Algebraic Number Theory 4 – Formal Groups 2 SS 2017

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Problem Sheet 9

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Problem 33

Let k be a field of positive characteristic p, and $G = (G_i)_i$ a p-divisible group over k. The Lie algebra of G is defined to be $\operatorname{Lie} G := \operatorname{Lie} G_1$, and the dimension of G is defined to be $\dim(G) := \dim_k(\operatorname{Lie} G)$. Show that

$$\dim(G) = \dim(\widehat{G^\circ}) = \log_p \operatorname{ord}(\operatorname{Ker}(F_G)),$$

where $\widehat{G^\circ}$ denotes the formal (Lie) group attached to the connected component of G.

Problem 34

Let k be a field of positive characteristic p, and G a p-divisible group over k with Cartier dual G^*. Show that $\operatorname{ht}(G) = \dim(G) + \dim(G^*)$.

Problem 35

Let p be a prime number. Recall the Witt polynomials as defined in the lectures:

$$\begin{aligned}
&w_0(X_0) = X_0;\\
&w_1(X_0, X_1) = X_0^p + pX_1;\\
&\quad\vdots\\
&w_n(X_0, X_1, \cdots, X_n) = X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^n X_n;\\
&\quad\vdots
\end{aligned}$$

Let $\Phi(X, Y) \in \mathbb{Z}[X, Y]$. Show that there exists a unique sequence of polynomials $(\phi_n)_{n\in\mathbb{N}}$ with $\phi_n \in \mathbb{Z}[X_0, \cdots, X_n, Y_0, \cdots, Y_n]$, such that

$$\begin{aligned}
&\Phi(w_n(X_0, \cdots, X_n), w_n(Y_0, \cdots, Y_n))\\
=&w_n(\phi_0(X_0, Y_0), \cdots, \phi_n(X_0, \cdots, X_n, Y_0, \cdots, Y_n)).
\end{aligned}$$

Hint: The existence and uniqueness of $(\phi_n)_{n\in\mathbb{N}}$ is clear in the ring of polynomials with coefficients in $\mathbb{Z}[1/p]$. Then it comes down to showing that the coefficients of the ϕ_n lie in $\mathbb{Z}$. We can use induction on n. Note that

$$\begin{aligned} w_{i+1}(X_0, \cdots, X_{i+1}) &= w_i(X_0^p, \cdots, X_i^p) + p^{i+1}X_{i+1} \\ &\equiv w_i(X_0^p, \ldots, X_i^p) \mod p^{i+1}. \end{aligned}$$

Problem 36

Let k be a field, and let A be a topological k-algebra. Assume that as a topological k-vector space, A is the inverse limit of finite dimensional k-vector spaces A_i, where we equip each A_i with the discrete topology. Let $N \subseteq A$ be an open k-subvector space. Let $\bar{m}$ be the composition $A \times A \to A \to A/N$ of the multiplication of A and the projection.

(1) Show that there exists an open k-subvector space $N' \subseteq A$ with $\bar{m}(A \times N') = \{0\}$.

(2) Conclude that there exists an open ideal of A which is contained in N.

(3) Show that A, as a topological k-algebra, is a profinite k-algebra, i.e. it is the inverse limit of finite k-algebras (where we equip finite k-algebras with the discrete topology).

Hint: (1) By continuity, we can find open k-subvector spaces M' and N' of A such that $M' \times N'$ is mapped to 0 under $\bar{m}$. Since A/M' is a finite-dimensional k-vector space, we get $A = (b_1 + M') \oplus \cdots \oplus (b_r + M')$ for some elements $b_1, \ldots, b_r \in A$. Note that $\bar{m}(b_i, 0) = 0$. Again using continuity, we can achieve $\bar{m}(\{b_i\} \times N') = \{0\}$ for each i by shrinking N', if necessary.