Algebraic Number Theory 3 – Formal groups WS 2016/17

Prof. Dr. Ulrich Görtz
Dr. Heer Zhao

Problem sheet 1

Due date: Nov. 2, 2016

Problem 1

Let p be a prime number, $R := \mathbb{F}_p[t]/(t^2)$, and

$$F(X,Y) := X + Y + tXY^p \in R[[X,Y]].$$

Show that F defines a non-commutative 1-dimensional formal group law over R.

Problem 2

Let k be a field of characteristic 0. Show that $\widehat{\mathbb{G}}_{\mathrm{a},k} \cong \widehat{\mathbb{G}}_{\mathrm{m},k}$.

Hint: Recall that the complex analytic exponential map

$$\exp : (\mathbb{C}, +) \to (\mathbb{C}^\times, \cdot), \quad x \mapsto \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is a covering map with kernel $2\pi i\mathbb{Z}$. In particular it is a local isomorphism. "Translate" this to the setting of formal groups.

Problem 3

Let k be a field of characteristic $p > 0$. Show that $\widehat{\mathbb{G}}_{\mathrm{a},k}$ is not isomorphic to $\widehat{\mathbb{G}}_{\mathrm{m},k}$. Conclude that $\widehat{\mathbb{G}}_{\mathrm{a}}$ and $\widehat{\mathbb{G}}_{\mathrm{m}}$ are not isomorphic to each other over $\mathbb{Z}$.

Problem 4

Let k be a field. Compute the endomorphism ring $\mathrm{End}_k(\widehat{\mathbb{G}}_{\mathrm{a},k})$ of $\widehat{\mathbb{G}}_{\mathrm{a},k}$. (You will need to distinguish cases depending on the characteristic of k.)