Algebraic Number Theory 4 – Formal Groups 2 SS 2017

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Problem Sheet 6

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Problem 21

Let k be either a finite field or the field of rational numbers $\mathbb{Q}$, let $\bar{k}$ be a fixed algebraic closure of k, and n a positive integer. Determine the number in $\mathbb{N} \cup \{\infty\}$ of finite étale group schemes G such that $G \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k} \cong \underline{\mathbb{Z}/n\mathbb{Z}}_{\operatorname{Spec} \bar{k}}$.

Problem 22

Let k be a field of positive characteristic p, and let G be one of the following group schemes: $\mathbb{G}_{\mathrm{m}}$, $\mathbb{G}_a$, μ_p, α_p and $\underline{\mathbb{Z}/p\mathbb{Z}}_{\operatorname{Spec} k}$. Compute the kernel of the relative Frobenius F_G. For the last three cases, compute also the kernel of the Verschiebung V_G.

Problem 23

Let k be a field of positive characteristic p, and G a finite commutative group scheme over k of order coprime to p. Show that G is étale.

Hint: Make use of Problem 12 to compute the tangent space of G. Use that $[\operatorname{ord}(G)]_G = 0$.

Problem 24

Let k be a field of positive characteristic p, G a finite group scheme over k. Let $G^{(p^n)} := (G^{(p^{n-1})})^{(p)}$ for $n > 1$, and F_G^n the composition

$$G \xrightarrow{F_G} G^{(p)} \xrightarrow{F_{G^{(p)}}} G^{(p^2)} \xrightarrow{F_{G^{(p^2)}}} \cdots \xrightarrow{F_{G^{(p^{n-1})}}} G^{(p^n)}.$$

Show that G is connected if and only if $F_G^n = 0$ for some positive integer n.

Hint: Express the affine coordinate ring A of G as $k[X_1, \cdots, X_r]/J$ such that the kernel of the counit $\epsilon : A \to k$ is generated by the

X_i's. Then try to understand the meaning of $F^n_G = 0$ in terms of this description.