

ALGEBRAIC NUMBER THEORY I

**Problem Set 10**

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**Exercise 1.** Let  $A$  be a Dedekind ring and let  $K$  be its fraction field. Let  $L/K$  be a finite extension and let  $B$  be the integral closure of  $A$  in  $L$ . Given an ideal  $\mathfrak{b} \subseteq B$ , let  $N_{L/K}(\mathfrak{b}) \subseteq A$  be the ideal generated by all the elements  $N_{L/K}(b)$ , where  $b \in \mathfrak{b}$ . The ideal  $N_{L/K}(\mathfrak{b})$  is called the *relative norm* of  $\mathfrak{b}$ .

- i) Show that  $N_{L/K}(bB) = N_{L/K}(b)A$  for all  $b \in B$ .
- ii) Let  $S \subseteq A$  be a multiplicative set. If  $\mathfrak{a} \subseteq A$  (resp.  $\mathfrak{b} \subseteq B$ ) is an ideal, denote by  $\mathfrak{a}_S$  (resp.  $\mathfrak{b}_S$ ) the ideal in  $S^{-1}A$  (resp.  $S^{-1}B$ ) generated by  $\mathfrak{a}$  (resp.  $\mathfrak{b}$ ). Prove that  $N_{L/K}(\mathfrak{b})_S = N_{L/K}(\mathfrak{b}_S)$ .
- iii) Show that  $N_{L/K}(\mathfrak{b}_1\mathfrak{b}_2) = N_{L/K}(\mathfrak{b}_1)N_{L/K}(\mathfrak{b}_2)$  for all ideals  $\mathfrak{b}_1, \mathfrak{b}_2 \in B$ .

*Hint: Check the equality of ideals locally. For a maximal ideal  $\mathfrak{p} \subseteq A$ , write  $S = A \setminus \mathfrak{p}$ . Note that  $A_S$  is a DVR and that  $B_S$  is a PID by Ex. 2 of PS6.*

**Exercise 2.** Let  $A$  be a DVR with uniformizer  $\pi$  and let  $K$  be its fraction field. Let  $L/K$  be a finite Galois extension and let  $B$  be the integral closure of  $A$  in  $L$ . Then  $B$  is a PID with only finitely many prime ideals  $(\Pi_1), \dots, (\Pi_q)$ , where  $\Pi_i \in B$  are such that

$$\pi = u \cdot \Pi_1^{e_1} \cdots \Pi_q^{e_q},$$

for some  $u \in A^\times$  and some integers  $e_i \geq 1$ . Show that  $\text{Gal}(L/K)$  acts transitively on the set  $\{(\Pi_1), \dots, (\Pi_q)\}$  and deduce that  $e_1 = \dots = e_q$  and that, up to units in  $A^\times$ ,  $N_{L/K}(\Pi_1) = \dots = N_{L/K}(\Pi_q)$ .

**Exercise 3.** Let  $A$  be a Dedekind ring and let  $K$  be its fraction field. Let  $L/K$  be a finite Galois extension and let  $B$  be the integral closure of  $A$  in  $L$ . Let  $\mathfrak{P}$  be a nonzero prime ideal of  $B$  and  $\mathfrak{p} = \mathfrak{P} \cap A$ . Prove that

$$N_{L/K}(\mathfrak{P}) = \mathfrak{p}^f,$$

where  $f$  is the residue degree of  $\mathfrak{P}$  over  $K$ .

**Exercise 4.** Let  $K$  be a number field. For an ideal  $\mathfrak{a} \subseteq \mathcal{O}_K$ , recall that we have defined the *absolute norm*

$$\mathcal{N}(\mathfrak{a}) = \#\mathcal{O}_K/\mathfrak{a}.$$

Show that there is an equality  $N_{K/\mathbb{Q}}(\mathfrak{a}) = (\mathcal{N}(\mathfrak{a}))$  of ideals of  $\mathbb{Z}$ . You may use that the statement of Exercise 3 holds without the condition that  $L/K$  is Galois.