Algebraic Geometry I Prof. Dr. Ulrich Görtz
WS 2018/19

Problem sheet 3

Due date: Oct. 30, 2018.

Problem 9

Let A be a ring. We call an element $e \in A$ *idempotent* if $e^2 = e$. Show that the following conditions are equivalent:

(i) $\operatorname{Spec} A$ is not connected.

(ii) There exists an idempotent element $e \in A$ different from 0 and 1.

(iii) There exists a ring isomorphism $A \cong A_1 \times A_2$ with non-zero rings A_1, A_2.

Problem 10

Let X be a topological space, and suppose that $X = \bigcup_{i=1}^n Z_i$, where the Z_i, $i = 1, \dots, n$, are closed irreducible subsets of X such that $Z_i \nsubseteq Z_j$ for $i \neq j$. Prove that the Z_i are precisely the irreducible components of X.

Problem 11

Let A be a ring, $X = \operatorname{Spec} A$. Show that every irreducible subset $Y \subseteq X$ contains at most one generic point. Give an example of a ring A and an irreducible subset of $\operatorname{Spec} A$ which does not contain a generic point.

Problem 12

Let A be a ring, $f \in A$, M an A-module. Consider the inductive system of A-modules (with index set $\mathbb{Z}_{\geqslant 0}$)

$$M \to M \to M \to \dots$$

where all transition maps are given by multiplication by f. Show that there exists a natural isomorphism between the colimit $\operatorname{colim}_i M$ and the localization M_f of the A-module M with respect to the element f.