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Problem sheet 7

Due date: Dec. 1, 2017

Problem 25

Let K be a complete non-archimedean field, and t a pseudouniformizer. For a t-adically complete and flat K^o-algebra R', we set $R := R' \otimes_{K^o} K$ and $|f| := \inf\{|\alpha|; \alpha \in K, f \in \alpha R'\}$ for every $f \in R$. Show that $(R, |.|)$ is a K-Banach algebra. Is this norm multiplicative?

Problem 26

Let K be as in the previous exercise. Show that $R \mapsto R^o$, where R^o are the powerbounded elements, defines a functor

$$\text{(uniform } K\text{-Banach algebras)} \to (t\text{-adically complete and flat } K^o\text{-algebras}).$$

Moreover, if we equip $R^o \otimes_{K^o} K$ with the norm of the previous exercise, then the evident map $R^o \otimes_{K^o} K \to R$ is an isomorphism of K-Banach algebras.

Problem 27

Let $f\colon A \to B$ be an adic homomorphism of Huber rings. Prove that the map f is bounded, i.e., for every bounded subset $S \subseteq A$, $f(S)$ is a bounded subset of B.

Problem 28

We call a topological space X *spectral*, if every closed irreducible subset has a unique generic point, X is quasi-compact, and the family of quasi-compact open subsets of X is stable under finite intersections and forms a basis of the topology of X.

We define the *constructible topology* on X as the coarsest topology such that all quasi-compact opens (in the sense of the original topology on X) and their complements are open, and denote the topological space X with the constructible topology by X_{cons}.

(1) Prove that the constructible topology is the coarsest topology such that all closed subsets of X and all quasi-compact open subsets of X are closed.

(2) Prove that X_{cons} is Hausdorff and quasi-compact.

Hint. For (2), this is what Hochster writes in his paper *Prime ideal structure in commutative rings*, Trans. AMS **142** (1969), 43-60, Proof of Theorem 1:

> It is certainly Hausdorff. Quasi-compactness will follow if every family of closed and quasi-compact open sets maximal with respect to having the finite intersection property intersects. But it is not difficult to see that the intersection of all the closed sets in such a family must also be in the family, and that it must be irreducible. Its generic point is then in the intersection.