

ALGEBRAIC NUMBER THEORY I

**Make Up Exam**

**Question 1.**

1. Give an example of a number field  $F \neq \mathbb{Q}$  such that 3 and 5 are ramified in  $F/\mathbb{Q}$ .
2. Give an example of a number field  $F \neq \mathbb{Q}$  such that 3 and 5 are inert in  $F/\mathbb{Q}$ .
3. Give an example of a number field  $F \neq \mathbb{Q}$  such that 3 and 5 are unramified, but not inert in  $F/\mathbb{Q}$ .

**Question 2.**

1. Let  $\alpha = \sqrt[3]{2}$ ,  $K = \mathbb{Q}(\alpha)$ . Let  $a, b, c \in \mathbb{Q}$ . Express the trace

$$\mathrm{Tr}_{K/\mathbb{Q}}(a + b\alpha + c\alpha^2)$$

in terms of  $a$ ,  $b$ , and  $c$ .

2. Give an example of an algebraic number  $\beta \in \mathbb{C}$  which is not integral over  $\mathbb{Z}$ , but such that  $N_{\mathbb{Q}(\beta)/\mathbb{Q}}(\beta) \in \mathbb{Z}$ .

**Question 3.** Let  $\alpha \in \mathbb{C}$  with  $\alpha^2 - \alpha + 9 = 0$ , and let  $K = \mathbb{Q}(\alpha)$ .

1. Determine the class group of  $K$ .
2. Determine all prime numbers  $p$  which are ramified in  $K/\mathbb{Q}$ .

**Question 4.** Let  $\alpha \in \mathbb{C}$  be a zero of  $f(X) = X^3 + 2X - 1$ . Let  $K = \mathbb{Q}(\alpha)$ .

1. Prove that the ring of integers of  $K$  is  $\mathcal{O}_K = \mathbb{Z}[\alpha]$ .
2. Prove that  $\alpha$  is a unit in the ring of integers  $\mathcal{O}_K$  of  $K$ .
3. Prove that the rank of  $\mathcal{O}_K^\times$  as a  $\mathbb{Z}$ -module is 1.

**Question 5.** Let  $L/K$  be a finite Galois extension of number fields, let  $\mathcal{O}_K$  and  $\mathcal{O}_L$  be the rings of integers of  $K$  and  $L$ , respectively, and let  $\mathfrak{P} \subset \mathcal{O}_L$  be a maximal ideal.

1. State the definition of the *decomposition group* of  $\mathfrak{P}$ .
2. Let  $E := L^{I_{\mathfrak{P}}}$  be the fix field of the inertia subgroup  $I_{\mathfrak{P}}$  of  $\mathfrak{P}$ . Let  $\mathfrak{P}' := \mathfrak{P} \cap \mathcal{O}_E$ . Prove that  $\mathfrak{P}'$  is unramified in the extension  $E/K$ . (*Hint.* You may use that the ramification index is multiplicative in a tower of field extensions.)

**Question 6.** Let  $\zeta \in \mathbb{C}$  be a primitive 8-th root of unity, let  $K = \mathbb{Q}(\zeta)$ , and let  $\mathcal{O}_K = \mathbb{Z}[\zeta]$  be the ring of integers of  $K$ .

1. Show that the minimal polynomial of  $\zeta$  over  $\mathbb{Q}$  is  $X^4 + 1$ .
2. Show that for every prime number  $p$ , the polynomial  $X^4 + 1$  is reducible in  $\mathbb{F}_p[X]$ .

**Question 7.**

1. Compute the  $p$ -adic absolute value  $\left|\frac{4}{5}\right|_p$  for all prime numbers  $p$ .
2. Show that the series

$$\sum_{i=0}^{\infty} 15^i$$

converges in  $\mathbb{Q}_5$  and compute its limit.

### Cheat sheet

**Quadratic number fields.** Let  $d \neq 0, 1$  be a square-free integer. Let  $K = \mathbb{Q}(\sqrt{d})$ .

- If  $d \equiv 1 \pmod{4}$ , then the absolute discriminant of  $K$  is  $d$ , and its ring of integers is  $\mathbb{Z}[\frac{1+\sqrt{d}}{2}]$ .
- If  $d \not\equiv 1 \pmod{4}$ , then the absolute discriminant of  $K$  is  $4d$ , and its ring of integers is  $\mathbb{Z}[\sqrt{d}]$ .

**Discriminant.** Let  $L/K$  be a separable field extension of degree 3, and assume that  $x \in L$  such that  $L = K(x)$  and  $x^3 + ax + b = 0$ ,  $a, b \in K$ . Then the discriminant  $D(1, x, x^2)$  is equal to  $-4a^3 - 27b^2$ .

**Minkowski bound.**

$$\left(\frac{4}{\pi}\right)^{r_2} \frac{n!}{n^n} \sqrt{|\text{disc}_K|}$$

**Prime ideals in an extension of number fields.** Let  $L/K$  be an extension of number fields and let  $\mathcal{O}_L, \mathcal{O}_K$  denote the rings of integers of  $L$  and  $K$ , respectively. A non-zero prime ideal  $\mathfrak{p} \subset \mathcal{O}_K$  is called

- *inert in  $L/K$* , if  $\mathfrak{p}\mathcal{O}_L$  is a prime ideal in  $\mathcal{O}_L$ ,
- *split in  $L/K$* , if  $\mathfrak{p}\mathcal{O}_L$  is a product of  $[L : K]$  pairwise different prime ideals of  $\mathcal{O}_L$ ,
- *ramified in  $L/K$* , if there exists a prime ideal  $\mathfrak{P} \subset \mathcal{O}_L$  with  $\mathfrak{p} \subseteq \mathfrak{P}^2$ .

