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ALGEBRAIC NUMBER THEORY I

Problem Set 3

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Exercise 1. Let K be a field and G a finite subgroup of K^\times . Show that G is cyclic.

Hint: Use that there exists $z \in G$ such that $\text{ord}(x) \mid \text{ord}(z)$ for every $x \in G$.

Exercise 2. Let \mathbb{F}_q be the finite field of cardinality q . Show that the set

$$\{(x, y, z) \in \mathbb{F}_q^3 \mid x^2 = yz\}$$

has cardinality q^2 .

Exercise 3. Let p be a prime and $a_1, \dots, a_{2p-1} \in \mathbb{Z}$. Show that there exists a subset $I \subseteq \{1, \dots, 2p-1\}$ of cardinality $|I| = p$ such that

$$\sum_{i \in I} a_i \equiv 0 \pmod{p}.$$

Hint: Consider the polynomials $\sum_{i=1}^{2p-1} x_i^{p-1}$, $\sum_{i=1}^{2p-1} a_i x_i^{p-1} \in \mathbb{F}_p[x_1, \dots, x_{2p-1}]$.

Exercise 4. Show that the ring $R = \mathbb{Q}[X, Y]/(Y^2 - X^3)$ is a domain. Show that there exists an element in $\text{Frac}(R)$ which is integral over R , but not contained in R .

Hint: Identify R with a subring of $\mathbb{Q}[T]$ by giving a ring homomorphism $\mathbb{Q}[X, Y] \rightarrow \mathbb{Q}[T]$ with kernel $(Y^2 - X^3)$.