R(G, H)$ the set of group scheme homomorphisms from G to H. In case that G is finite locally free over R and commutative, we denote by G^* the Cartier dual of G. For a morphism $f : X \to \mathrm{Spec}A$ of schemes, $f^\sharp$ denotes the corresponding ring homomorphism $A \to \Gamma(X, \mathcal{O}_X)$.

Problem 9

Let $\mathcal{C}$ be an abelian category, and f a morphism in $\mathcal{C}$.

(1) Prove that the following are equivalent:

  (a) f is a kernel;

  (b) f is a monomorphism;

  (c) $\mathrm{Ker}(f) = 0$.

(2) Prove that the following are equivalent:

  (a) f is a cokernel;

  (b) f is an epimorphism;

  (c) $\mathrm{Coker}(f) = 0$.

(3) Prove that the following are equivalent:

  (a) f is an isomorphism;

  (b) f is a monomorphism and an epimorphism.

Problem 10
Let $R := \mathbb{F}_p[u]$ with p a prime number, $G = \alpha_{p,R} = \mathrm{Spec}A$ with $A := R[T]/(T^p)$. Show that $T \mapsto u \cdot T$ induces an endomorphism f of the group scheme G over R such that $\mathrm{Ker}(f)$ is not flat over R.

Problem 11
Let $p > 2$ be a prime number, and let $G = \mu_{p,\mathbb{Q}}$. Determine the number of points of the topological space $|G|$. Show that G is not a constant group scheme, but that there exists a finite Galois extension $K/\mathbb{Q}$ such that $G \otimes_{\mathbb{Q}} K$ is constant.

Problem 12
Let G be a (affine) group scheme over a field k. The tangent space $T_{G,e}$ of G at its identity e can be defined as $\mathrm{Ker}(G(k[t]/(t^2)) \to G(k))$.

Now assume that G is finite. Show that $T_{G,e} = \mathrm{Hom}_k(G^*, \mathbb{G}_a)$.

Hint: Let $G = \mathrm{Spec}A$, and μ (resp. ε) the comultiplication (resp. counit) of A. Let $G^* = \mathrm{Spec}A^*$, and μ^* (resp. ε^*) the comultiplication (resp. counit) of $A^* = \mathrm{Hom}_{k-\mathrm{Mod}}(A, k)$. Identify any element α of $T_{G,e}$ with an element λ of A^* satisfying certain properties. Note that λ determines a morphism $G^* \to \mathbb{A}^1_k$ of schemes over k, and such properties are exactly the ones making this morphism into a homomorphism $G^* \to \mathbb{G}_a$ of group schemes over k.