This semester we will study the wonderful compactification. Let $G$ be a semisimple linear algebraic group of adjoint type, let $\sigma : G \to G$ be an automorphism of order 2, and let $H = G^\sigma$ be the subgroup of fix points of $\sigma$. The homogeneous space $G/H$ is called a symmetric variety. The wonderful compactification is a smooth projective variety $X$ with a $G$-action and a $G$-equivariant open embedding $G/H \to X$, such that the boundary $X \setminus G$ is a divisor with normal crossings whose irreducible components are smooth and such that the closure relations among the $G$-orbits admit a simple combinatorial description.

The most important example is the following: Let $H$ be a semisimple linear algebraic group of adjoint type, let $G = H \times H$, and define $\sigma$ by $\sigma(h, h') := (h', h)$. Then $G^\sigma = H$ (embedded diagonally), and $G/H = H$. So in this case we obtain a compactification of the group $H$.

Another interesting example (to which we will come back in the seminar) is $G = SL_{n+1}, \sigma(A) = tA^{-1}$. (Even though this $G$ is not of adjoint type, the theory still applies with suitable modifications.)

All speakers are encouraged to specialize the results to these examples, whenever appropriate.

After developing the basic theory, following de Concini and Procesi [dCP1], we will turn to the situation in positive characteristic, consider an example, and finally study an application of the wonderful compactification to representation theory of $p$-adic groups, following a recent paper by Bezrukavnikov and Kazhdan [BeK].

Several other sources for (part of) the theory are listed in the references.

1. Overview, distribution of talks.

2. Representation theory of reductive groups. Recall basic facts about the representation theory of reductive algebraic groups and their Lie algebras, as in the books by Humphreys [H], and Fulton and Harris [FH]. The main results form [H] Ch. 20, 21 should be explained. The passage from Lie algebras to groups is discussed in [FH] Ch. 8.

   Discuss a few examples (e.g., a selection of $SL_2$ ([FH] Ch. 12), $SL_3$ ([FH] Ch. 13), $SL_n$ ([FH] Ch. 15), $PGL_n$, $SO_n$ ([FH] Ch. 19)). See also Jantzen’s book [J] (especially for the theory in positive characteristic — but this should be treated very briefly, or omitted in this talk).

3. Preliminaries. [dCP1] Section 1. The main result is Proposition 1.7; it is the basis for everything that follows. (We will eventually construct the compactification as the closure of a $G$-orbit in $\mathbb{P}(V)$, where $V$ is a suitable representation. Using
the machinery of the previous talk, Proposition 1.7 gives us the representation and the element whose orbit we will take.) Practically all the results established before are needed, too. Probably some of the proofs can only be sketched. For a proof of Proposition 1.1, compare [Steinberg, Endomorphisms of linear algebraic groups, Mem. AMS 80 (1968)] §7. The discussion after the proof of Proposition 1.7 can be omitted, if necessary.

4. Construction of the compactification. [dCP1] Section 2. Using the results of the previous talk, it is easy to construct the compactification we are interested in (Section 2.1). The main task of this talk is to analyze its local structure: Proposition 2.3 implies that it is smooth. Proposition 2.8 clarifies the situation at the boundary.

5. Properties of the compactification. [dCP1] Sections 3, 4, 5. In Theorem 3.1, the results proved so far are summarized. In 3.2 and Section 4, various kinds of independence statements are shown. Finally, Section 5 provides a geometric description of the $G$-stable pieces of the boundary, in terms of partial flag varieties and wonderful compactifications of Levi subgroups.

6. Positive characteristic, line bundles. [BrK] 6.1.A, 6.1.B In this talk we study wonderful compactifications in positive characteristic. For simplicity we follow [BrK], which only considers the compactification of $G = (G \times G)/G$. (The general case could be found in [dCS], but we will stick to the other reference.) First recall, why representations of $G$ do not form a semisimple category (e.g. [J, Introduction]), and indicate why [BrK, Lemma 6.1.1] suffices to carry over the construction of wonderful compactifications even in positive characteristic [BrK, Theorem 6.1.8]. (If you need some background [J, Appendix E] explains how to construct tilting representations, in any case you should at least give the example of the Steinberg module (say of $SL_2$) to have an explicit example, the proof is then completely parallel to what we have seen, so we can be very brief.)

The second part of the talk should then explain how to compute the Picard group of the wonderful compactification [BrK, 6.1.B].

There should be time left to state the results about line bundles proved in [dCP1] 7, 8 in the more general situation of compactifying a symmetric homogeneous space - the proof is not very different, so we don’t need to repeat it. (Again, [dCS] show the results in positive characteristic.)

7. Frobenius splitting. In positive characteristic, Frobenius splittings allow to prove vanishing results for line bundles, which we want to see from [BrK, 6.1.C] (see Corollary 6.1.13).

The talk should start by explaining general facts about Frobenius splittings, e.g. [MR] Section 1, 2, see also a selection of [BrK] Ch. 1: The main points for us are: Vanishing results for (semi-)ample bundles ([BrK, 1.4.12], but [MR, Proposition 3] may be easier to explain) and the relation of Frobenius splittings an sections of $\omega^{1-p}$ [MR, Proposition 6]. This can then be applied to prove Corollary 6.1.13. (If you have time, you can tell us about the application to construct good filtrations, which is nice, but we will not need this.)

8. An example: Counting quadrics. In this talk we would like to see a classical application of the theory: Schubert’s count of quadrics tangent to 9 quadrics in general position (666841088). This is computed in [dCP1] Section 10. To do this we first need to relate the problem with the compactifications and then understand
the underlying algorithm [dCP1, Section 9] - but you could immediately specialize it to the case considered in Section 10, which is the quotient $\text{SL}_n/\text{SO}_n$ (or even $n = 4$).

9. **Reductive embeddings.** The wonderful compactification is only constructed for adjoint groups. In this talk we want to study how one could compactify semisimple or even reductive groups. The first approach to this is to study embeddings of $G$ that cover the wonderful compactification of the adjoint quotient of $G$. This is described in [BrK, Section 6.2A], the main result is Proposition 6.2.4.

   6.2B should be skipped, in order to leave some time to explain another point of view on wonderful compactifications, which is due to Renner and Vinberg and uses partial, compactifications $G \subset M$ (or rather $G \times T \subset M$) such that $M$ is a monoid. It would be very nice if the second half of the talk could be used to explain this notion (Definition 6.2.10) and give Theorem 6.2.13. The case that is used most in the literature is Vinberg’s semigroup, which is explained in Exercise 6.2.E(3) using 6.1.E. (See the comments at the end of Section 6.2.)

10. **Geometry of second adjointness I.** This is the first talk of four on the new proof of Bernstein’s second adjointness theorem due to Bezrukavnikov-Kazhdan in [BeK] using the wonderful compactification. The first talk should review the smooth representation theory of $p$-adic reductive groups on $\mathbb{C}$-vector spaces. Definition, induction, compact induction, modulus character, admissibility, properties of the contragredient representation, parabolic induction, Jacquet functor, proof the the first (or Frobenius adjointness), statement of the second adjointness. Prove the equivalence of categories of the smooth representations and non-degenerate modules over the Hecke algebra. If time permits explain the statement of Bernstein’s center.

11. **Geometry of second adjointness II.** [BeK, §3] without §3.2. The main point is to explain the notion of admissible analytic map as in Definition 3.1 and admissible family in Definition 3.3 and its properties. We will apply the results of this talk to $D = G$ and $W$ the wonderful compactification of $G$. The speaker should illustrate what is going on with the Example 2.8 ($G = \text{PGL}_2$) throughout the talk. We will need the existence of admissible families to define the map $B_I$ in 5.1 in the next talk.

12. **Geometry of second adjointness III.** The main point is to prove [BeK, Prop 5.3]. We may assume for simplicity that $G$ is adjoint. It seems that the proof needs Claim 2.7 which does not seem to be proved in the paper...

13. **Geometry of second adjointness IV.** In this talk the second adjointness, [BeK, Thm 6.1] should be proved. It seems to me that it should follow fairly straightforwardly from Prop 5.3. The speaker should explain Lemma 5.4, if there is enough time for it, although this doesn’t seem to be needed for the second adjointness.

**References**


[FH] W. Fulton, J. Harris, Representation Theory, A First Course, Springer GTM


[H] J. Humphreys, Lie algebras and representation theory, Springer GTM


