Perfectoid spaces
Adic spaces
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PD Dr. A. Chatzistamatiou
Prof. Dr. Ulrich Görtz

Problem sheet 5

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Problem 17

Help with the details in Example (3.5) and (3.6) from the perfectoid spaces lecture. Let R be the t-adic completion of $\mathbb{F}_p[t^{\frac{1}{p^\infty}}]$ (or the p-adic completion of $\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]$), and let K be the field of fractions. For $p \neq 2$, we consider $L = K(t^{\frac{1}{2}})$ (or $L = K(p^{\frac{1}{2}})$, respectively). Show that L^o is not a finitely generated R-algebra. However,

$$L^o = \bigcup_m R \oplus R \cdot t^{\frac{1}{2p^m}}, \quad (L^o = \bigcup_m R \oplus R \cdot p^{\frac{1}{2p^m}}, \text{ respectively})$$

and L^o is an almost (finitely presented) projective R^a-module.)

Problem 18

Let (R, I) be as in the almost mathematics setup. For $t \in I$, show that $\mathrm{Mod}_R^a \to \mathrm{Mod}_{R[t^{-1}]}$ induces a functor from almost finite étale R^a-algebras to finite étale $R[t^{-1}]$-algebras.

Problem 19

Let R be a valuation ring.

(1) Prove that every radical ideal in R is prime. (Equivalently: Arbitrary intersections of prime ideals in R are prime ideals.)

(2) Prove that the following are equivalent:

  (a) There exists $a \in R \setminus \{0\}$ such that $\bigcap_{n \geqslant 0}(a^n) = 0$.
  (b) There exists a prime ideal $\mathfrak{p} \subset R$ of height 1 (i.e., $\mathfrak{p} \neq 0$ is prime, and does not contain any prime ideals $\neq 0, \mathfrak{p}$).

(c) The intersection of all non-zero prime ideals in R is $\neq 0$.

Hint. In part (2), to show that (a) $\Rightarrow$ (b), consider the ideal $\sqrt{(a)}$. For the implication (b) $\Rightarrow$ (a), take $a \in \mathfrak{p} \smallsetminus \{0\}$ and show that $\bigcap_n (a^n)$ is a radical ideal. For both these implications, make then use of part (1).

Problem 20

Let A be a ring, and let $X = \mathrm{Spv}(A)$ be the set of equivalence classes of absolute values on A. We equip X with the topology generated by the subsets

$$X\left(\frac{f}{g}\right) := \{|\cdot| \in X;\ |f| \leqslant |g| \neq 0\}.$$

(Note that $\frac{f}{g}$ is just a formal symbol, not the fraction fg^{-1} (in fact, g need not be invertible, it could even be 0).)

Prove that the map

$$\mathrm{Spv}(A) \to \mathrm{Spec}(A), \quad |\cdot| \mapsto \mathrm{supp}(|\cdot|),$$

is continuous.