

ALGEBRAIC NUMBER THEORY I

Problem Set 13

Delivery: 9/2/2016

Exercise 1. Let $L = \mathbb{Q}(\sqrt{5}, \sqrt{-1})$.

- i) Show that the ring of integers of L is $\mathbb{Z} \left[\sqrt{-1}, \frac{1+\sqrt{5}}{2} \right]$. Compute the absolute discriminant of L .
- ii) Show that the only primes that ramify in L are 2 and 5, and that the corresponding ramification indices are both 2.
- iii) Compute the Frobenius automorphism $\left(\frac{L/\mathbb{Q}}{p} \right)$ for every prime p distinct from 2 and 5. Determine the inertia and decomposition groups of 2 and 5.
- iv) Show that no prime ideal of $\mathbb{Q}(\sqrt{-5})$ ramifies in L .

Exercise 2. Let p be a prime and $n > 2$ an integer such that $p \nmid n$. Prove that (p) splits completely in $\mathbb{Q}(\zeta_n)$ if and only if $p \equiv 1 \pmod{n}$.

Exercise 3. .

- i) Let k be a finite field and $|\cdot|: k \rightarrow \mathbb{R}_{\geq 0}$ an absolute value. Prove that $|x| = 1$ for every $x \in k^\times$.
- ii) Let k be a field of characteristic p . Show that there does not exist an archimedean absolute value on k .
- iii) Give two non-equivalent archimedean absolute values on $\mathbb{Q}(\sqrt{2})$.

Exercise 4. Let k be a field with a non-archimedean absolute value $|\cdot|$. For $x, y \in k$, define $d(x, y) = |x - y|$.

- i) Show that if $|x| \neq |y|$, then $|x + y| = \max\{|x|, |y|\}$.
- ii) For $a \in k$ and $r \in \mathbb{R}_{>0}$, let $D(a, r) = \{x \in k \mid d(x, a) \leq r\}$ be the “closed” disc of center a and radius r . Show that $D(a, r)$ is open and closed in k .
- iii) Show that two discs D and D' are either disjoint or concentric (that is, there exists $a \in k$ and $r, r' \in \mathbb{R}_{>0}$ such that $D = D(a, r)$ and $D' = D(a, r')$).
- iv) Show that every triangle is isosceles: if for $x, y, z \in k$ one has $d(x, z) < d(y, z)$, then $d(y, z) = d(x, y)$.