Algebraic Number Theory 4 – Formal Groups 2 SS 2017

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Problem Sheet 5

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Problem 17

Let k be a field of characteristic $p > 0$, and n a positive integer. Let G and H be one of the three group schemes $\alpha_p, \mu_p, \underline{\mathbb{Z}/n\mathbb{Z}}$ over k. Compute $\mathrm{Hom}_k(G, H)$.

Problem 18

(1) Let $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \langle \tau \rangle$ act on $\mathrm{Mat}_2(\mathbb{C})$ by

$$\tau \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix},$$

where $\bar{\cdot}$ $(= \tau(\cdot))$ denotes complex conjugation. Show that $\mathbb{H} := \mathrm{Mat}_2(\mathbb{C})^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}$ is the Hamilton quaternion algebra, in particular it is a 4-dimensional $\mathbb{R}$-algebra which is not isomorphic to $\mathrm{Mat}_2(\mathbb{R})$, but such that $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{Mat}_2(\mathbb{C})$.

(2) Let p be an odd prime number, and $K := \mathbb{Q}[\zeta_p]$ with ζ_p a primitive p-th root of unity. Specify an action of $\mathrm{Gal}(K/\mathbb{Q})$ on $\underline{\mathbb{Z}/p\mathbb{Z}}_{\mathrm{Spec} K}$ such that the group scheme over $\mathbb{Q}$ obtained via Galois descent is $\mu_{p,\mathbb{Q}}$.

Problem 19

Let R be a ring of characteristic $p > 0$. Let E be the closed subgroup scheme of GL_2 over R defined by

$$E(B) = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(B) \,\middle|\, x^p = 1, y^p = 0 \right\} \subset \mathrm{GL}_2(B)$$

for any R-algebra B. Show that E fits into a short exact sequence

$$0 \to \alpha_p \to E \to \mu_p \to 0$$

and E is non-commutative of order p^2. Here by "short exact sequence" we mean that α_p is a closed subgroup scheme of E such that μ_p is the quotient (as in the lectures).

Remark: Recall that an abstract group of order p^2 must be commutative. This problem shows that this is not the case for group schemes.

Problem 20

Let G be a finite étale group scheme of order p^2 over a ring R. Show that G is commutative.