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ALGEBRAIC NUMBER THEORY I

Problem Set 7

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Exercise 1. Let K be a number field and let $2r_2$ be the number of complex non-real embeddings of K. If $x_1, \dots, x_n$ is a $\mathbb{Q}$-basis of K, show that

$$\operatorname{sign}(D(x_1, \dots, x_n)) = (-1)^{r_2}.$$

Hint: Let $\sigma_1, \dots, \sigma_n$ denote the embeddings of K into $\mathbb{C}$. What is the complex conjugate of $\det(\sigma_j(x_i))$?

Exercise 2. (Minkowski's theorem on linear forms). Let

$$\lambda_i(x_1, \dots, x_n) = \sum_{j=1}^{n} a_{ij} x_j, \qquad i = 1, \dots, n,$$

be real linear forms such that $\det(a_{ij}) \neq 0$, and let $c_1, \dots, c_n$ be positive real numbers such that $c_1 \cdots c_n > |\det(a_{ij})|$. Show that there exist integers $m_1, \dots, m_n \in \mathbb{Z}$ such that

$$\lambda_i(m_1, \dots, m_n) < c_i, \qquad i = 1, \dots, n.$$

Exercise 3. Show that $\sum_{j\geq 1} 10^{-(j!)} \in \mathbb{R}$ is a transcendental number.

Hint: Apply Liouville's theorem.

Exercise 4. Let $K = \mathbb{Q}(\sqrt{-23})$ and let $\mathcal{O}_K$ denote its ring of integers.

i) Determine prime ideals $\mathfrak{p}, \overline{\mathfrak{p}} \subset \mathcal{O}_K$ such that $\mathfrak{p}\overline{\mathfrak{p}} = (2) \subset \mathcal{O}_K$. Deduce that $\mathfrak{p}$ is not a principal ideal.

ii) Show that $\mathfrak{p}^3$ is a principal ideal.

Hint: Consider the prime ideal factorization of $\left(\frac{3+\sqrt{-23}}{2}\right) \subset \mathcal{O}_K$.