Prof. Dr. U. Görtz **SS 2017**

Seminar on Rapoport-Zink spaces

In this seminar, we want to understand (part of) the book [RZ] by Rapoport and Zink. More precisely, we will study the definition and properties of the *period morphism* (loc. cit. 5.16)

$$\breve{\pi}^1 \colon \breve{\mathcal{M}}^{\mathrm{rig}} \longrightarrow \breve{\mathcal{F}}^{\mathrm{wa}}.$$

Before we come to its definition (Talk 9), we will have to take our time to understand the definitions of the source and the target, though:

- The space $\mathcal{M}$ is a formal scheme representing a moduli functor of p-divisible groups (which is very close to a natural generalization of the formal schemes we have seen in the course on formal groups), Talks 4, 5, 7.
- The superscript $-^{\mathrm{rig}}$ means that we pass from a formal scheme to its "generic fiber", a "rigid analytic space", Talk 8.
- The weakly admissible locus $\breve{\mathcal{F}}^{\mathrm{wa}}$ is another rigid analytic space which arises as a subspace of a flag variety, Talk 10.

Along the way, we will see methods to analyze the étale-local structure of the formal schemes (so-called local models, Talk 6), and at the end we will discuss the image of the period morphism, Talk 11. (If there are enough people who are interested in a continuation, then next term we could look at the uniformization results for Shimura varieties presented in the final chapter of [RZ].)

Some of the descriptions of the talks are a little sketchy, so feel free to ask for further details. Most of the references are freely available electronically (see the links on the web page of the seminar). If you have difficulties finding any of the others, please contact me

ECTS points: If applicable, the seminar counts as a Master's seminar (9 ECTS points).

1. Introduction

2. p-divisible groups

We start with some basics about p-divisible groups. Give the definition of a p-divisible group (over a scheme S, and over a formal scheme $\mathfrak{X}$). Define and discuss the notion of quasi-isogeny. Explain (/sketch) how the notion of p-divisible group is related to the notion of formal group law which we studied last term.

References: [Wa] 1.1, 1.2, 2.1; [RZ] 2.1–2.10; see also [F].

3. Crystals and Grothendieck-Messing theory

The one most important theorem about the deformation theory of p-divisible groups which we will use in several places is the Theorem of Grothendieck and Messing.

We can only sketch the theory here (so it is particularly important that you give clear statements of the main theorems in a form which we can use later on). We do not want to go into the definition of the crystalline site and of crystalline cohomology.

We exclude the case of residue chaacteristic 2 here to avoid certain technical difficulties.

Generally, Weinstein's survey ([We] pp. 5 (bottom) – 12) seems quite useful.

Give the definition of the crystal $\mathbf{D}(G)$ of a p-divisible group G (it seems feasible to give both variants that Weinstein gives (p. 6 and p. 12); we ignore the question why they coincide). State the definition of the universal vector extension, but do not prove its existence.

You should explain the *crystalline nature* of the construction ([We], Lemma 1, Theorem 3 and its application on p. 8/9 to show that $M(\mathscr{G}_0)$ is independent of the choice of lift).

The main theorems you should state are

Dieudonné theory: Theorem 4 in [We] (p. 10).

Grothendieck-Messing theory: Let A be a local Artin ring with residue class field of characteristic > 2, and let $A_0 = A/I$, where $I \subseteq A$ is an ideal with $I^2 = 0$. Let G_0 be a p-divisible group over A_0. We have the free A_0-module $\mathbf{D}(G_0)(A_0)$ which comes with a surjection

$$\mathbf{D}(G_0)(A_0) \to \mathrm{Lie}(G_0)$$

of free A_0-modules. Let us denote its kernel by $\mathscr{F}_0$.

Then there exists a lift G of G_0 to A (i.e., $G \otimes_A A_0 \cong G$), and the free A-module $\mathbf{D}(G)(A)$ is independent of the choice of lift (this is another instance of the crystalline nature of this construction; in fact, $\mathbf{D}(G)(A)$ is equal to $\mathbf{D}(G_0)(A)$, the crystal of G_0 evaluated at A).

We have an equivalence of categories

$$\{\text{lifts } H \text{ of } G_0 \text{ to } A\} \longrightarrow \{\mathscr{F} \subseteq \mathbf{D}(G_0)(A) \text{ a direct } A\text{-module summand; } \mathscr{F} \otimes_A A_0 = \mathscr{F}_0\}$$
$$H \mapsto \ker(\mathbf{D}(G_0)(A) = \mathbf{D}(H)(A) \to \mathrm{Lie}(H)).$$

It would be nice if you point out how Grothendieck-Messing theory gives an easy solution to computing the tangent space of the deformation functor of a p-divisible group (in particular, of a formal group of finite height — the ad hoc computation of this tangent space took up quite some time in the course on formal groups).

References: [We], [Wa] 1.3. For further details, you could look at the Springer Lecture Notes in Math. volumes by Messing (vol. 264), Mazur and Messing (vol. 370), Berthelot, Breen and Messing (vol. 930).

4. Moduli of quasi-isogenies

In this talk we prove the fundamental representability result for moduli spaces of quasi-isogenies of p-divisible groups ([RZ] Thm. 2.16). For the proof, you will need to explain some of the notions from the beginning of loc. cit., Ch. 1 (unless they were already explained in previous talks). The representability by some formal scheme is explained in 2.22; but the finiteness statements in the theorem are of course an essential part, and are proved in 2.23–2.29, using the preparations in 2.11–2.15.

If there is enough time, then you should explain how this construction relates to the representability results of the course on formal groups, see [RZ] Prop. 3.79. (This could also be done in Talk 7.)

Also mention the result of Prop. 2.32.

References: [RZ] 2.11 – 2.32; [Wa] Thm, 2.2.1.

5. Rapoport-Zink spaces

The moduli space constructed in the previous talk is a particular case of what is nowadays usually called a *Rapoport-Zink space*. In this talk, we will generalize the concept, however, to avoid the notation becoming too cumbersome, we will not handle the most general case of [RZ]. Specifically, instead of considering general (EL) and (PEL) data ([RZ] 1.38, 1.39) and, correspondingly multi-chains of lattices as in the beginning of [RZ] Ch. 3, we will "drop the L(evel structure)" for the moment and only consider the cases (E) and (PE) as in [Wa] 2.2.

Explain how to reduce the proof of representability to the main theorem of the previous talk, [RZ] Thm. 3.25. Explain the Kottwitz condition Def. 3.21 (iv), see loc. cit. 3.23.

References: [Wa] 2.2, [RZ] Ch. 3, in particular Def. 3.21, Thm. 3.25.

6. Local models

Local models are schemes which are defined in terms of linear algebra (i.e., which are closed subschemes of products of Grassmannians, defined by "simple" conditions), and whose formal completion is étale-locally isomorphic to the corresponding Rapoport-Zink space.

First, explain the principle of relating a given Rapoport-Zink space to its local model. (In [RZ], the result is phrased in an equivalent but slightly different way.) So fix a RZ space $\mathcal{M} = \{(X,\rho)\}$ (as in the previous talk) and consider the corresponding local model $\mathbf{M}^{\mathrm{loc}}$ (as in [RZ] Def. 3.27). There is a small mistake in that definition: correspondingly to the periodicity isomorphisms of the lattice chain, one needs to impose compatibility isomorphisms on the t_Λ. However, at this point in the talk the definition should be simplified anyway, in order to match our setup where there is only a single Λ (up to homothety). As in [RZ], we denote by $\widehat{\mathbf{M}}^{\mathrm{loc}}$ the p-adic completion of $\mathbf{M}^{\mathrm{loc}} \otimes O_{\breve{E}}$, and define

$$\mathcal{N} = \{(X,\rho,\gamma\colon M \xrightarrow{\cong} \mathcal{O}_S^h)\}$$

as in [RZ] Def. 3.28. We obtain a forgetful morphism $\mathcal{N} \to \mathcal{M}$ which is a torsor under (a certain subgroup of) GL_h (which is smooth). Here we use loc. cit., Thm. 3.11 and Thm. 3.16; note that they are easy in our simplified situation.

On the other hand, we have a morphism

$$\mathcal{N} \to \widehat{\mathbf{M}}^{\mathrm{loc}}, \quad (X,\rho,\gamma) \mapsto (\mathcal{O}_S^h \cong M \to \mathrm{Lie}(X))$$

which is smooth by Grothendieck-Messing theory. We obtain the *local model diagram*

$$\breve{\mathcal{M}} \longleftarrow \mathcal{N} \longrightarrow \widehat{\mathbf{M}}^{\mathrm{loc}}.$$

(The flatness conjecture right before loc. cit. 3.36 has turned out to be too optimistic in the general case; see [PRS] for a detailed survey.)

In the second part, give some examples of local models. To illustrate some interesting points of the theory, we will not aim at being completely consistent with the previous talk, i.e., we will look at local models which do incorporate the (L)evel structure which was omitted. Suitable examples to look at would be [PRS] 2.1, 2.2 and 2.4. Maybe you can sketch the connection to the affine flag variety.

Another interesting case would be the local model in the Drinfeld case, [RZ] 3.76.

References: [RZ], [PRS]

7. Examples: The Lubin-Tate case and the Drinfeld case

If it has not yet been discussed in Talk 4, then you should explain how this construction relates to the representability results of the course on formal groups, see [RZ] Prop. 3.79. (The *Lubin-Tate case.*)

The largest part of the talk should be dedicated to explaining the *Drinfeld case*, [RZ], 3.54 – 3.77. (You will also need to look up some notation from Chapter 1 there, but we will postpone the computation of the weakly admissible locus.)

Discuss the notion of special formal O_D-module and show that the RZ space in this case coincides with Drinfeld's moduli space (except for the point that Drinfeld restricts to quasi-isogenies of height 0). After that, you could discuss some of the special features of this case:

The formal scheme $\breve{\mathcal{M}}$ is p-adic (unlike, for instance, the Lubin-Tate case), [RZ] Cor. 3.63. You should quote Prop. 3.62 without proof (since it relies on Cartier theory which we have not covered, and would take up too much time anyway). Instead, it would be good to explain the proof of the corollary, and to explain why rigidity of quasi-isogenies ([RZ], (2.1)) is not enough to get the statement of Prop. 3.62.

Furthermore, in the Drinfeld case it is possible to describe the formal scheme $\breve{\mathcal{M}}$ very explicitly ([RZ] Thm. 3.72, see also [BC] I, II). Again it will not be possible to give the proof of Drinfeld's theorem. For $d = 2$, draw a picture. Another point would be that the local structure coincides with the local structure of the corresponding local model ([RZ] 3.69, in particular (3.13), and 3.76).

8. Rigid Geometry

To define the period morphism, we will need to pass to the "generic fiber" of the formal schemes arising as Rapoport-Zink spaces. Here, by the generic fiber of a formal scheme one means a rigid analytic space, so we will need to familiarize ourselves with this notion, and with the construction of the generic fiber of a formal scheme after Raynaud and Berthelot. For now, we will work in the setting of "classical" rigid analytic spaces, because that is the simplest version of the theory.

References: [Wa] 2.3, [Bo], [Co] (but we do not want to go into the theory of Berkovich spaces), [RZ] 5.1–5.12.

9. The period morphism

Now we can construct the period morphism, a morphism from the generic fiber of our moduli space to a certain Grassmannian variety (or more generally a partial flag variety), [RZ] 5.13–5.16.

You should cover the definition of the period morphism and, if possible, say something about it being étale ([RZ] Prop. 5.17).

For some general background, including the classical notion of period domain, see [R1] and [DOR], Introduction.

References: [RZ] Ch. 5, see also [Wa] 2.4.

10. The image of the period morphism

In this talk we will define the so-called weakly admissible locus, and see that the image of the period morphism is contained inside it. The weakly admissible locus is a very explicit (rigid analytic) subspace of the target space of the period morphism, and should be seen as a "easy uppder bound of the image of the period morphism". The most famous example is Drinfeld's upper half space:

$$\mathbb{P}^n_{\mathbb{C}_p} \setminus \bigcup_H H,$$

where the union runs through all $\mathbb{Q}_p$-rational hyperplanes H. (E.g., if $n = 1$, we obtain $\mathbb{P}^1_{\mathbb{C}_p} \setminus \mathbb{P}^1(\mathbb{Q}_p)$.) Since there are infinitely many such hyperplanes, it is clear that this construction does not make sense for schemes. In the Lubin-Tate case, the period domain is all of projective space.

Give the definition of the weakly admissible locus. As an example, discuss at least the Drinfeld case, [RZ] 1.44–1.46. Cf. also [BC] I.2.

In this talk you should *not* touch the more delicate questions related to the *admissible locus* (Fontaine's rings, Galois representation, etc.); this will be the topic of the following talk.

References: [RZ] Ch. 1, [DOR] XI.4, [Wa] 3.2

11. The admissible locus

One can show that all "classical points" of the weakly admissible locus are in the image of the period morphism. Nevertheless, the image of the period morphism does not usually coincide with the weakly admissible locus! This is due to the subtleties of the theory of rigid analytic spaces, and at this point it actually is more appropriate to use another variant of rigid geometry (Berkovich spaces, or Huber's adic spaces: they have "more points" and therefore allow one to identify "non-classical" points inside the weakly admissible locus which are not in the image of the period morphism).

In this talk, some results on the admissible locus (which is the image of the period morphism in the strict sense) should be discussed.

References: [RZ] Ch. 1, [DOR] XI.4 and the references given there (in particular to the papers of Faltings and of Hartl). See also [R2], [Wa] Ch. 4.

References

[Bo] S. Bosch, *Lectures on Formal and Rigid Geometry*, Springer Lect. Notes in Math. 2105, 2014.

[BC] J.-F. Boutot, H. Carayol, *Uniformisation p-adique de courbes de Shimura: les théorèmes de Čerednik et Drinfeld*, in: Courbes modulaires et courbes de Shimura, Astérisque **196–197**, 1997.

[Co] B. Conrad, *Several approaches to non-archimedean geometry*, AWS Lecture notes, available online.

[DOR] J.-F. Dat, S. Orlik, M. Rapoport, *Period domains over finite and p-adic fields*, Cambridge Univ. Press, 2010.

[F] L. Fargues, *An introduction to Lubin-Tate spaces and p-divisible groups*, available online.

[PRS] G. Pappas, M. Rapoport, B. Smithling, *Local models of Shimura varieties I. Geometry and combinatorics*, in: Handbook of moduli (eds. G. Farkas and I. Morrison), vol. III, 135–217, Adv. Lect. in Math. 26, International Press, 2013.

[R1] M. Rapoport, *Non-Archimedean Period Domains*, Proc. of the International Congress of Mathematicians (Zürich, 1994), 423–434, Birkhäuser, 1995.

[R2] M. Rapoport, *Accessible and weakly accessible period domains*, Appendix to: P. Scholze, On the p-adic cohomology of the Lubin-Tate tower, `arXiv:1506.04022`, 2015.

[RZ] M. Rapoport, Th. Zink, *Period Spaces for p-divisible Groups*, Annals of Math. Studies, Princeton Univ. Press, 1996.

[Wa] H. Wang, Moduli spaces of p-divisible groups and period morphisms, Master's thesis Univ. Leiden, 2009.
`https://www.math.leidenuniv.nl/scripties/wang.pdf`

[We] J. Weinstein, *The Geometry of Lubin-Tate spaces*, available online.