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Problem sheet 6

Due date: Nov. 24, 2017.

Problem 21

Let R be the t-adic completion of $\mathbb{F}_p[t^{\frac{1}{p^\infty}}]$ (or the p-adic completion of $\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]$), and let K be the field of fractions. For $p \neq 2$, we consider $L = K(t^{\frac{1}{2}})$ (or $L = K(p^{\frac{1}{2}})$, respectively). Show that L° is an almost finite étale R-algebra.

Problem 22

In this exercise, $\widehat{B}$ means the p-adic completion of B. Let $R := \mathbb{Z}_p[p^{\frac{1}{p^\infty}}]$ and $A := R[x^{\frac{1}{p^\infty}}]$. Show that $B := A \mathbin{\widehat{\otimes}_{R[x]}} A$ is not integrally closed in $B[p^{-1}]$. Is the integral closure of B contained in $p^{-1}B$?

Problem 23

Let A be an adic ring, and let I, J be ideals in A. Prove that:

(1) If I and J are ideals of definition, then $\sqrt{I} = \sqrt{J}$ (where $\sqrt{I} = \{f \in A;\ f^n \in I \text{ for some } n\}$ denotes the radical of an ideal I).

(2) If I and J are finitely generated, $\sqrt{I} = \sqrt{J}$ and I is an ideal of definition, then J is an ideal of definition.

(3) Can the hypothesis that I be finitely generated be dropped in part (2)?

Problem 24

Let A be a topological ring. We call $a \in A$ a *power-bounded* element, if $\{a^n; n \geqslant 0\}$ is a bounded subset of A. Denote by A° the subset of power-bounded elements of A.

(1) Let A be a topological ring such that 0 has a fundamental system of open neighborhoods consisting of subgroups of $(A, +)$ (e.g., a Huber ring). Let $S \subseteq A$ be bounded. Show that the additive subgroup of A generated by S is bounded.

(2) Let A be a topological ring such that 0 has a fundamental system of open neighborhoods consisting of subgroups of $(A, +)$. Show that A° is a subring of A. What if A is an arbitrary topological ring?

(3) Let $A = \mathbf{Q}_p\langle t\rangle/(t^2)$ (with the topology induced by the residue norm of the Gauss norm). Show that A is Huber, and give a ring of definition A_0. Show that A_0 is bounded and that A° is not bounded.