

Problem sheet 2

Due date: Nov. 9, 2016

Problem 5

Let A be a ring such that there exists $a \in A \setminus \{0\}$ satisfying $n \cdot a = a^m = 0$ for some positive integers m, n . Construct a 1-dimensional non-commutative FGL over A .

Hint: Construct an element $t \in A$ using a , such that $t^2 = pt = 0$ for some prime number p , then make use of Problem 1 from “Problem sheet 1”.

Problem 6

Let G be an n -dimensional FGL over a ring A , and let $\mathcal{N}il_A$ (resp. $\mathcal{G}p$) be the category of nilpotent A -algebras (resp. groups). Recall that G defines a functor $F : \mathcal{N}il_A \rightarrow \mathcal{G}p$ given by: for any $N \in \mathcal{N}il_A$, $F(N)$ is the group with underlying set N^n and group structure defined by $\underline{x} +_G \underline{y} := \underline{G}(\underline{x}, \underline{y})$ for any $\underline{x}, \underline{y} \in N^n$. Show that G is completely determined by the functor F .

Hint: Consider the nilpotent A -algebras $(\underline{X})/(\underline{X})^r$, where r varies in the set of positive integers, and (\underline{X}) denotes the ideal of $A[[\underline{X}]]$ generated by X_1, \dots, X_n .

Problem 7

Let G be a FGL over a ring A . Let $\mu : A[[\underline{X}]] \rightarrow A[[\underline{X}, \underline{Y}]]$ be the comultiplication morphism of G , and let p_1 (resp. p_2) be the morphism $A[[\underline{X}]] \rightarrow A[[\underline{X}, \underline{Y}]]$ defined by $p_1(X_i) = X_i$ (resp. $p_2(X_i) = Y_i$). Show that an element $\omega \in \Omega_{A[[\underline{X}]]}^1$ is G -invariant if and only if $\mu_\bullet \omega = p_{1\bullet} \omega + p_{2\bullet} \omega$

Problem 8

A Lie algebra over a field k is a k -vector space \mathfrak{g} together with a binary operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying:

- (1) $[\cdot, \cdot]$ is a bilinear;
- (2) (Skew-symmetry) $[x, x] = 0$ for any $x \in \mathfrak{g}$;
- (3) (Jacobi's Identity) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

Let G be a FGL over a field k . Show that both $\text{Der}_k(k[[\underline{X}]])$ and $\text{Der}_G(k[[\underline{X}]])$ are Lie algebras with respect to the operation $[D_1, D_2] := D_1D_2 - D_2D_1$ for any $D_1, D_2 \in \text{Der}_k(k[[\underline{X}]])$.