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Problem sheet 12

Due date: January 19, 2018

Problem 45

Let K be a perfect non-archimedean characteristic p field. Show that

$$\operatorname{Spa}(K\langle t^{\frac{1}{p^\infty}}\rangle, K^o\langle t^{\frac{1}{p^\infty}}\rangle) \to \operatorname{Spa}(K\langle t\rangle, K^o\langle t\rangle).$$

is a homeomorphism. Give a counterexample in characteristic 0.

Problem 46

Let R be a perfectoid K-algebra with tilt $R^\flat$. Is $\sharp : R^\flat \to R$ surjective? (Find a counterexample!)

Problem 47

Let K be a field which is complete with respect to a non-archimedean absolute value of rank 1. Denote by $\mathcal{O}$ its valuation ring and by ϖ an element with $0 < |\varpi| < 1$. Let $R = K[T, T^{-1}, Z]/(Z^2)$. Let R_0 be the $\mathcal{O}$-submodule of R with $\mathcal{O}$-basis

$$\varpi^{|n|}T^n, n \in \mathbb{Z}, \quad \varpi^{-|n|}T^n Z, n \in \mathbb{Z}.$$

(Here $|\cdot|$ denotes the usual, archimedean absolute value on $\mathbb{Z}$.)

Check that R_0 is an $\mathcal{O}$-subalgebra of R with $KR_0 = R$. Show that R_0 is not noetherian (consider the ideal $ZR \cap R_0$).

Problem 48

We continue with the notation of Problem 47. We give R the topology where $\varpi^N R_0$ is a fundamental system of open neighborhoods of 0, and obtain a Huber pair (R, R°). Let $X = \operatorname{Spa}(R, R^\circ)$.

We let

$$U = \{x \in X; |T(x)| \leqslant 1\},$$
$$V = \{x \in X; |T(x)| \geqslant 1\}.$$

(1) Show that U and V are rational subdomains of X and that $U \cup V = X$.

(2) Show that $Z \neq 0$ in $\mathcal{O}_X(X)$, but that $Z = 0$ in $\mathcal{O}_X(U)$ and in $\mathcal{O}_X(V)$. (In particular, the map $\mathcal{O}_X(X) \to \mathcal{O}_X(U) \oplus \mathcal{O}_X(V)$ given by restriction of sections to U and V is not injective. Therefore the presheaf $\mathcal{O}_X$ is not a sheaf.)

Reference for Problems 47/48 (and further results along these lines): K. Buzzard, A. Verberkmoes, *Stably uniform affinoids are sheafy*, to appear in J. Reine Angew. Math., `arxiv:1404.7020`.