Algebraic Number Theory 4 – Formal Groups 2 SS 2017

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Problem Sheet 4

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Problem 13

Give an example of a finite flat group scheme over R with connected special fiber and non-connected generic fiber over R, in each of the following two cases:

(1) $R = \mathbb{Z}_p$, the ring of p-adic integers.

(2) $R = \mathbb{F}_p[[t]]$.

Problem 14

Let k be a field of characteristic $p > 0$. Let $G := \mathrm{GL}_{n,k} = \operatorname{Spec} A$ with $A := k[T_{i,j}, U]/(U \cdot \det((T_{i,j})_{i,j}) - 1)$, I the kernel of the counit homomorphism $\varepsilon_G : A \to k$, and $[p]_G$ the morphism (not a group homomorphism unless $n = 1$!) given by $g \mapsto g^p$ on points. Denote by $[p]_G^\sharp$ the corresponding ring homomorphism.

(1) Show that $[p]_G^\sharp(I) \subset I^p$.

(2) Let $H = \operatorname{Spec} B$ be a finite group scheme over k, and J the kernel of the counit homomorphism $\varepsilon_H : B \to k$. Show that $[p]_H^\sharp(J) \subset J^p$.

Hint: For part (1), make use of the equality $(g+1)^p = g^p + 1$ for g any point of G. By Problem 6, H can be embedded into $\mathrm{GL}_{r,k}$ for some r. Make use of part (1) for part (2).

Problem 15

Let k be a field of positive characteristic p, $G = \operatorname{Spec} A$ a connected group scheme over k. Let $N = \operatorname{ord}(G)$. Show that $[N]_G = 0$ (where $[N]_G(g) := g^N$, assuming G is written multiplicatively).

Hint: You may use without proof that N is a p-power, say $N = p^n$. Let I be the kernel of the counit homomorphism $\varepsilon_G : A \to k$. Note that $I^{p^n} = 0$. Deduce $[p^n]_G^\sharp(I) = 0$ using Problem 14.

Problem 16

Let k be a field, and let $k' = k(\sqrt[p]{u})$ be a purely inseparable extension of degree p, $u \in k$. For $i \in \mathbb{F}_p$, let

$$G_i = \operatorname{Spec} k[T]/(T^p - iu),$$

and $G = \coprod_{i \in \mathbb{F}_p} G_i$. Then G is a group scheme via

$$G_i \times G_j \to G_{i+j}, \quad (t, t') \mapsto t + t'.$$

Show that $G^0 \cong G_0 \cong \alpha_{p,k}$, and that $G/G^0 \cong \underline{\mathbb{Z}/p\mathbb{Z}}$, but that $G \not\cong \alpha_{p,k} \times \underline{\mathbb{Z}/p\mathbb{Z}}$.