Program of the workshop

Riemann-Roch for Deligne-Mumford stacks

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Introduction

The Riemann-Roch theorem and its generalizations such as the Hirzebruch-Riemann-Roch and the Grothendieck-Riemann-Roch theorems are among the most important results in algebraic geometry. After briefly recalling several versions of the theorem and the required input such as Chern classes and Todd classes, K-groups and Chow groups, the main part of the workshop will focus on the equivariant theory, i.e., on the theory in the presence of the action of a (linear algebraic) group. We will study equivariant K-theory (and localization theorems in equivariant K-theory) and equivariant intersection theory. Most of this is contained in the series [EG1], [EG2], [EG3], [EG4] of Edidin and Graham. The recent preprint [E] by Edidin will serve as a guide.

Prerequisites We assume some familiarity with basic intersection theory (as in the first 6 chapters of [Fu]) and with the general definition of algebraic stacks.

Setting We assume that all schemes are of finite type over a fixed algebraically closed field of characteristic zero. Of course many of the results hold much more generally and every speaker should feel free to work in a more general setting.

1 Equivariant K-theory and equivariant Chow groups (1st day)

1.1 The Riemann-Roch theorem of Baum, Fulton, and MacPherson (60 minutes)

The goal of the talk is the explanation of [E] Theorem 2.4. A brief summary of all notions can be found in [E] Section 6. See also the references given there, especially [Fu].

(0) We assume that every participant is familiar with Chow groups and with the intersection product on smooth varieties over a field. But feel free to recall some properties as in [E] 6.2.

(1) Recall the notion of Chern classes, Chern characters and Todd classes.

(2) Recall the Grothendieck ring $K_0(X)$ of vector bundles and the Grothendieck group $G_0(X)$ of coherent sheaves, their functorialities, and their behaviour with respect to localization.
(3) State [E] Theorem 2.4 and explain why Grothendieck-RR and RR are special cases of this result.

1.2 Equivariant K-theory (60 minutes)

The goal of this talk is the definition of equivariant K-theory ([Tho1]). There is also a quick review in [Tho3]. Note that for the statement of Riemann-Roch theorems we need only K_0 (and G_0) thus the emphasis should lie here. But some proofs (such as the localization theorem in the next talk) need also the higher K-groups. For an additional reference for K_0 see also [CG] Chapter 5 and 6.

(1) Define $K_0(G, X)$ and $G_0(G, X)$. In particular introduce $R(G)$, give some examples of $R(G)$ (G finite group, $G = GL_n$) and define its action on $K_0(G, X)$ and $G_0(G, X)$. Moreover explain the λ-ring structure on $K_0(G, X)$.
(2) Recall the resolution property for stacks ([To]) and its G-equivariant analogue ([Tho1]). Explain when $K_0(G, X) \to G_0(G, X)$ is an isomorphism.
(3) Define very roughly higher K-groups and explain the long localization exact sequence of $G_*(G, \cdot)$ ([Tho1] Theorem 2.7, [Tho4] 0.3).
(4) Define K_0 and G_0 also for algebraic stacks. Recall the definition of a quotient stack $[X/G]$ and show that $K_0([X/G]) = K_0(G, X)$ and $G_0([X/G]) = G_0(G, X)$. (See e.g. [E] 3.2.1.)

1.3 Localization in equivariant K-theory (75 minutes)

Explain the statement of the localization theorem [E] Theorem 4.6 and give a proof following [Tho4] Section 3 (in particular Lemma 3.2 and Lemma 3.3). State and explain its extension to actions by not necessarily diagonalizable groups ([E] Theorem 4.14, [EG4]).

1.4 Equivariant Chow groups (60 minutes)

The goal is the introduction of equivariant Chow groups $\mathrm{Ch}^*_G(X)$. The main reference is [EG1].

(1) Define equivariant Chow groups and their functorial properties ([EG1] 2.2, 2.3), equivariant Chern classes ([EG1] 2.4), equivariant Chern character and Todd classes ([EG2]), and equivariant exterior products and intersection product ([EG1] 2.5).
(2) Give an example (e.g. the $\mathbb{G}_m$-equivariant Chow ring of $\mathbb{P}^n$, [EG1] 3.3).

2 The equivariant Riemann-Roch theorem (2nd day)

2.1 Chow groups for quotient stacks (60 minutes)

The goal of this talk is the definition and some properties for Chow groups of separated Deligne-Mumford stacks $\mathscr{X}$ which are quotient stacks, i.e., of the form $\mathscr{X} = [X/G]$ for an algebraic space X and a linear algebraic group G.

(1) Define the Chow groups of $\mathscr{X} = [X/G]$ as the equivariant Chow groups $\mathrm{Ch}^*_G(X)$, see [EG1] (we do not need Chow groups for general algebraic stacks as defined by Kresch in [Kre], and Remark 2.1.7 of loc. cit. ensures that Kresch's general definition agrees with this definition in the case of quotient stacks). Remark that this depends only on $\mathscr{X}$ and explain the relation of $\mathrm{Ch}^*(\mathscr{X})$ and the Chow group of the coarse moduli space of $\mathscr{X}$ (if it exists).

(2) Explain and prove (as much as possible of) the following results of [EG1]: Proposition 8(a), Theorem 3(a), Theorem 4.

2.2 Riemann-Roch for representable morphisms of quotient stacks (75 minutes)

The goal of this talk is [E] Theorem 3.5. Explain why this follows from [E] Theorem 3.1. Then explain as much as possible of the proof of [E] Theorem 3.1 (see [EG2]).

2.3 Riemann-Roch for quotient Deligne-Mumford stacks (60 minutes)

The goal of this talk is a Hirzebruch-Riemann-Roch theorem for quotient Deligne-Mumford stacks.

(1) Consider first the case of a diagonalizable group: State and prove [E] Theorem 4.10.

(2) State and prove the general case: [E] Theorem 4.16.

(3) If time remains, explain the reformulation in terms of the inertia stack: [E] Theorem 4.19.

2.4 Examples (60 minutes)

Give as many as, as instructive as possible examples for the Riemann Roch theorem for quotient Deligne-Mumford stacks. Possible examples are: BG for a finite group G ([E] Example 4.2), the weighted projective line ([E] 4.2.1, 4.2.2, 4.2.5), $[(\mathbb{P}^2)^3/(\mathbb{Z}/3\mathbb{Z})]$ ([E] 4.2.6), or the moduli space $\mathcal{M}_g$ of curves of genus g (no good reference known to the organizers).

References

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[Tho1] R.W. Thomason, *Algebraic K-theory of group scheme actions*, Algebraic topology and algebraic K-theory (Princeton, N.J., 1983), Ann. of Math. Stud. **113**, Princeton Univ. Press (1987), 539–563.

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