Algebraic Number Theory 4 – Formal Groups 2 SS 2017

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Problem Sheet 7

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Problem 25

Let k be a perfect field of positive characteristic p. Show that the group scheme $\alpha_{p^n,k} := \operatorname{Ker}(\mathbb{G}_{a,k} \xrightarrow{F^n_{\mathbb{G}_{a,k}}} \mathbb{G}_{a,k}^{(p^n)}) = \operatorname{Spec} k[X]/(X^{p^n})$ is of local-local type.

Hint: One possible approach is to show that there is a short exact sequence $0 \to \alpha_p \to \alpha_{p^n} \to \alpha_{p^{n-1}} \to 0$ and to do induction.

Problem 26

Let $G = \operatorname{Spec} A$ be an affine group scheme over a ring R. Show that there is a canonical isomorphism

$$\Omega^1_{A/R} = (\Omega^1_{A/R} \otimes_A R) \otimes_R A,$$

where the inner tensor product is taken with respect to the map $A \to R$ giving the neutral element of G. In particular, if R is a field, then $\Omega^1_{A/R}$ is a free A-module.

Hint: Consider the diagram

$$\begin{array}{ccccccc}
G & \xrightarrow{(\mathrm{id},e)} & G \times G & \xrightarrow{(\mathrm{id},m)} & G \times G & \xrightarrow{\mathrm{pr}_2} & G \\
 & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow \\
 & & G & \xrightarrow{\mathrm{id}} & G & \longrightarrow & \operatorname{Spec} R
\end{array}$$

and use that the module of Kähler differentials is compatible with base change and stable under isomorphisms.

Problem 27

Let k be a field of positive characteristic p, and let $G = \operatorname{Spec} A$ be a connected finite group scheme over k. Show that the order $\mathrm{ord}(G)$ of G is a power of p.

Hint: First reduce to the case that $F_G = 0$ by considering the short exact sequence

$$0 \to \mathrm{Ker}(F_G) \to G \to H \to 0$$

and using Problem 24. If $F_G = 0$, then $A \cong k[X_1, \cdots, X_n]/J$ such that $(X_1^p, \ldots, X_n^p) \subset J$ and $X_1 \bmod m^2, \cdots, X_n \bmod m^2$ form a basis of the cotangent space m/m^2 of A. (Here m denotes the maximal ideal of A.) Now use that

$$\Omega^1_{A/k} = \bigoplus_{i=1}^{n} A \cdot dX_i/(df, f \in J)$$

is free of rank n over A by Problem 26 to show that $J = (X_1^p, \ldots, X_n^p)$.

Problem 28

Let E and R be as in Problem 19, and suppose that $R = \mathbb{F}_p$. Show that E cannot be lifted to a finite flat group scheme over $\mathbb{Z}_p$.

Hint: Use Problem 20.