Algebraic Number Theory 4 – Formal Groups 2 SS 2017

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Problem Sheet 8

Due date: June. 22, 2017

Problem 29

Let k be a field, and k^{s} a separable closure of k. Let G be a p-divisible group over k of height h. The Tate module of G is defined to be

$$T_pG = \varprojlim_i G[p^i](k^s).$$

Show that:

(1) T_pG is a free $\mathbb{Z}_p$-module of rank at most h, endowed with a continuous $\mathrm{Gal}(k^{\mathrm{s}}/k)$-action;

(2) the association of T_pG to G induces an equivalence from the category of étale p-divisible groups over k to the category of finite rank free $\mathbb{Z}_p$-modules endowed with a continuous $\mathrm{Gal}(k^{\mathrm{s}}/k)$-action.

Problem 30

Let k be a field of positive characteristic p, and $G = (G_i)_i$ a p-divisible group over k with $G_i = \operatorname{Spec} A_i$. We can associate to G a commutative formal group $\operatorname{Spf} A$ over k with $A := \varprojlim_i A_i$ in a functorial way. Show that the functor associating $\operatorname{Spf} A$ to G is fully faithful, and has essential image the commutative formal groups $\mathscr{G}$ over k satisfying: (1) $[p]_{\mathscr{G}}$ is an epimorphism; (2) $\mathscr{G} = \varinjlim \operatorname{Ker}([p^i]_{\mathscr{G}})$; (3) $\operatorname{Ker}([p]_{\mathscr{G}})$ is a finite group scheme over k.

Problem 31

Let R be a complete local noetherian ring, $\mathfrak{m}$ the maximal ideal of R and k the residue field of R. Suppose that k is of positive characteristic p. Let X be a 1-dimensional commutative formal group law over R. Show that the following are equivalent:

(1) the ring homomorphism $[p]_X^* : R[[T]] \to R[[T]]$ makes $R[[T]]$ a free module of finite rank over itself;

(2) $X_k := X \otimes_R k$ as a formal group law over k has finite height.

Hint: Apply the Weierstrass Preparation Theorem to the formal power series $[p]_X(T)$. (The Weierstrass Preparation Theorem for the formal power series ring $R[[T]]$ over R can be proven by the same way as in Problem 17 of Algebraic Number Theory III.)

Problem 32

Let R, $\mathfrak{m}$, and X be as in Problem 31. Let $\mathfrak{p}$ be a prime ideal of R with residue field $\kappa(\mathfrak{p})$. Assume that X is a p-divisible connected formal group, so that it corresponds to the p-divisible group $X[p^\infty] = (X[p^i])_i$. Show that

$$(X[p^\infty] \otimes_R \kappa(\mathfrak{p}))^\circ = (X \otimes_R \kappa(\mathfrak{p}))[p^\infty].$$

Hint: Apply the Weierstrass Preparation Theorem to the formal power series $[p^i]_X(T)$.