Perfectoid spaces PD Dr. A. Chatzistamatiou
Adic spaces Prof. Dr. Ulrich Görtz
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Problem sheet 10

Due date: Dec. 22, 2017

Problem 37

Let R be a perfectoid K-algebra. Show that $(R^o)^\flat \cong \varprojlim_{x\mapsto x^p} R^o$ extends to a bijection $R^\flat \cong \varprojlim_{x\mapsto x^p} R$.

Problem 38

Let R be a perfectoid K-algebra with tilt $R^\flat$. Show that R is a field if and only if $R^\flat$ is a field.

Problem 39

We call a field k with an absolute value *spherically complete*, if for every descending chain $D_0 \supseteq D_1 \supseteq \cdots$ of disks, the intersection $\bigcap_i D_i$ is non-empty. Here the D_i are supposed to be closed disks of the form $D = \{x \in k;\ |x - z| \leqslant r\}$ for some fixed $z \in k$ and r in the value group $|k^\times|$.

(1) Assume that k is locally compact. Show that k is spherically complete.

(2) Prove that $\mathbb{C}_p$, the completion of an algebraic closure of $\mathbb{Q}_p$, is not spherically complete. (*Hint:* One approach is as follows: Show there exist $r \in \mathbb{R}_{>0}$ and a strictly decreasing sequence $r_i \to r$ converging to r, where $r_i \in \overline{\mathbb{Q}}_p$. Let D_0 be the disk around 0 with radius r_0, choose disjoint disks D_1 and D_1' of radius r_1 contained in D_0 (why is this possible?), then choose two disjoint disks of radius r_2 in each of D_1 and D_1', etc. We obtain a "tree structure" of disks, and for each path in the tree, we can form the intersection. Show that each such intersection is either empty or a disk of radius r, and conclude the proof from this point.)

Problem 40

Let k be an algebraically closed non-archimedean field which is complete with respect to a rank 1 valuation. Let D_i be a descending sequence of disks (as in Problem 39) with $\bigcap_i D_i = \varnothing$, where D_0 is contained in the closed unit disk. Show that

$$k\langle T\rangle \to \mathbb{R}_{\geqslant 0}, \quad f \mapsto \inf_i \sup_{x\in D_i} |f(x)|$$

is a point in $\mathrm{Spa}(k\langle T\rangle, k^\circ\langle T\rangle)$.