Antispecial cycles on the Drinfeld upper half plane and degenerate Hirzebruch-Zagier cycles

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Abstract

We define the notion of antispecial cycles on the Drinfeld upper half plane in analogy to the notion of special cycles in [KR1]. We determine equations for antispecial cycles and calculate the intersection multiplicity of two antispecial cycles. The result is applied to calculate the intersection multiplicity of certain degenerate Hirzebruch-Zagier cycles. Finally we compare this intersection multiplicity to certain representation densities.

Introduction

Let k be an algebraically closed field of characteristic $p > 2$, and let $W = W(k)$ be its ring of Witt vectors. Let B be a division quaternion algebra over $\mathbb{Q}_p$, and let O_B be its ring of integers. According to Drinfeld ([D]), a *special formal O_B-module* over a W-scheme S is a p-divisible formal group X over S of dimension 2 and height 4, with an O_B-action $\iota : O_B \to \mathrm{End}_S(X)$ such that the induced $\mathbb{Z}_{p^2} \otimes \mathcal{O}_S$-module $\mathrm{Lie}\, X$ is, locally on S, free of rank 1. Let $\widehat{\Omega}$ be the formal model of the Drinfeld upper half plane for $\mathbb{Q}_p$, and let $\mathcal{M} = \widehat{\Omega} \times_{\mathrm{Spf}\ \mathbb{Z}_p} \mathrm{Spf}\ W$. Recall that $\mathcal{M}$ represents the following functor on the category Nilp of W-schemes S such that p is locally nilpotent in $\mathcal{O}_S$ (comp. [BC] or [KR1]). Let $\mathbb{X}$ be a fixed special formal O_B-module over $\mathrm{Spec}\, k$. The functor associates to a scheme $S \in$ Nilp the set of isomorphism classes of pairs (X, ϱ), where X is a special formal O_B-module over S and where

$$\varrho : \mathbb{X} \times_{\mathrm{Spec}\, k} \overline{S} \to X \times_S \overline{S}$$

is an O_B-linear quasi-isogeny of height 0. Here, $\overline{S} = S \times_{\mathrm{Spec}\, W} \mathrm{Spec}\, k$.

We write

$$\mathbb{Z}_{p^2} = \mathbb{Z}_p[\delta]/(\delta^2 - \Delta),$$

for some $\Delta \in \mathbb{Z}_p^\times$ which is not a square and write

$$O_B = \mathbb{Z}_{p^2}[\Pi]/(\Pi^2 - p),\ \Pi a = a^\sigma \Pi\ (a \in \mathbb{Z}_{p^2}).$$

Following [KR1] we call an element $j \in \mathrm{End}(\mathbb{X}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ *special* if it commutes with the O_B-action and its trace is 0.

Let $*$ be a $\mathbb{Q}_p$-linear automorphism of order 2 of B. By the theorem of Skolem-Noether there is some $b_* \in B$ such that $* = \mathrm{Int}(b_*)$, i.e. $x^* = b_* x b_*^{-1}$. The element b_* is unique up to multiplication by some element of $\mathbb{Q}_p^\times$. We say that an element $j \in \mathrm{End}(\mathbb{X}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is $*$-*special* if $j\iota(a)j^{-1} = \iota(a^*)\ \forall a \in B$, and if $s_j := \iota(b_*^{-1})j$ has trace 0. (This means that s_j is special, since it follows from the first condition that s_j commutes with the O_B-action.)

Since $*$ has order 2, it follows that $b_*^2 \in \mathbb{Q}_p$. We will choose b_* in such a way that b_*^2 has valuation 0 or 1. Precisely one of these possibilities can be realized. In the first case we call $*$ unramified (since $\mathbb{Q}_p(b_*)$ is an unramified quadratic extension of $\mathbb{Q}_p$ in this case), in the second case we call $*$ ramified (since $\mathbb{Q}_p(b_*)$ is a ramified quadratic extension of $\mathbb{Q}_p$ in this case). Hence b_* is a unit in O_B if $*$ unramified, and $b_* = \Pi \cdot \varepsilon_*$, where ε_* is a unit in O_B, if $*$ is ramified. In the latter case we write $b_*^2 = \eta_* p$ for some $\eta_* \in \mathbb{Z}_p^\times$. Note that since ε_* is unique up to multiplication by some element of $\mathbb{Z}_p^\times$, it follows that the quadratic residue character $\chi(\eta_*)$ is well defined.

Let $V[*]$ be the space of $*$-special endomorphisms. It is a quadratic $\mathbb{Q}_p$-vector space via the quadratic form

$$q(j) = (ps_j)^2.$$

(The scaling by p is done to facilitate the comparison with special cycles, cf. Theorem 0.1. Note that the ambiguity in the choice of b_* leads to an ambiguity in s_j and hence in q. But q is unique up to multiplication by some element of $\mathbb{Z}_p^{\times,2}$.) The quadratic form q induces a bilinear form β on $V[*]$ given by $\beta(j_1, j_2) = q(j_1 + j_2) - q(j_1) - q(j_2)$. If $j \in \mathrm{End}(\mathbb{X}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, we define the cycle $Z(j)$ to be the closed formal subscheme of $\mathcal{M}$ consisting of all pairs (X, ϱ) such that $\varrho \circ j \circ \varrho^{-1}$ lifts to an endomorphism of X. If j is special, resp. $*$-special, we call $Z(j)$ a special resp. $*$-special cycle.

The content of the first six sections of [KR1] is the description of special cycles by equations and the determination of the intersection product of two special cycles. Our first aim in this paper is to do the same for $*$-special cycles. For unramified $*$ and a $*$-special j we have $Z(j) = Z(s_j)$, which is a special cycle. For arbitrary unramified $*$ the notion of $*$-special cycles is the same as the notion of special cycles. Hence from now on we assume that $*$ is ramified. In this case we call a $*$-special cycle also an *antispecial cycle*. The notion of antispecial cycles does not depend on the particular $*$, i.e. all ramified $*$ induce the same $*$-special cycles. We now state our main results on antispecial cycles.

Theorem 0.1 (See Corollary 2.10 and (for $p = 3$) Remark 3.2.)
Let $p > 3$, and let $j \in V[*]$ with $q(j) \neq 0$ and $Z(j) \neq \emptyset$. Then $Z(j)$ is a divisor in $\mathcal{M}$. We have

$$Z(j) = Z(ps_j)^{\mathrm{pure}},$$

where the upper index pure *means the associated divisor of $Z(ps_j)$, i.e. the subscheme defined by the ideal sheaf of local sections with finite support. If $j^2 \equiv 0 \mod p$ then this statement is also true in case $p = 3$.*

For the proof we show first by considering Dieudonné modules that $Z(j)$ and $Z(ps_j)$ have the same k-valued points. Then we use the known equations for $Z(ps_j)$ on the one hand and Grothendieck-Messing theory on the other hand to give equations for $Z(j)$. The application of Grothendieck-Messing theory uses infinitesimal thickenings which carry a nilpotent *pd*-structure. To establish this *pd*-structure we need the assumption $p \neq 2$. Also the application of the results of [KR1] uses this assumption.

We define the intersection product $(Z(j_1), Z(j_2))$ of two antispecial cycles $Z(j_1)$ and $Z(j_2)$ using the general definition in [KR1]:

$$(Z(j_1), Z(j_2)) := \chi(\mathcal{O}_{Z(j_1)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(j_2)}),$$

where χ denotes the *Euler-Poincaré* characteristic and $\otimes^{\mathbb{L}}$ denotes the derived tensor product.

Theorem 0.2 (See Theorem 3.1 and (for $p = 3$) Remark 3.2.) *Let $j_1, j_2 \in V[*]$ such that j_1, j_2 span a 2-dimensional nondegenerate quadratic $\mathbb{Z}_p$-submodule of $V[*]$. Let*

$$T := \begin{pmatrix} q(j_1) & \frac{1}{2}\beta(j_1, j_2) \\ \frac{1}{2}\beta(j_2, j_1) & q(j_2) \end{pmatrix}.$$

We suppose that T is $GL_2(\mathbb{Z}_p)$-equivalent to $diag(\eta_1 p^{\beta_1}, \eta_2 p^{\beta_2})$, where $\eta_1, \eta_2 \in \mathbb{Z}_p^\times$ and $1 \leq \beta_1 \leq \beta_2$ for $p > 3$, resp. $1 < \beta_1 \leq \beta_2$ for $p = 3$. Define $\varepsilon_i \in \mathbb{Z}_p^\times$ and $\alpha_i \in \mathbb{N}$ by $\eta_* \eta_i p^{\beta_i - 1} = \varepsilon_i p^{\alpha_i}$. Then the intersection multiplicity of $Z(j_1)$ and $Z(j_2)$ is finite and is given by

$$(Z(j_1), Z(j_2)) = \\ \alpha_1 + \alpha_2 + 3 - \begin{cases} p^{(\alpha_1+1)/2} + 2\frac{p^{(\alpha_1+1)/2} - 1}{p-1} & \text{if } \alpha_1 \text{ is odd and } \chi(\eta_* \varepsilon_1) = -1 \\ (\alpha_2 - \alpha_1 + 1) p^{(\alpha_1+1)/2} + 2\frac{p^{(\alpha_1+1)/2} - 1}{p-1} & \text{if } \alpha_1 \text{ is odd and } \chi(\eta_* \varepsilon_1) = 1 \\ 2\frac{p^{\alpha_1/2+1} - 1}{p-1} & \text{if } \alpha_1 \text{ is even,} \end{cases}$$

where χ denotes the quadratic residue character on $\mathbb{Z}_p^\times$.

(Note that by the earlier remarks the values for β_i, α_i and $\chi(\eta_* \varepsilon_i)$ do not depend on the choice of b_*.) This theorem is in fact a simple consequence of Theorem 0.1 and the formula for the intersection product of special cycles given in [KR1] which only depends on the divisors associated to the special cycles.

Our second aim in this paper is to apply Theorem 0.2 to compute the intersection product of certain degenerate intersections of arithmetic Hirzebruch-Zagier cycles. We consider the following moduli problem. Fix a supersingular formal p-divisible group $\mathcal{A}$ over k of height 4 and dimension 2 which is equipped with an action

$$\iota_0 : \mathbb{Z}_{p^2} \to \mathrm{End}(\mathcal{A}),$$

such that $\mathcal{A}$ is special with respect to this $\mathbb{Z}_{p^2}$-action. We further suppose that $\mathcal{A}$ is equipped with a polarization

$$\lambda : \mathcal{A} \xrightarrow{\sim} \hat{\mathcal{A}},$$

such that for the Rosati involution $\iota_0(a)^* = \iota_0(a)$. We consider the following functor $\mathcal{M}^{HB}$ on the category Nilp. It associates to a scheme $S \in$ Nilp the set of isomorphism classes of the following data.

(1) A p-divisible group X over S, with an action

$$\iota_0 : \mathbb{Z}_{p^2} \to \mathrm{End}(X),$$

such that X is special with respect to this $\mathbb{Z}_{p^2}$-action.

(2) A quasi-isogeny of height zero

$$\varrho : \mathcal{A} \times_{\mathrm{Spec}\, k} \overline{S} \to X \times_S \overline{S},$$

which commutes with the action of $\mathbb{Z}_{p^2}$ such that the following condition holds. Let $\lambda_{\overline{S}} : \mathcal{A}_{\overline{S}} \to \hat{\mathcal{A}}_{\overline{S}}$ be the map induced by λ. Then we require the existence of an isomorphism $\tilde{\lambda} : X \to \hat{X}$ such that for the induced map $\tilde{\lambda}_{\overline{S}} : X_{\overline{S}} \to \hat{X}_{\overline{S}}$ we have $\lambda_{\overline{S}} = \hat{\varrho} \circ \tilde{\lambda}_{\overline{S}} \circ \varrho$.

This functor is representable by a formal scheme which we also call $\mathcal{M}^{HB}$. (We note that by [RZ] the scheme $\mathcal{M}^{HB}$ can be used to uniformize the completion along the supersingular locus of the Hilbert-Blumenthal moduli surface at an inert prime of the real quadratic field.) On the isocrystal N of $\mathcal{A}$ we have a perfect symplectic form. We define in this context the space of special endomorphisms

$$V' = \{j \in \mathrm{End}(N; F); j\iota_0(a) = \iota_0(a^\sigma)j \ \text{ and } \ j^* = j\},$$

where $*$ denotes the adjoint with respect to the symplectic form. Then V' is a 4-dimensional vector space over $\mathbb{Q}_p$ with quadratic form

$$Q(j) = j^2.$$

(Compare [KR2], §5.) For a special endomorphism $j \in V'$ we define the special cycle $Z(j)$ as a closed formal subscheme of $\mathcal{M}^{HB}$ as above. In analogy to Theorem 0.1 we have the following

Proposition 0.3 (See Proposition 4.5.) *Let $j \in V'$ be such that $Q(j) \neq 0$ and $Z(j) \neq \emptyset$. Then $Z(j)$ is a divisor in $\mathcal{M}^{HB}$.*

Now we fix a special endomorphism $j_1 \in V'$ with $j_1^2 = \varepsilon_1 p$ for some $\varepsilon_1 \in \mathbb{Z}_p^\times$. Let

$$V'[j_1] = \{j \in V' \mid j \perp j_1 \text{ with respect to the bilinear form associated to } Q\}.$$

Let $j_2, j_3 \in V'[j_1]$ be such that the $\mathbb{Z}_p$-span $\mathbf{j} = \mathbb{Z}_p j_2 + \mathbb{Z}_p j_3$ has rank 2 as a submodule of V', and such that Q induces a nondegenerate bilinear form on $\mathbf{j}$. We further suppose that the matrix of the bilinear form on $\mathbf{j}$ associated to Q with respect to the basis $j_2,\ j_3$ is equivalent to $\mathrm{diag}(\varepsilon_2 p^{\beta_2}, \varepsilon_3 p^{\beta_2})$ with $\varepsilon_i \in \mathbb{Z}_p^\times$ and $1 \le \beta_2 \le \beta_3$. We define the intersection product by the *Euler-Poincaré* characteristic of the derived tensor product,

$$(Z(j_1), Z(j_2), Z(j_3)) := \chi(\mathcal{M}^{HB}, \mathcal{O}_{Z(j_1)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(j_2)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(j_3)}),$$

which is well defined since $Z(j_1) \cap Z(j_2) \cap Z(j_3)$ is proper over $\mathrm{Spec}\, k$.

Proposition 0.4 (See Corollary 4.3 and Proposition 4.7.) *$Z(j_1)$ is isomorphic to $\mathcal{M}$, such that $Z(j_i) \cap \mathcal{M}$ for $i = 2, 3$ can be identified with the cycle associated to a $*$-special endomorphism for $* = \mathrm{Int}(\delta j_1)$ (identifying B with $\mathbb{Q}_{p^2}[j_1]$). Furthermore*

$$(Z(j_1), Z(j_2), Z(j_3)) = (Z(j_2) \cap \mathcal{M}, Z(j_3) \cap \mathcal{M}).$$

The latter is given explicitly by Theorem 0.2 in case $* = \mathrm{Int}(\delta j_1)$.

Our third aim is to compare the intersection multiplicity $(Z(j_1), Z(j_2), Z(j_3))$ to certain representation densities. To formulate the result, recall that, for $S \in \mathrm{Sym}_m(\mathbb{Z}_p)$ and $T \in \mathrm{Sym}_n(\mathbb{Z}_p)$ with $\det(S) \neq 0$ and $\det(T) \neq 0$, the representation density is defined as

$$\alpha_p(S, T) = \lim_{t \to \infty} p^{-tn(2m-n-1)/2} \mid \{x \in M_{m,n}(\mathbb{Z}/p^t\mathbb{Z});\ S[x] - T \in p^t \mathrm{Sym}_n(\mathbb{Z}_p)\} \mid .$$

For S as above, let

$$S_r = \begin{pmatrix} S & & \\ & 1_r & \\ & & -1_r \end{pmatrix}.$$

Then there is a rational function $A_{S,T}(X)$ of X such that

$$\alpha_p(S_r, T) = A_{S,T}(p^{-r}).$$

Let

$$\alpha_p'(S, T) = \frac{\partial}{\partial X}(A_{S,T}(X))|_{X=1}.$$

(Compare [KR1], §7.) Recall our assumption that the matrix of the bilinear form on $\mathbf{j}$ associated to Q with respect to the basis $j_2,\ j_3$ is equivalent to $\mathrm{diag}(\varepsilon_2 p^{\beta_2}, \varepsilon_3 p^{\beta_2})$ where $\varepsilon_i \in \mathbb{Z}_p^\times$ and $1 \le \beta_2 \le \beta_3$. Now let $T = \mathrm{diag}(\varepsilon_1 p, \varepsilon_2 p^{\beta_2}, \varepsilon_3 p^{\beta_3})$. (Then T equals the matrix of the bilinear form associated to the quadratic form Q restricted to the $\mathbb{Z}_p$-submodule $\mathbf{j}' := \mathbb{Z}_p j_1 \oplus \mathbb{Z}_p j_2 \oplus \mathbb{Z}_p j_3$ of V' for a suitable basis of $\mathbf{j}'$ consisting of a suitable basis of $\mathbf{j}$ together with j_1.) Let $S = \mathrm{diag}(1, -1, 1, -\Delta)$. We show the following

Theorem 0.5 (See Theorem 5.1.)

$$(Z(j_1), Z(j_2), Z(j_3)) = -\frac{p^4}{(p^2+1)(p^2-1)} \alpha_p'(S, T).$$

The proof is by explicit calculation of the r.h.s. and comparing it to the expression given by Theorem 0.2 for the l.h.s.. The calculation of the r.h.s. uses a combination of a lemma of Shimura ([S]) and a formula for $\alpha_p(\tilde{S}_r, T)$ in case $\tilde{S} = \mathrm{diag}(1, -1, 1, -1)$ given by Katsurada ([Ka]) which together allow us to calculate $\alpha_p^{'}(S, T)$ for our S and T explicitly. Here again our assumption $p \neq 2$ enters: even though Katsurada's formula extends to $p = 2$, the application of Shimura's lemma in case $p = 2$ seems problematic.

In [KR2] an analogous formula for $(Z(j_1), Z(j_2), Z(j_3))$ is proved in case $Q(j_1) \in \mathbb{Z}_p^\times$. A conjecture of Kudla and Rapoport states that this formula holds in general provided that $\mathbf{j}^{'} = \mathbb{Z}_p j_1 \oplus \mathbb{Z}_p j_2 \oplus \mathbb{Z}_p j_3$ is of rank 3 and Q induces a nondegenerate bilinear form on $\mathbf{j}^{'}$. Theorem 0.5 confirms a special case of this conjecture.

The paper is divided into five sections. The first section introduces some linear algebra concerning Dieudonné modules and the notion of $*$-special endomorphisms. The second section introduces $\mathcal{M}$ and investigates antispecial cycles and their local equations. In the third section we investigate intersection products of antispecial cycles and prove Theorem 0.2. In the fourth section we discuss the application to arithmetic Hirzebruch-Zagier cycles, and in the fifth section we prove Theorem 0.5.

I want to conclude this introduction by thanking those people who helped me to write this paper. In particular I thank M. Rapoport for suggesting this topic and for his stimulating support during the work. My deep thanks go to U. Görtz for many hours of patient help. Thanks are also due to Prof. Katsurada who gave the crucial hint how to calculate the representation densities and to Prof. Messing for helpful comments on p-divisible groups. Finally, I thank Prof. Kudla for alerting me to a mistake in a first version of this paper.

1 O_B-lattices and $*$-special endomorphisms

As in the introduction, let k be an algebraically closed field of characteristic $p > 2$, let $W = W(k)$ be its ring of Witt vectors with fraction field $W_{\mathbb{Q}}$, and let σ be the Frobenius automorphism of W. As in the introduction, we write

$$\mathbb{Z}_{p^2} = \mathbb{Z}_p[\delta]/(\delta^2 - \Delta) \text{ for some } \Delta \in \mathbb{Z}_p^\times \setminus \mathbb{Z}_p^{\times,2}.$$

We will regard $\mathbb{Z}_{p^2}$ as a subset of W (the set of elements fixed by σ^2). Let B be a quaternion division algebra over $\mathbb{Q}_p$, and let O_B be its ring of integers, which we identify with

$$O_B = \mathbb{Z}_{p^2}[\Pi]/(\Pi^2 - p), \Pi a = a^\sigma \Pi \ \forall a \in \mathbb{Z}_{p^2}.$$

Let (M, F, V) be a Dieudonné module of height 4 and dimension 2. It is a free W-module of rank 4 with a σ-linear resp. σ^{-1}-linear endomorphism F resp. V satisfying $VF = FV = p$, and for which the k-vector space M/VM has dimension 2. Now we assume that M is equipped with an O_B-operation, i.e. an action $\iota : O_B \to \mathrm{End}(M)$ commuting with F and V. From the action of $\mathbb{Z}_{p^2}$ we obtain a $\mathbb{Z}/2$-grading,

$$M = M_0 \oplus M_1,$$

where

$$\begin{aligned} M_0 &= \{m \in M \mid \iota(a)m = am \ \forall a \in \mathbb{Z}_{p^2}\}, \\ M_1 &= \{m \in M \mid \iota(a)m = a^\sigma m \ \forall a \in \mathbb{Z}_{p^2}\}. \end{aligned}$$

If we denote by N the isocrystal of M (i.e. the $W_{\mathbb{Q}}$-vector space $M \otimes_W W_{\mathbb{Q}}$ equipped with the induced notions for F and V and the O_B-action), we also obtain a $\mathbb{Z}/2$-grading,

$$N = N_0 \oplus N_1.$$

Definition 1.1 A *special Dieudonné module with O_B-action* is a Dieudonné module (M, F, V) over W of height 4 and dimension 2 which is equipped with an O_B-action such that the k-vector spaces M_0/VM_1 and M_1/VM_0 are one dimensional.

If (M, F, V) is a special Dieudonné module with O_B-action and if $i \in \mathbb{Z}/2$, we say that the index i is O_B-*critical* (for M), if $VM_i = \iota(\Pi)M_i$. Further M is called O_B-*superspecial*, if both indices $0, 1$ are O_B-critical. Finally, M is called O_B-*ordinary*, if only one index is O_B-critical.

Definition 1.2 Let (M, F, V) be a special Dieudonné module with O_B-action, and let N be its isocrystal. An *O_B-lattice in N* is a free W-submodule L of rank 4 in N, which is spanned by a basis of N and which is stable under F and V and under ι, and for which the k-vector spaces L_0/VL_1 and L_1/VL_0 have dimension 1. (Here, as usual, $L = L_0 \oplus L_1$ is the $\mathbb{Z}/2$- grading obtained from the action of $\mathbb{Z}_{p^2}$.)

Let us fix M and hence also fix N for this section. Then an O_B-lattice L is a special Dieudonné module (with O_B-operation) together with an isomorphism of isocrystals

$$L \otimes_W W_{\mathbb{Q}} \to N,$$

i.e. an isomorphism of vector spaces which commutes with the endomorphisms F, V and the operation of O_B.

Lemma 1.3 *Let L be an O_B-lattice, and let $i \in \mathbb{Z}/2$. Then*

(i) the inclusions $pL_i \subset VL_{i+1} \subset L_i$ are both of index 1.

(ii) the inclusions $pL_i \subset \iota(\Pi)L_{i+1} \subset L_i$ are both of index 1.

Proof. The first statement is essentially the assumption that L is special, for the second see [BC], chapitre II, Proposition 5.1. □

Lemma 1.4 *(i) Any O_B-lattice L has an O_B-critical index.*

(ii) There exists an O_B-superspecial O_B-lattice.

Proof. *i*) See [KR1], p. 165.

ii) Let L be an O_B-lattice, and let $i \in \mathbb{Z}/2$ be O_B-critical. By the theorem of Dieudonné ([Z], Satz 6.26) we can choose a basis e_1, e_2 of L_i satisfying $\Pi^{-1}Ve_i = e_i$. Then the O_B-lattice spanned by $e_1, e_2, e_3 = Ve_1$ and $e_4 = p^{-1}Ve_2$ is superspecial. □

Definition 1.5 A basis e_1, e_2, e_3, e_4 of N is called a *standard basis,* if $e_1, e_2 \in N_0$, and if the relations $Ve_i = \Pi e_i$ for $i = 1, 2$ and $e_3 = Ve_1$ and $e_4 = p^{-1}Ve_2$ hold.

It follows from the proof of Lemma 1.4 (*ii*) that there is a standard basis of N.

Lemma 1.6 *Any O_B-superspecial O_B-lattice is the W-span of some standard basis of N.*

Proof. Let L be a superspecial O_B-lattice. Then $L_i = L_i^{\Pi^{-1}V} \otimes_{\mathbb{Z}_p} W$ for $i = 0, 1$ (see the proof of Lemma 1.4 (*ii*)). It follows from Lemma 1.3 that the inclusion $VL_0^{\Pi^{-1}V} \subset L_1^{\Pi^{-1}V}$ has index 1. By the elementary divisor theorem we can choose a basis e_3, e_4 of $L_1^{\Pi^{-1}V}$ such that e_3, pe_4 is a basis of $VL_0^{\Pi^{-1}V}$. Setting $e_1 = V^{-1}e_3$ and $e_2 = V^{-1}pe_4$ we get the desired basis e_1, e_2, e_3, e_4. □

Denote by $\mathrm{End}(N, V)$ the space of endomorphisms of N which commute with V. Following [KR1] we call an element $j \in \mathrm{End}(N, V)$ *special* if it commutes with the O_B-action and its trace is 0. We denote the space of special endomorphisms by V. Note that, by restricting an element $y \in V$ to the fixed ($\mathbb{Q}_p$-)module $N_0^{V^{-1}\Pi}$, we can identify V with $M_2(\mathbb{Q}_p)^0$, the space of traceless matrices in $M_2(\mathbb{Q}_p)$, cf. [KR1], (2.2).

Let $*$ be a fixed $\mathbb{Q}_p$-linear automorphism of order 2 of B. By the theorem of Skolem-Noether there is some $b_* \in B$ such that $* = \mathrm{Int}(b_*)$, i.e. $x^* = b_* x b_*^{-1}$. The element b_* is unique up to multiplication by some element of $\mathbb{Q}_p^\times$.

Definition 1.7 An element $j \in \mathrm{End}(N, V)$ is $*$-*special* if the following conditions are satisfied.

1.) $j\iota(a)j^{-1} = \iota(a^*)\ \forall a \in B$,

2.) $s_j := \iota(b_*^{-1})j$ has trace 0.

Note that this definition is equivalent to the condition that s_j is special since the first condition is equivalent to the condition that s_j commutes with the O_B-action. Note also that the second condition is independent of the choice of b_* since b_* is unique up to multiplication by an element of $\mathbb{Q}_p^\times$. We denote the space of $*$-special endomorphisms by $V[*]$.

Since $*$ has order 2, it follows that $b_*^2 \in \mathbb{Q}_p$. We will always choose b_* in such a way that b_*^2 has valuation 0 or 1. Precisely one of these possibilities can be realized. In the first case we call $*$ unramified (since $\mathbb{Q}_p(b_*)$ is an unramified quadratic extension of $\mathbb{Q}_p$ in this case), in the second case we call $*$ ramified (since $\mathbb{Q}_p(b_*)$ is a ramified quadratic extension of $\mathbb{Q}_p$ in this case). Hence b_* is a unit in O_B if $*$ is unramified, and $b_* = \Pi \cdot \varepsilon_*$, where ε_* is a unit in O_B, if $*$ is ramified. In the latter case we write $b_*^2 = \eta_* p$ for some $\eta_* \in \mathbb{Z}_p^\times$.

2 Antispecial cycles

Before we come to the definition of antispecial cycles we recall some definitions and statements of [KR1], §1.

A *special formal O_B-module* over a W-scheme S is a p-divisible formal group X over S of dimension 2 and height 4, with an O_B-action $\iota : O_B \to \mathrm{End}_S(X)$ such that the induced $\mathbb{Z}_{p^2} \otimes \mathcal{O}_S$-module $\mathrm{Lie}\, X$ is, locally on S, free of rank 1. We fix a special formal O_B-module $\mathbb{X}$ over $\mathrm{Spec}\, k$. Let us consider the following functor $\mathcal{M}$ on the category Nilp of W-schemes S such that p is locally nilpotent in $\mathcal{O}_S$. It associates to a scheme $S \in$ Nilp the set of isomorphism classes of pairs (X, ϱ) consisting of a special formal O_B-module X over S and an O_B-linear quasi-isogeny of height zero,

$$\varrho : \mathbb{X} \times_{\mathrm{Spec}\, k} \overline{S} \to X \times_S \overline{S},$$

where $\overline{S} = S \times_{\mathrm{Spec}\, W} \mathrm{Spec}\, k$. The functor $\mathcal{M}$ is representable by the Deligne-Drinfeld formal scheme $\widehat{\Omega} \times_{\mathrm{Spf}\ \mathbb{Z}_p} \mathrm{Spf}\ W$. Denote by $\mathcal{B} = \mathcal{B}(PGL_2(\mathbb{Q}_p))$ the Bruhat-Tits building of $PGL_2(\mathbb{Q}_p)$. The formal scheme $\widehat{\Omega}$ is obtained by glueing formal open subschemes $\widehat{\Omega}_\Delta$, where Δ runs over the simplices of $\mathcal{B}$. We will only need to know $\widehat{\Omega}_\Delta$ for $\Delta =$ standard vertex and $\Delta =$ standard edge. So let $\Delta = [\Lambda_0]$ be the homothety class of the standard lattice

$$\Lambda_0 = [e_1, e_2],$$

where $[e_1, e_2]$ here denotes the $\mathbb{Z}_p$-span of the standard basis in $\mathbb{Q}_p^2$. Then

$$\widehat{\Omega}_{\Lambda_0} = \mathrm{Spf}\ \ \mathbb{Z}_p[T, (T^p - T)^{-1}]^\wedge. \tag{2.1}$$

Here, $^\wedge$ denotes the p-adic completion. If $\Delta = \Delta_0 = ([\Lambda_0], [\Lambda_1])$ is the standard edge corresponding to $\Lambda_0 = [e_1, e_2]$ and $\Lambda_1 = [pe_1, e_2]$, then

$$\widehat{\Omega}_{\Delta_0} = \mathrm{Spf}\ \ \mathbb{Z}_p[T_0, T_1, (1 - T_0^{p-1})^{-1}, (1 - T_1^{p-1})^{-1}]^\wedge / (T_0 T_1 - p). \tag{2.2}$$

Any k-valued point of $\mathcal{M}$ corresponds to a special Dieudonné module with an O_B-action (as defined in section 1). We may choose $\mathbb{X}$ in its isogeny class so that its Dieudonné module $\mathbb{L}$ is O_B-superspecial. By Lemma 1.6 we therefore find a standard basis $e_1, e_2, e_3 = \Pi e_1, e_4 = p^{-1}\Pi e_2$ of $\mathbb{L}$. We suppose that the isocrystal N considered in the preceding section equals the isocrystal of $\mathbb{L}$. Then any k-valued point of $\mathcal{M}$ corresponds to an O_B-lattice (in N) defined as above. The superspecial points (i.e. those k-valued points of $\mathcal{M}$ whose O_B-lattice is O_B-superspecial) are in one-to-one correspondence with the edges in $\mathcal{B}$. This correspondence can be chosen in such a way that $\mathbb{L}$ corresponds to the standard edge Δ_0 defined above. In the formal scheme $\widehat{\Omega} \times_{\mathrm{Spf}\ \mathbb{Z}_p} \mathrm{Spf}\ W$ the point pt_{Δ_0} corresponding to Δ_0 lies in $\widehat{\Omega}_{\Delta_0} \times_{\mathrm{Spf}\ \mathbb{Z}_p} \mathrm{Spf}\ W$ and is given there by the equations

$T_0 = T_1 = 0$. Any ordinary k-valued point of $\mathcal{M}$ (i.e. whose O_B-lattice is O_B-ordinary) corresponds in $\widehat{\Omega} \times_{\mathrm{Spf}\ \mathbb{Z}_p} \mathrm{Spf}\ W$ to a k-valued point of some $\widehat{\Omega}_\Lambda \times_{\mathrm{Spf}\ \mathbb{Z}_p} \mathrm{Spf}\ W$ for some vertex $[\Lambda] \in \mathcal{B}$.

Since the isocrystal of $\mathbb{X}$ equals N we can make the following

Definition 2.1 Let $j \in \mathrm{End}(N, V)$. Then the *cycle* $Z(j)$ *associated to* j is the closed formal subscheme of $\mathcal{M}$ consisting of all points (X, ϱ) such that $\varrho \circ j \circ \varrho^{-1}$ lifts to an endomorphism of X. If j is a special endomorphism, then $Z(j)$ is called a *special cycle* cf. [KR1], Definition 2.1. Let $*$ be a $\mathbb{Q}_p$-linear automorphism of order 2 of B. A $*$*-special cycle* is a cycle of the form $Z(j)$ for some j which is $*$-special. An *antispecial cycle* is a cycle of the form $Z(\iota(\Pi)y)$ for some special endomorphism y.

The fact that $Z(j)$ is a closed formal subscheme of $\mathcal{M}$ follows from [RZ], Proposition 2.9, see also [KR1], p. 167.

Remark 2.2 Let $\varepsilon \in O_B^\times$, and let $j, \tilde{j} \in \mathrm{End}(N; V)$ be such that $j = \iota(\varepsilon)\tilde{j}$. Then since $\iota(\varepsilon)$ is invertible, it follows that $Z(\tilde{j}) = Z(j)$.

Suppose we are given a $\mathbb{Q}_p$-linear automorphism $*$ of order 2 of B. If $*$ is unramified and j is $*$-special, then it follows from Remark 2.2 that $Z(j) = Z(s_j)$, hence $Z(j)$ is a special cycle. (Recall that $s_j = \iota(b_*^{-1})j$ and b_* is a unit in O_B.) Conversely, it follows again from Remark 2.2 that every special cycle is $*$-special for arbitrary unramified $*$. Since equations for special cycles are known from [KR1], in the sequel we will exclude the case that $*$ is unramified. Therefore, in the sequel we assume that $*$ is ramified. Recall that $b_* = \Pi \cdot \varepsilon_*$ where $\varepsilon_* \in O_B^\times$ in this case. It follows again from Remark 2.2 that for arbitrary ramified $*$ a cycle is $*$-special if and only if it is of the form $Z(\iota(\Pi)y)$ for some special endomorphism y, i.e. it is an antispecial cycle. In particular it follows that the notion of $*$-special cycles does not depend on the particular choice of the (ramified) $*$. We now fix a ramified $*$ for this section.

Let $j \in V[*]$. By Remark 2.2 we have $Z(j) = Z(\iota(\Pi)s_j)$. A k-valued point of $\mathcal{M}$ belongs to $Z(j)$ if and only if the O_B-lattice corresponding to that point is mapped by j (or, equivalently, by $\iota(\Pi)s_j$) into itself. If L is an O_B-lattice, this means $\iota(\Pi)s_jL \subset L$ and implies $ps_j^2L \subset L$. From this we see $\nu_p(\det(s_j)) \geq -1$, where ν_p is the valuation associated to p. So, in investigating $Z(j)$ we may assume $\nu_p(\det(s_j)) \geq -1$, otherwise $Z(j) = \emptyset$.

Now, given a $*$-special endomorphism j and an O_B-lattice $L = L_0 \oplus L_1$, we want to investigate under which conditions the inclusion $jL \subset L$ holds. Since $\Pi N_0 = N_1$ and $\Pi N_1 = N_0$, we see that $jL \subset L$ holds if and only if $jL_0 \subset L_1$ and $jL_1 \subset L_0$.

If 0 is O_B-critical we set $A_0 = L_0$ and $A_1 = VL_1 \subset A_0$. These are lattices in N_0. The condition $jL \subset L$ then translates into the conditions $V\iota(\Pi)s_jL_0 \subset VL_1$ and $\iota(\Pi)s_jVL_1 \subset VL_0$, hence

$$jL \subset L \iff ps_jA_0 \subset A_1 \text{ and } s_jA_1 \subset A_0. \tag{2.3}$$

If 1 is O_B-critical we set $A_0 = VL_0$ und $A_1 = L_1$. We then get an analogous condition to (2.3) where the roles of 0 and 1 are interchanged.

Now we assume, for example, that 0 is O_B-critical. By the elementary divisor theorem, there is a W-basis f_1, f_2 of A_0 for which pf_1, f_2 is a basis of A_1.

If $ps_jA_0 \subset A_1$, we can write

$$ps_jf_1 = paf_1 + cf_2 \text{ where } a, c \in W, \tag{2.4}$$

and if $s_jA_1 \subset A_0$, we can write

$$s_jf_2 = bf_1 + df_2 \text{ where } b, d \in W. \tag{2.5}$$

Conversely, if (2.4) and (2.5) are satisfied, so is the condition of (2.3). So, given L (where 0 is O_B-critical) and hence given A_0 and A_1, we have $jL \subset L$ if and only if there is a W-basis f_1, f_2 of $A_0 = L_0$ such that (2.4) and (2.5) hold. The case where 1 is O_B-critical is treated in the same way. (One just has to replace A_0 by A_1 and conversely in the above reasoning.)

Thus we have shown the following

Lemma 2.3 *Assume that $i \in \mathbb{Z}/2$ is O_B-critical for the O_B-lattice L. Then we have $jL \subset L$ if and only if there is a W-basis f_0, f_1 of L_i for which the matrix expressing s_j in f_0, f_1 has the form*

$$s_j = \begin{pmatrix} a & b \\ p^{-1}c & d \end{pmatrix} \text{ where } a, b, c, d \in W. \tag{2.6}$$

□

Proposition 2.4 *Let j be an $*$-special endomorphism, where $\nu_p(\det(s_j)) \geq -1$. Then, regarding ps_j as a special endomorphism, we have an equality of k-valued points*

$$Z(j)(k) = Z(ps_j)(k).$$

Proof. It is clear that $Z(j) = Z(\Pi s_j) \subset Z(ps_j)$, hence $Z(j)(k) \subset Z(ps_j)(k)$.

Conversely, given a k-valued point (X, ϱ) of $Z(ps_j)$, let $L = L_0 \oplus L_1$ be its Dieudonné module which we can regard as an O_B-lattice in N. Assume (for example) that 0 is O_B-critical. Since $ps_jL \subset L$, we have inclusions

$$ps_jL_0 \subset L_0 \text{ and } ps_jVL_1 \subset VL_1. \tag{2.7}$$

Since the inclusion $VL_1 \subset L_0$ has index 1, there is a W-basis f_1, f_2 of L_0 for which pf_1, f_2 is a W-basis of VL_1. Because of (2.7) we see that in the basis f_1, f_2 the endomorphism ps_j has matrix

$$ps_j = \begin{pmatrix} \tilde{a} & pb \\ c & -\tilde{a} \end{pmatrix} \text{ where } \tilde{a}, b, c \in W,$$

and hence $\det(s_j) = -p^{-2}\tilde{a}^2 - p^{-1}bc$. But $\nu_p(\det(s_j)) \geq -1$, so we conclude that $\tilde{a}$ is divisible by p. Therefore the matrix of s_j can be written in the form

$$s_j = \begin{pmatrix} a & b \\ p^{-1}c & -a \end{pmatrix} \text{ where } a, b, c \in W.$$

From Lemma 2.3 we therefore see that (X, ϱ) also belongs to $Z(j)$. □

Given an $*$-special endomorphism j, we want to determine local equations for $Z(j)$. This will be done with the help of the Grothendieck-Messing theory. We summarize below the facts from this theory which we will need.

Let A be a local Artin ring with residue field k (which is algebraically closed). Then A is a W-algebra in a unique way and $p \in A$ is nilpotent. Assume that the maximal ideal I of A (which is nilpotent) carries a nilpotent *pd*-structure (in the sense of [M], chapter III, Definition 1.1). Let X_0 be a p-divisible group over k of height 4 and dimension 2. Denote by L the Dieudonné module of X_0, and define $P(X_0)_k = L \otimes_W k$ and $P(X_0)_A = L \otimes_W A$. Let $\mathcal{F}_k = VL/pL \subset P(X_0)_k$. This gives us the Hodge filtration

$$0 \to \mathcal{F}_k \to P(X_0)_k \to L/VL \to 0. \tag{2.8}$$

To a lifting X over A of X_0 corresponds a lifting of the Hodge filtration (of A-modules),

$$0 \to \mathcal{F} \to P(X_0)_A \to \mathrm{Lie}(X) \to 0, \tag{2.9}$$

where $\mathcal{F}$ is a direct summand of $P(X_0)_A$ of rank 2, and where (2.9) lifts (2.8).

This establishes an equivalence of categories between the category of liftings of X_0 over k to some X over A and the category of liftings of the Hodge filtration (2.8) to filtrations of the form (2.9).

An endomorphism $\phi : X_0 \to X_0$ gives rise to an endomorphism $\Phi : L \to L$. Then ϕ lifts to an endomorphism of X if and only if the endomorphism induced by Φ on $P(X_0)_A$ maps the submodule $\mathcal{F}$ into itself. (In this situation we will denote the induced endomorphism by Φ as well. Note that by rigidity for p-divisible groups the lift of ϕ to an endomorphism of X is unique if it exists.) Now assume that X_0 is equipped with an O_B-action $\iota : O_B \to \mathrm{End}_k(X_0)$. Then we get an O_B-action $\iota : O_B \to \mathrm{End}(L)$. We apply the equivalence of categories just mentioned to get the following

Proposition 2.5 *Using the same notations as above, the following categories are equivalent*

(i) The category of liftings of X_0 to A, also lifting the O_B-action (i.e. an object in this category is given by a p-divisible group X over A with an O_B-action together with an isomorphism $\alpha : X \times_A k \to X_0$, and a morphism between two objects is given by a morphism of the corresponding p-divisible groups over A respecting α.)

(ii) The category of liftings of the Hodge filtration (2.8) to $P(X_0)_A$ which are stable under the induced O_B-action. (I.e. an object in this category is given by a direct

summand $\mathcal{F}$ of rank 2 of $P(X_0)_A$ which is stable under the O_B-action and such that $\mathcal{F} \otimes_A k$ equals $\mathcal{F}_k$. A morphism between two such filtrations $\mathcal{F}_1 \subset P(X_0)_A$ and $\mathcal{F}_2 \subset P(X_0)_A$ is an endomorphism of $P(X_0)_A$ compatible with the identification of the special fibers and which maps $\mathcal{F}_1$ into $\mathcal{F}_2$.)

□

Lemma 2.6 *The rings $A = W/p^n$ for $n \in \mathbb{N}$ and if $p > 3$ also the rings $A = W[x]/(x^2 - p\varepsilon, x^r)$, where $\varepsilon \in W^\times$ and $r \geq 2$ satisfy the requirements of the Grothendieck-Messing theory mentioned above, i.e., they are local Artin rings with residue field k, and their maximal ideals carry a nilpotent pd-structure.*

Proof. i) In case $A = W/p^n$ this is well known, see for example [G], p. 71.

ii) $A = W[x]/(x^2 - p\varepsilon, x^r)$ clearly is a local Artin ring with residue field k. We show that the ideal (x) of $W[x]/(x^2 - p\varepsilon)$ carries a *pd*-structure which induces a nilpotent *pd*-structure on the ideal (x) of $A = W[x]/(x^2 - p\varepsilon, x^r)$.

Let ν_x denote the valuation of $W[x]/(x^2 - p\varepsilon)$ associated to x and denote by ν_p the valuation of W associated to p. To see that the ideal $(x) \subset W[x]/(x^2 - p\varepsilon)$ carries a *pd*-structure we consider $\nu_x(i!)$ for any positive integer i. Now, $\nu_x(i!) = 2 \cdot \nu_p(i!)$, and since $p \geq 5$,

$$\nu_x(i!) = 2 \cdot \nu_p(i!) = 2 \cdot \sum\nolimits_{k=1}^{\infty} [\frac{i}{p^k}] < 2 \cdot \sum\nolimits_{k=1}^{\infty} \frac{i}{5^k} = \frac{i}{2}.$$

Here, $[\ ]$ denotes Gauss-brackets. This shows that for any $z \in (x) \subset W[x]/(x^2 - p\varepsilon)$ we have a well-defined expression $\frac{z^k}{k!}$ and that there is some N such that, whenever $n_1 + ... + n_l \geq N$, we have $\frac{z^{n_1}}{n_1!} \cdot ... \cdot \frac{z^{n_l}}{n_l!} \equiv 0 \mod x^r$ for all $z \in I$. It follows that this *pd*-structure induces a nilpotent *pd*-structure on the ideal (x) of A. □

Proposition 2.7 *Let $n \in \mathbb{N}$, and let j be a $*$-special endomorphism with $\nu_p(\det(s_j)) \geq -1$. Then $Z(j)$ and $Z(ps_j)$ have the same W/p^n-valued points,*

$$Z(j)(W/p^n) = Z(ps_j)(W/p^n).$$

Proof. The case $n = 1$ is treated in Proposition 2.4, so we may assume that $n \geq 2$. Since $Z(j) = Z(\Pi s_j) \subset Z(ps_j)$, it is enough to show that $Z(ps_j)(W/p^n) \subset Z(\Pi s_j)(W/p^n)$. We fix a k-valued point (X_k, ϱ_k) of $Z(ps_j)$ and consider liftings to W/p^n with the Grothendieck-Messing theory. Let L be the Dieudonné module of (X_k, ϱ_k). Let $P = L/p^nL$. Then the Hodge filtration of (X_k, ϱ_k) is given by $VL/pL \subset L/pL$. By Proposition 2.5 and Lemma 2.6 to a lifting (X, ϱ) of (X_k, ϱ_k) over W/p^n (within $Z(ps_j)$) corresponds a lifting of the Hodge filtration $\mathcal{F} \subset P$ stable under the O_B-action and under ps_j.

From the action of $\mathbb{Z}_{p^2}$ we obtain $\mathbb{Z}/2$-gradings,

$$\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$$

and

$$P = P_0 \oplus P_1.$$

Let $\mathcal{F}_0 =< \overline{f_0} >$ and $\mathcal{F}_1 =< \overline{f_1} >$ for suitable elements $\overline{f_i} \in P_i$.

Claim: *For some units $\varepsilon_0, \varepsilon_1$ we have either*

$$\Pi\overline{f_0} = \varepsilon_1 p\overline{f_1} \text{ and } \Pi\overline{f_1} = \varepsilon_0\overline{f_0}, \tag{2.10}$$

or

$$\Pi\overline{f_0} = \varepsilon_1\overline{f_1} \text{ and } \Pi\overline{f_1} = \varepsilon_0 p\overline{f_0}. \tag{2.11}$$

To see this, choose preimages $f_0 \in L_0$ of $\overline{f_0}$ and $f_1 \in L_1$ of $\overline{f_1}$. Since $\Pi L_i \subset L_{i+1}$, we have

$$\Pi f_0 = \varepsilon_1 p^{\nu_1} f_1 + p^n \xi_1 \text{ and } \Pi f_1 = \varepsilon_0 p^{\nu_0} f_0 + p^n \xi_0,$$

for some integers ν_0, ν_1, some units $\varepsilon_0, \varepsilon_1$ and some $\xi_0, \xi_1 \in L$. Therefore, for some $\Xi \in L$, we have

$$pf_0 = \varepsilon_0\varepsilon_1 p^{\nu_0+\nu_1} f_0 + p^n \Xi$$

and hence (using $n \geq 2$) we get $\nu_0 + \nu_1 = 1$, i.e. either $\nu_0 = 0$ and $\nu_1 = 1$ or $\nu_0 = 1$ and $\nu_1 = 0$. This confirms the claim.

Because of the symmetry of (2.10) and (2.11), we may, for example, assume that (2.10) holds. By replacing f_0 by Πf_1 and thereby changing $\overline{f_0}$ only by a unit and not changing $\mathcal{F}$ we can assume that

$$\Pi f_0 = pf_1 \text{ and } \Pi f_1 = f_0.$$

We choose $\overline{l_0} \in P_0$ with preimage $l_0 \in L_0$ and $\overline{l_1} \in P_1$ with preimage $l_1 \in L_1$ such that $P =< \overline{l_0}, \overline{f_0}, \overline{l_1}, \overline{f_1} >$. By Nakayama's lemma we also have $L =< l_0, f_0, l_1, f_1 >$. Let

$$\Pi l_0 = al_1 + bf_1 \quad \text{and} \quad \Pi l_1 = cl_0 + df_0 \quad \text{where} \quad a, b, c, d \in W,$$

then

$$pl_0 = a(cl_0 + df_0) + bf_0,$$

hence $ac = p$ and $ad + b = 0$.

Claim: *a is a unit.*

To see this, assume that a is divisible by p. Then b is also divisible by p. Therefore,

$$\Pi L_0 \ =< \Pi f_0, \Pi l_0 > = < pf_1, p(\frac{a}{p} l_1 + \frac{b}{p} f_1) > .$$

But this contradicts the fact that $L_1/\Pi L_0$ is a k-vector space of dimension 1. Therefore a is a unit which proves the claim.

By Proposition 2.4 we have $\Pi s_j L \subset L$. Writing

$$s_j f_0 = rl_0 + sf_0$$

we may assume $\nu_p(r), \nu_p(s) \geq -1$, where ν_p is the valuation associated to p. (This follows from $\Pi s_j L \subset L$ and hence $ps_j L \subset L$.) We now want to show that $\Pi s_j \mathcal{F} \subset \mathcal{F}$. If $l \in L$, denote by $\bar{l}$ its image in P. First,

$$\Pi s_j \overline{f_0} = \overline{\Pi s_j f_0} = \overline{s_j \Pi f_0} = \overline{s_j p f_1} = ps_j \overline{f_1} \in \mathcal{F},$$

since $ps_j\mathcal{F} \subset \mathcal{F}$. Therefore,

$$\mathcal{F} \ni \Pi s_j \overline{f_0} = \overline{\Pi s_j f_0} = \overline{\Pi(rl_0 + sf_0)} = \overline{r(al_1 + bf_1) + psf_1}.$$

Now, $r(al_1 + bf_1) + psf_1 \in L$, and a is a unit, hence $\nu_p(r) \geq 0$ and further, since $\overline{r(al_1 + bf_1) + psf_1} \in \mathcal{F}$, and a is a unit, we even have $r \equiv 0 \bmod p^n$. Now, $L \ni \Pi s_j f_1 = s_j f_0$, so $\nu_p(s) \geq 0$. Since $r \equiv 0 \bmod p^n$, we get

$$\overline{\Pi s_j f_1} = \overline{s_j \Pi f_1} = \overline{s_j f_0} = \overline{sf_0} \in \mathcal{F}.$$

This completes the proof. □

In the following statement we denote by an upper index "ord" the intersection with the ordinary locus of $\mathcal{M}$ resp. $\widehat{\Omega}$, i.e. the open formal subscheme formed by the complement of the superspecial points. (It is open since the superspecial points are the $\mathbb{F}_p$-valued points which form a discrete set of points.)

Proposition 2.8 *Let j be a $*$-special endomorphism with $j^2 \neq 0$, where $\nu_p(\det(s_j)) \geq -1$. Then*

$$Z(j)^{\text{ord}} = Z(ps_j)^{\text{ord}}.$$

Proof. It is clear that $Z(j)^{\text{ord}} \subset Z(ps_j)^{\text{ord}}$. Let $[\Lambda]$ be a lattice in $\mathbb{Q}_p^2$. In order to show $Z(j) \cap (\widehat{\Omega}_{[\Lambda]} \times_{\text{Spf } \mathbb{Z}_p} \text{Spf } W) = Z(ps_j) \cap (\widehat{\Omega}_{[\Lambda]} \times_{\text{Spf } \mathbb{Z}_p} \text{Spf } W)$ we may assume that $[\Lambda] = [\Lambda_0]$ is the standard lattice. (This can be shown by the same reasoning as in the proof of [KR1], Proposition 3.2, using the action of $PGL_2(\mathbb{Q}_p)$ on $\hat{\Omega}$ described in loc. cit.) We write in the basis e_1, e_2 of N_0

$$ps_j = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = p^m \cdot \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & -\overline{a} \end{pmatrix},$$

where $\overline{a}, \overline{b}, \overline{c} \in \mathbb{Z}_p$ are not simultaneously divisible by p. The equation for $Z(ps_j) \cap (\widehat{\Omega}_{[\Lambda]} \times_{\text{Spf } \mathbb{Z}_p} \text{Spf } W)$ can be written as

$$p^m \cdot (\overline{b}T^2 - 2\overline{a}T - \overline{c}) = 0,$$

see [KR1], Proposition 3.2. Let $Z(j) \cap (\widehat{\Omega}_{[\Lambda]} \times_{\text{Spf } \mathbb{Z}_p} \text{Spf } W)$ be described by the ideal I of $W[T, (T^p - T)^{-1}]^\wedge$ as a closed subscheme of $\widehat{\Omega}_{[\Lambda]} \times_{\text{Spf } \mathbb{Z}_p} \text{Spf } W = \text{Spf } W[T, (T^p - T)^{-1}]^\wedge$ (comp. (2.1)).

Claim: *Every element of I is divisible by $(\overline{b}T^2 - 2\overline{a}T - \overline{c})$.*

If

$$\text{Spf } W[T, (T^p - T)^{-1}]^\wedge / (\overline{b}T^2 - 2\overline{a}T - \overline{c})$$

is not empty, then (by [KR1], Proposition 3.2) it meets the special fibre of $Z(ps_j)$ in two different ordinary points. From the form of the matrix in (2.6) (for some basis of Λ), we see that the rank of the matrix obtained from ps_j by reduction mod p is at most 1. On the other hand, in [KR1], Proposition 2.3 it is shown that if $Z(ps_j)(k) \cap (\widehat{\Omega}_{[\Lambda]} \times_{\text{Spf } \mathbb{Z}_p} \text{Spf } W) \neq$

$\emptyset$, then this rank equals 0 or 2. Hence we conclude that ps_j is divisible by p. Hence $s_j(\Lambda) \subset \Lambda$. Therefore,

$$\mathrm{Spf}\ W[T,(T^p-T)^{-1}]^\wedge/(\bar{b}T^2-2\bar{a}T-\bar{c}) \subset Z(s_j) \subset Z(j),$$

as asserted.

Claim: *Every element of I is divisible by $p^m \cdot (\bar{b}T^2-2\bar{a}T-\bar{c})$.*

Suppose there is an element $Q \in I$ which is not divisible by $p^m \cdot (\bar{b}T^2-2\bar{a}T-\bar{c})$. By multiplying with a suitable power of p we may suppose that Q is divisible by $p^{m-1}(\bar{b}T^2-2\bar{a}T-\bar{c})$ and write

$$Q = q \cdot p^{m-1} \cdot (\bar{b}T^2-2\bar{a}T-\bar{c}),$$

where we may suppose that q is a polynomial in T in which no coefficient is divisible by p. (This assumption is allowed, since

$$(p^m \cdot (\bar{b}T^2-2\bar{a}T-\bar{c})) \subset I,$$

which follows from the inclusion $Z(j) \subset Z(ps_j)$.)

Let $\tilde{q}$ be the image of

$$q \cdot (\bar{b}T^2-2\bar{a}T-\bar{c}) \cdot (T^p-T)$$

in $k[T]$. By Gauss' lemma $\tilde{q} \neq 0$, and we can find $\tau \in k^\times$ with $\tilde{q}(\tau) \neq 0$. Choose a preimage $t \in W/p^m$ of τ. Then $q(t) \cdot (\bar{b}t^2-2\bar{a}t-\bar{c}) \cdot (t^p-t)$ is a unit in W/p^m.

Now we are ready to apply Proposition 2.7, which says that any W/p^m-valued point

$$\phi : W[T,(T^p-T)^{-1}]^\wedge/(p^m \cdot (\bar{b}T^2-2\bar{a}T-\bar{c})) \longrightarrow W/p^m$$

factors through $W[T,(T^p-T)^{-1}]^\wedge/I$.

Define ϕ by $\phi(T)=t$ and to be W-linear. Then $\phi(Q)=Q(t)\neq 0$. But then ϕ does not factor through $W[T,(T^p-T)^{-1}]^\wedge/I$. This contradiction proves the claim.

Altogether we get

$$(p^m \cdot (\bar{b}T^2-2\bar{a}T-\bar{c})) \ \subset\ I\ \subset\ (p^m \cdot (\bar{b}T^2-2\bar{a}T-\bar{c})),$$

and therefore $Z(j)^{\mathrm{ord}} = Z(ps_j)^{\mathrm{ord}}$. □

The next proposition gives local equations of $Z(j)$ for a $*$-special endomorphism j in a neighborhood of a superspecial point. Any superspecial point x corresponds to a simplex $\Delta = ([\Lambda],[\tilde{\Lambda}])$. We may suppose that $\Delta = \Delta_0 = ([\Lambda_0],[\Lambda_1])$ is the standard simplex (again by the same reasoning as in the beginning of the proof of [KR1], Proposition 3.2.) We then have $x \in Z(ps_j)$ and, with respect to the basis e_1, e_2 of N_0, we write again

$$ps_j = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = p^m \cdot \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & -\bar{a} \end{pmatrix}.$$

Let g be a special endomorphism with $g^2 \neq 0$ and $\nu_p(\det(g)) \geq 0$. In [KR1], Proposition 3.3 local equations for the special cycle $Z(g)$ are given, where the following three cases are distinguished:

(i) $[\Lambda_0]$ is strictly closer than $[\Lambda_1]$ to the fixed point set $\mathcal{B}^g$ in the Bruhat-Tits-building $\mathcal{B}$.

(ii) The fixed point set $\mathcal{B}^g$ is the midpoint of Δ, i.e. $[\Lambda_0]$ and $[\Lambda_1]$ both have distance $1/2$ to $\mathcal{B}^g$.

(iii) Δ lies in the fixed apartment $\mathcal{B}^g$, i.e. $[\Lambda_0]$ and $[\Lambda_1]$ both have distance 0 to $\mathcal{B}^g$.

In the following proposition we follow these cases for the special endomorphism ps_j to give local equations for $Z(j)$ (in case $j^2 \neq 0$ with $\nu_p(\det(s_j)) \geq -1$) in a neighborhood of the superspecial point $x = pt_\Delta$. Using these notations and the description for $\widehat{\Omega}_{\Delta_0}$ in (2.2) we state

Proposition 2.9 *Let $p > 3$. Let $x = pt_{\Delta_0}$ be the superspecial point as above.*

(i) Suppose $[\Lambda_0]$ is strictly closer than $[\Lambda_1]$ to the fixed point set $\mathcal{B}^{s_j}$. We then have $m \geq 1$, and $Z(j)$ is (Zariski-)locally around x given by the equation

$$T_0 \cdot p^{m-1} = 0.$$

(ii) Suppose $\mathcal{B}^{s_j}$ is the midpoint of Δ_0. Then $\overline{b}$ is divisible by p, and $Z(j)$ is locally around x given by the equation

$$p^m \cdot (b_0 T_0 - 2\overline{a} - \overline{c} T_1) = 0,$$

where $\overline{b} = p \cdot b_0$.

(iii) Suppose Δ_0 lies in the fixed apartement $\mathcal{B}^{s_j}$. Then $m \geq 1$, and $Z(j)$ is locally around x given by the equation

$$p^m = 0.$$

Proof of i). We have $m \geq 1$, and there is an (affine) neighborhood of x in which $Z(ps_j)$ is given by the equations

$$T_0^2 p^{m-1} = p^m = 0,$$

see [KR1], Proposition 3.3. (All references to [KR1] in this proof refer to Proposition 3.3.) Let this neighborhood be given by

$$\mathrm{Spf}\ W[T_0, T_1, \gamma^{-1}]^\wedge/(T_0T_1 - p),$$

for some

$$\gamma \in W[T_0, T_1]^\wedge \setminus (T_0, T_1, T_0T_1 - p)$$

which is divisible by $(1 - T_0^{p-1})(1 - T_1^{p-1})$. Then $Z(j)$ is in this neighborhood given by an ideal I of the ring

$$R := W[T_0, T_1, \gamma^{-1}]^\wedge/(T_0T_1 - p, T_0^2 p^{m-1},\ p^m)$$

as a closed subscheme of $Z(ps_j)$.

Suppose there is an element Q of I which is not divisible by p^{m-1}. By multiplying with a suitable power of p we may suppose then that Q is divisible by p^{m-2} and write

$$Q \;=\; p^{m-2} \cdot (\sum_{i=0}^{l_0} a_i T_0^i \;+\; \sum_{j=1}^{l_1} b_j T_1^j) \;+\; p^{m-1} \cdot \eta,$$

where the a_i and b_j are either units in W or zero, and where $\eta \in R$. Let $q = Q/p^{m-2}$.

Claim: $a_i = 0\ \forall i$.

To see this, suppose there is an $a_i \neq 0$. Let $\tilde{q}$ be the image of $q \cdot \gamma$ in $k[T_0, T_1]$. Since $\gamma \notin (T_0, T_1, T_0T_1 - p) = (T_0, T_1, p)$, we find $\tau \in k^\times$ such that $\tilde{q}(\tau, 0) \neq 0$. Let t be a lifting of τ in W/p^{m-1}. We are going to apply Proposition 2.7 again which here says that any W/p^{m-1}-valued point

$$\phi : R \longrightarrow W/p^{m-1}$$

factors through R/I. Now define ϕ by $\phi(T_0) = t,\ \phi(T_1) = p \cdot t^{-1}$ and to be W-linear. Then ϕ is well defined ($\phi(\gamma)$ is a unit) but $\phi(Q) = Q(t, p \cdot t^{-1}) \neq 0$, since the image of $Q(t, p \cdot t^{-1})/p^{m-2}$ in k equals $\tilde{q}(\tau, 0)$. Therefore ϕ does not factor through R/I. This contradiction confirms the claim.

In the same way one sees that $b_j = 0\ \forall j$. This shows that any element of I is divisible by p^{m-1}. Now we show that any element of I is divisible by $T_0 \cdot p^{m-1}$. Any $Q \in I$ can be written as

$$Q \;=\; a \cdot p^{m-1} T_0 \;+\; p^{m-1} \cdot \sum_{j=0}^{l} b_j T_1^j,$$

where a and all b_j are either units in W or zero. Let $q = Q/p^{m-1}$.

Claim: $b_j = 0\ \forall j$.

Assuming there is a $b_j \neq 0$, we find again $\tau \in k^\times$ such that $(q \cdot \gamma)(0, \tau)$ does not vanish in k. We lift τ to $t \in W/p^m$ and define a W-linear W/p^m-valued point $\phi : R \to W/p^m$ by $\phi(T_0) = p \cdot t^{-1}$ and $\phi(T_1) = t$. We see as above that it does not factor through R/I. This contradiction shows that all b_j vanish. Hence either $I = (p^{m-1} \cdot T_0)$ or $I = 0$. To confirm the claim of *(i)* it is thus enough to show that $I \neq 0$.

In order to show that $I \neq 0$, we show the existence of an $W[z]/(z^2 - p,\ p^m)$-valued point of $Z(ps_j)$ whose underlying k-valued point is x, and which does not belong to $Z(j)$. By the Grothendieck-Messing theory, this can be done by constructing a lifting of the Hodge filtration of x over $W[z]/(z^2 - p,\ p^m)$ which is stable under ps_j, but not stable under j.

Let L be the Dieudonné module of x. Then a basis of L is given by $e_1, e_2, e_3 = \Pi e_1, e_4 = p^{-1}\Pi e_2$, since x corresponds to the standard simplex Δ_0, i.e $L = \mathbb{L}$. Let

$$L_z = L \otimes_W W[z]/(z^2 - p),$$

and let

$$P = L \otimes_W W[z]/(z^2 - p, p^m) = L_z/p^m L_z.$$

Define

$$\begin{aligned} f_0 \;&=\; e_2 + z(e_1 + e_2) \;\in L_z, \\ f_1 \;&=\; e_3 + pe_4 + ze_4 \;\in L_z. \end{aligned}$$

Denote by $\overline{f_0}$ resp. $\overline{f_1}$ the image of f_0 resp. f_1 in P, and define a filtration $\mathcal{F} \hookrightarrow P$ by $\mathcal{F} = <\overline{f_0}, \overline{f_1}>$. To see that it lifts the Hodge filtration of x we note that the image of $\mathcal{F}$ in $L \otimes_W k$ equals the span of the images of e_2, e_3, and this is the image of $\Pi L = VL$. Further we have $\Pi f_0 = z f_1$ and $\Pi f_1 = z f_0$ which shows that $\mathcal{F}$ is O_B-stable.

The map ps_j induces on P the zero map, in particular $\mathcal{F}$ is stable under it. Now let us show that $\mathcal{F}$ is not stable under Πs_j. For $l \in L_z$ let us denote its image in P by $\bar{l}$. A short calculation shows

$$\overline{\Pi s_j f_0} = \overline{(p^{m-1}\overline{b} + zp^{m-1}\overline{a} + zp^{m-1}\overline{b})e_3}.$$

Let us suppose this is contained in $\mathcal{F}$. Then we can find $r \in W[z]/(z^2 - p)$ with $\overline{rf_1} = \overline{\Pi s_j f_0}$, hence $\overline{re_3} = \overline{\Pi s_j f_0}$ and $\overline{r(p+z)e_4} = 0$ and hence,

$$\begin{aligned} r &\equiv p^{m-1}\overline{b} + zp^{m-1}\overline{a} + zp^{m-1}\overline{b} \bmod p^m, \\ (p+z)r &\equiv 0 \bmod p^m. \end{aligned}$$

But if we multiply the first congruence by $p + z$, we see that the second is not fulfilled, since $\overline{b}$ is not divisible by p, see [KR1]. Hence indeed $\Pi s_j \mathcal{F} \not\subseteq \mathcal{F}$.

Proof of *ii*). We consider the (affine) neighborhood of x given by (2.2)

$$\mathrm{Spf}\ W[T_0, T_1, \gamma^{-1}]^\wedge/(T_0T_1 - p),$$

where

$$\gamma = (1 - T_0^{p-1})(1 - T_1^{p-1}).$$

By [KR1] in this neighborhood $Z(ps_j)$ is described by the equations

$$p^m T_0(b_0T_0 - 2\overline{a} - \overline{c}T_1) = p^m T_1(b_0T_0 - 2\overline{a} - \overline{c}T_1) = 0.$$

Then $Z(j)$ is given by an ideal I of the ring

$$R := W[T_0, T_1, \gamma^{-1}]^\wedge/(T_0T_1 - p,\ p^m T_0(b_0T_0 - 2\overline{a} - \overline{c}T_1),\ p^m T_1(b_0T_0 - 2\overline{a} - \overline{c}T_1))$$

as a closed subscheme of $Z(ps_j)$ in that neighborhood.

Claim: *Any element of I is divisible by* $b_0T_0 - 2\overline{a} - \overline{c}T_1$.

(In case $m \geq 1$ this is clear, since then $Z(s_j) \subset Z(j)$ and the ideal of $Z(s_j)$ is contained in $(b_0T_0 - 2\overline{a} - \overline{c}T_1)$.) Let

$$S = W[T_0, T_1, \gamma^{-1}]^\wedge/(T_0T_1 - p, b_0T_0 - 2\overline{a} - \overline{c}T_1).$$

Denoting the image of I in S by $\overline{I}$, we must show that $\overline{I} = 0$. From the relations $T_0T_1 - p = 0$ and $b_0T_0 - 2\overline{a} - \overline{c}T_1 = 0$ and the fact that b_0 and $\overline{c}$ are units (see [KR1]), we get $T_1 = c^{-1}(b_0T_0 - 2\overline{a})$ and $T_0^2 = 2ab_0^{-1}T_0 + pcb_0^{-1}$, showing that any element in S and so in particular any $Q \in \overline{I}$ can be written in the form $Q = rT_0 + s$, where $r, s \in W$. Now let $\varepsilon = (\frac{\overline{a}}{pb_0})^2 p + \frac{c}{b_0}$ (note that $\overline{a}$ is divisible by p, see [KR1]) and define for any $n \in \mathbb{N}$ a W-linear homomorphism

$$\phi_n : S \longrightarrow W[z]/(z^2 - p\varepsilon, p^n), \quad T_0 \mapsto \frac{\overline{a}}{b_0} + z, \quad T_1 \mapsto \frac{p}{\frac{\overline{a}}{b_0} + z}.$$

One easily checks that ϕ_n is well defined. Let $\pi : R \longrightarrow S$ be the canonical projection. Consider for any n the $W[z]/(z^2 - p\varepsilon, p^n)$-valued point of $Z(ps_j)$,

$$\alpha_n = \phi_n \circ \pi : R \longrightarrow W[z]/(z^2 - p\varepsilon, p^n).$$

These maps are compatible with varying n. We want to show that α_n is also a $W[z]/(z^2 - p\varepsilon, p^n)$-valued point of $Z(\Pi s_j)$. Let $L = \mathbb{L}$ be the Dieudonné module of x, let $L_z = L \otimes_W W[z]/(z^2 - p\varepsilon)$, and let $\mathcal{F}_n \hookrightarrow L_z/(p^n L_z)$ be the Hodge filtration associated to α_n. Then $\mathcal{F}_n = \mathcal{F}_{n+1}/(p^n L_z)$, since the p-divisible group corresponding to α_{n+1} lifts the one corresponding to α_n. Let $\mathcal{F}_{n+1} =< \overline{f_0}, \overline{f_1} >$, where $\overline{f_i}$ is contained in the index i-part of $\mathcal{F}_{n+1}$. Further, for $i \in \mathbb{Z}/2$, let f_i be a lifting of $\overline{f_i}$ in L_z. Since $ps_j \mathcal{F}_{n+1} \subset \mathcal{F}_{n+1}$, we have $ps_j f_i = \lambda_i f_i + p^{n+1}\xi$ for some $\lambda_i \in W[z]/(z^2 - p\varepsilon)$ and some $\xi \in L_z$. Therefore $p\Pi s_j f_i = \tilde{\lambda} f_{i+1} + p^{n+1}\tilde{\xi}$ for some $\tilde{\lambda_i} \in W[z]/(z^2 - p\varepsilon)$ and some $\tilde{\xi} \in L_z$. Because of Proposition 2.4 we have $\Pi s_j L_z \subset L_z$, hence $p\Pi s_j f_i \in pL_z$. Since f_i belongs to a basis of L_z (Nakayama's lemma), we conclude that $\tilde{\lambda}$ is divisible by p, hence for some $\mu \in W[z]/(z^2 - p\varepsilon)$ we have $\Pi s_j f_i = \mu f_{i+1} + p^n \tilde{\xi}$. Therefore $\Pi s_j \mathcal{F}_n \subset \mathcal{F}_n$, which shows that α_n is a $W[z]/(z^2 - p\varepsilon, p^n)$-valued point of $Z(\Pi s_j)$.

It follows that ϕ_n must factor through $S/\overline{I}$. Assume now that $\overline{I} \neq 0$. Then we find $0 \neq rT_0 + s \in \overline{I}$ and a sufficiently large n such that $r(\frac{a}{b_0} + z) + s$ does not vanish in $W[z]/(z^2 - p\varepsilon, p^n)$. Therefore for such n the homomorphism ϕ_n will not factor through $S/\overline{I}$. This contradiction shows $\overline{I} = 0$, hence any element of I is divisible by $b_0 T_0 - 2\overline{a} - \overline{c}T_1$.

Claim: *Any element of I is divisible by $p^m(b_0 T_0 - 2\overline{a} - \overline{c}T_1)$.*

The proof is the same as the proof for the first step in part i) and will therefore be omitted. It follows that either $I = (p^m(b_0 T_0 - 2\overline{a} - \overline{c}T_1))$ or $I = 0$.

Claim: $I \neq 0$.

Proceeding as in part i) this will be done by constructing a lifting of the Hodge filtration of x over $W[z]/(z^2 - p,\ p^{m+1})$ which is stable under ps_j, but not stable under j. Again let $L = \mathbb{L}$ be the Dieudonné module of x. Then a basis of L is given by $e_1, e_2, e_3 = \Pi e_1, e_4 = p^{-1}\Pi e_2$. Let

$$L_z = L \otimes_W W[z]/(z^2 - p),$$

and let

$$P = L \otimes_W W[z]/(z^2 - p, p^{m+1}) = L_z/p^{m+1}L_z.$$

Since $\overline{c}$ and b_0 are units (see [KR1]), we can choose $t \in W^\times$ such that t^2 is not congruent to $\frac{\overline{c}}{b_0}$ modulo p. Define

$$\begin{aligned} f_0 &= te_2 + z(e_1 + e_2) \ \in L_z, \\ f_1 &= e_3 + pe_4 + zte_4 \ \in L_z. \end{aligned}$$

Denote again by $\overline{f_0}$ resp. $\overline{f_1}$ the image of f_0 resp. f_1 in P and define a Hodge filtration $\mathcal{F} \hookrightarrow P$ by $\mathcal{F} = < \overline{f_0}, \overline{f_1} >$. As above we see that it lifts the Hodge filtration of x, in particular that $\Pi f_0 = z f_1$ and $\Pi f_1 = z f_0$ holds. Using $p \mid \overline{a}$ and $p \mid \overline{b} = pb_0$ we calculate in L_z:

$$ps_j f_0 \equiv p^m z\overline{c}t^{-1} f_0 \bmod p^{m+1} \quad \text{and} \quad ps_j f_1 \equiv p^m z b_0 t f_1 \bmod p^{m+1},$$

showing that $ps_j\mathcal{F} \subset \mathcal{F}$. Writing $\overline{a} = pa_0$ we also calculate

$$\Pi s_j f_1 \equiv p^m z t b_0 \cdot e_1 + (-p^m z t a_0 + p^m \overline{c}) \cdot e_2 \bmod p^{m+1}.$$

We claim now that $\Pi s_j \mathcal{F} \nsubseteq \mathcal{F}$. Otherwise we would have in particular $\Pi s_j \overline{f_1} \in \mathcal{F}$. We then find $r \in W[z]/(z^2 - p)$ with $\Pi s_j f_1 \equiv r f_0 \bmod p^{m+1}$, hence,

$$r(t+z) \equiv -p^m z t a_0 + p^m \overline{c} \bmod p^{m+1},$$
$$rz \equiv p^m z t b_0 \bmod p^{m+1}.$$

Subtracting the second congruence from the first and multiplying the result by z we get

$$rtz \equiv p^m z\overline{c} \bmod p^{m+1},$$

and on the other hand multiplying the second congruence by t we get

$$rtz \equiv p^m z t^2 b_0 \bmod p^{m+1}.$$

We conclude that $\overline{c} - t^2 b_0 \equiv 0 \bmod z$ and hence that $\overline{c} - t^2 b_0 \equiv 0 \bmod p$, which contradicts the assumption we made on t. Hence $\Pi s_j \mathcal{F} \nsubseteq \mathcal{F}$. This confirms the claim and ends the proof of *ii*).

Proof of *iii*). In this case $\overline{a}$ is a unit and $\overline{b}$ is divisible by p, see [KR1]. Since $\nu_p(\det(ps_j)) \geq 1$, it follows that $m \geq 1$. We start as in part *i*) by showing first that in some affine neighborhood the ideal I of $Z(j)$ in $Z(ps_j)$ contains the ideal p^m. The proof is the same, so it will be omitted. Using the equations for $Z(ps_j)$, which are now given in a neighborhood of x by $T_0 p^m = T_1 p^m = 0$ (see [KR1]), it follows that I is either the zero ideal or equals (p^m). Thus we must show again that $I \neq 0$, which will be done by constructing a lifting of the Hodge filtration of x over $W[z]/(z^2 - p,\ zp^{m+1})$ which is stable under ps_j, but not stable under j. Again let $L = \mathbb{L}$ be the Dieudonné module of x, and consider the basis $e_1, e_2, e_3 = \Pi e_1, e_4 = p^{-1}\Pi e_2$ of L. Let

$$L_z = L \otimes_W W[z]/(z^2 - p),$$

and let

$$P = L \otimes_W W[z]/(z^2 - p, zp^m) = L_z / zp^m L_z.$$

Define

$$f_0 = e_2 + z(e_1 + e_2) \ \in L_z,$$
$$f_1 = e_3 + pe_4 + ze_4 \ \in L_z.$$

Denote by $\overline{f_0}$ resp. $\overline{f_1}$ the image of f_0 resp. f_1 in P, and define a filtration $\mathcal{F} \hookrightarrow P$ by $\mathcal{F} = <\overline{f_0}, \overline{f_1}>$. By the same reasons as in part i) it lifts the Hodge filtration of x, and we have the relations $\Pi f_0 = zf_1$ and $\Pi f_1 = zf_0$. By [KR1], in this case $\overline{b}$ is divisible by p and $\overline{a}$ is a unit. We calculate

$$ps_j f_0 \equiv -p^m \overline{a} f_0 \bmod zp^m \quad \text{and} \quad ps_j f_1 \equiv p^m \overline{a} f_1 \bmod zp^m,$$

showing that $ps_j\mathcal{F} \subset \mathcal{F}$. We also calculate,

$$\Pi s_j f_1 \equiv p^m \overline{a} \cdot e_1 + (-p^{m-1} z\overline{a} + p^m \overline{c} - p^m \overline{a}) \cdot e_2 \bmod p^m.$$

Proceeding as above we assume $\Pi s_j \mathcal{F} \subset \mathcal{F}$ and have in particular $\Pi s_j \overline{f_1} \in \mathcal{F}$. Hence we find $r \in W[z]/(z^2 - p)$ with $\Pi s_j f_1 \equiv r f_0 \bmod zp^m$ and hence

$$\begin{aligned} (1+z)r &\equiv -p^{m-1} z\overline{a} + p^m \overline{c} - p^m \overline{a} \bmod zp^m, \\ zr &\equiv p^m \overline{a} \bmod zp^m. \end{aligned}$$

Subtracting the second congruence from the first and multiplying the result by z, we get

$$zr \equiv -p^m \overline{a} \bmod zp^m,$$

which is a contradiction to the second congruence since $\overline{a}$ is a unit and $p \neq 2$. Hence $\Pi s_j \mathcal{F} \not\subseteq \mathcal{F}$. This completes the proof of the proposition. □

Let $g \in \mathrm{End}(N, V)$. Following [KR1] §4, we denote by $Z(g)^{\mathrm{pure}}$ the closed subscheme of $Z(g)$ defined by the ideal sheaf of local sections with finite support.

Corollary 2.10 *If $p > 3$, and j is $*$-special with $j^2 \neq 0$ and $\nu_p(\det(s_j)) \geq -1$, the antispecial cycle $Z(j)$ is a divisor and equals $Z(ps_j)^{\mathrm{pure}}$.*

Proof. It follows from the proof of Proposition 2.8 and from Proposition 2.9 that

$$Z(j)^{\mathrm{pure}} = Z(j) = Z(ps_j)^{\mathrm{pure}}. \tag{2.12}$$

□

3 Intersection calculus of antispecial cycles

We keep our fixed ramified automorphism $*$ of order 2 of B. In this section we calculate the intersection number of two antispecial cycles.

On the space $\mathbb{Q}_p$-vector space $V[*]$ we have a quadratic form

$$q(j) = (ps_j)^2 = -p^2 \det(s_j).$$

(s_j is special and hence $s_j^2 \in \mathbb{Q}_p$, see [KR1], p. 167.) Recall that $s_j = \iota(b_*^{-1})j$ and $b_*^2 = \eta_* p$. We also consider the quadratic form

$$Q(j) = j^2 = p^{-1} \eta_* q(j),$$

Suppose we are given two $*$-special endomorphisms j_1 and j_2. We assume that $j_i^2 \neq 0$ and $\nu_p(\det((s_j)_i)) \geq -1$ for $i = 1, 2$. Following the general definitions in [KR1], §4, we define the intersection number of their associated cycles by

$$(Z(j_1), Z(j_2)) = \chi(\mathcal{O}_{Z(j_1)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(j_2)}),$$

where χ denotes the *Euler-Poincaré* characteristic and $\otimes^{\mathbb{L}}$ denotes the derived tensor product.

By [KR1], Lemma 4.3,

$$(Z(j_1), Z(j_2)) = (Z(j_1)^{\text{pure}}, Z(j_2)^{\text{pure}}). \tag{3.1}$$

We define $\mathbf{j}$ to be the $\mathbb{Z}_p$-span of j_1 and j_2 in $V[*]$ and we assume that $\mathbf{j}$ has rank 2. Let β be the bilinear form on $\mathbf{j}$ associated to the quadratic form q,

$$\beta(x, y) = q(x + y) - q(x) - q(y).$$

If $\mathbf{j}$ is nondegenerate with respect to this bilinear form, it follows from [KR1], Theorem 5.1 that $(Z(j_1), Z(j_2))$ depends only on $\mathbf{j}$ (write $j_i = \iota(b_*)(s_j)_i$ and use that $(s_j)_i$ is special).

On the other hand, since $p \neq 2$, we can choose a $\mathbb{Z}_p$-basis $j, j^{'}$ of $\mathbf{j}$, for which the matrix of β is diagonal,

$$T := \begin{pmatrix} q(j) & \frac{1}{2}\beta(j, j^{'}) \\ \frac{1}{2}\beta(j, j^{'}) & q(j^{'}) \end{pmatrix} = \operatorname{diag}(\eta_1 p^{\beta_1}, \eta_2 p^{\beta_2}) \tag{3.2}$$

where $\eta_1, \eta_2 \in \mathbb{Z}_p^\times$ and $\beta_1 \leq \beta_2$.

Define $\varepsilon_i \in \mathbb{Z}_p^\times$ and $\alpha_i \in \mathbb{N}$ by $\eta_* \eta_i p^{\beta_i - 1} = \varepsilon_i p^{\alpha_i}$. Then $\alpha_1 \leq \alpha_2$. If j_1, j_2 is a basis of $\mathbf{j}$ for which T has the form $\operatorname{diag}(\eta_1 p^{\beta_1}, \eta_2 p^{\beta_2})$ then $Q(j_i) = j_i^2 = \varepsilon_i p^{\alpha_i}$. The numbers β_i and hence the α_i are invariants of $\mathbf{j}$. In case $\beta_1 < \beta_2$, the units η_i and ε_i are up to a square uniquely determined by $\mathbf{j}$ as well. Using (3.1) and (2.12) we get from Theorem 6.1 of [KR1] directly an expression for $(Z(j_1), Z(j_2))$ using the invariants β_i and η_i of $\mathbf{j}$ defined by the quadratic form q. For later use we translate this formula into an expression depending on the invariants α_i and ε_i defined by the quadratic form Q.

Theorem 3.1 *Let $p > 3$, let $j_1, j_2 \in V[*]$, and let $\mathbf{j}$ be their $\mathbb{Z}_p$-span in* $\operatorname{End}(N, V)$. *Assume that $\mathbf{j}$ is of dimension 2 and nondegenerate. Using the same notations as above the following formula holds*

$$(Z(j_1), Z(j_2)) =$$
$$\alpha_1 + \alpha_2 + 3 - \begin{cases} p^{(\alpha_1+1)/2} + 2\frac{p^{(\alpha_1+1)/2-1}}{p-1} & \text{if } \alpha_1 \text{ is odd and } \chi(\eta_*\varepsilon_1) = -1 \\ (\alpha_2 - \alpha_1 + 1)p^{(\alpha_1+1)/2} + 2\frac{p^{(\alpha_1+1)/2}-1}{p-1} & \text{if } \alpha_1 \text{ is odd and } \chi(\eta_*\varepsilon_1) = 1 \\ 2\frac{p^{\alpha_1/2+1}-1}{p-1} & \text{if } \alpha_1 \text{ is even,} \end{cases}$$

where χ denotes the quadratic residue character on $\mathbb{Z}_p^\times$.

(Note that in case $\alpha_1 = \alpha_2$ this expression does not depend on ε_1. Note also that ε_* is well defined up to multiplication by a unit of $\mathbb{Z}_p$ and hence $\chi(\eta_*)$ is well defined.) □

Remark 3.2 Theorem 3.1 is still valid for $p = 3$ provided that $\alpha_1 \geq 1$. Indeed, the only point where we used $p > 3$ is in Lemma 2.6 for the nilpotence of the *pd*-structure of the maximal ideals of rings of the form $A = W[x]/(x^2 - p\varepsilon, x^r)$. These rings are only needed in the proof of Proposition 2.9. Looking back at this proof and using the same notation, we see that in case (i) and in case (iii) we do not need these rings if we only want to determine $Z(j)^{\mathrm{pure}}$ in a neighborhood of the superspecial point x we are considering. In case (ii) rings of the above form are also needed to determine $Z(j)^{\mathrm{pure}}$ in a neighborhood of x if and only if $m = 0$. But in this case we have $\nu_p((ps_j)^2) = 1$, since $\overline{a}$ is divisible by p and b_0 and c are units. Hence, if $p = 3$ and $\alpha_1 \geq 1$ (and hence $\beta_1 > 1$), we see that $Z(j_1)^{\mathrm{pure}}$ and $Z(j_2)^{\mathrm{pure}}$ are given by the same equations as in case $p > 3$.

4 An application to Arithmetic Hirzebruch-Zagier cycles

In this section we compute the intersection multiplicity of (a class of examples of) three Hirzebruch-Zagier cycles defined on a certain formal scheme $\mathcal{M}^{HB}$. This formal scheme is a formal moduli space of p-divisible groups that can be used to uniformize the completion along the supersingular locus of the Hilbert-Blumenthal moduli surface at an inert prime of the real quadratic field (see [KR2]). Again we have a quadratic $\mathbb{Q}_p$-space V' of so called special endomorphisms and for each $j \in V'$ we define a special cycle (or Hirzebruch-Zagier cycle) $Z(j)$ similarly to the above notion of special cycles. We consider three endomorphims j_1, j_2, j_3 such that $j_1^2 = \varepsilon_1 \cdot p$, where $\varepsilon_1 \in \mathbb{Z}_p^\times$ and such that some additionl conditions on j_2, j_3 are satisfied. In order to compute the intersection multiplicity of $Z(j_1), Z(j_2)$ and $Z(j_3)$ we first show that $Z(j_1)$ can be identified with $\mathcal{M}$ and show then that the intersection product $(Z(j_1), Z(j_2), Z(j_3))$ equals $(Z(j_2) \cap \mathcal{M}, Z(j_3) \cap \mathcal{M})$. This last intersection product can be computed with the help of Theorem 3.1.

We consider a supersingular formal p-divisible group $\mathcal{A}$ over k of height 4 and dimension 2 which is equipped with an action

$$\iota_0 : \mathbb{Z}_{p^2} \to \mathrm{End}(\mathcal{A}),$$

such that $\mathcal{A}$ is special with respect to ι_0. We further assume that $\mathcal{A}$ is equipped with a polarization

$$\lambda : \mathcal{A} \xrightarrow{\sim} \hat{\mathcal{A}},$$

such that for the Rosati involution $\iota_0(a)^* = \iota_0(a)$.

We consider the following functor $\mathcal{M}^{HB}$ on the category Nilp of W-schemes S such that p is locally nilpotent in $\mathcal{O}_S$. It associates to a scheme $S \in$ Nilp the set of isomorphism classes of the following data.

(1) A p-divisible group X over S, with an action

$$\iota_0 : \mathbb{Z}_{p^2} \to \mathrm{End}(X),$$

such that X is special with respect to this $\mathbb{Z}_{p^2}$-action.

(2) A quasi-isogeny of height zero

$$\varrho : \mathcal{A} \times_{\mathrm{Spec}\, k} \overline{S} \to X \times_S \overline{S},$$

which commutes with the action of $\mathbb{Z}_{p^2}$ such that the following condition holds. Let $\lambda_{\overline{S}} : \mathcal{A}_{\overline{S}} \to \hat{\mathcal{A}}_{\overline{S}}$ be the map induced by λ. Then we require the existence of an isomorphism $\tilde{\lambda} : X \to \hat{X}$ such that for the induced map $\tilde{\lambda}_{\overline{S}} : X_{\overline{S}} \to \hat{X}_{\overline{S}}$ we have $\lambda_{\overline{S}} = \hat{\varrho} \circ \tilde{\lambda}_{\overline{S}} \circ \varrho$.

(Here, as in the preceding sections, a p-divisible group X over S with $\mathbb{Z}_{p^2}$-action is said to be special if the induced $\mathbb{Z}_{p^2} \otimes \mathcal{O}_S$-module $\mathrm{Lie}\, X$ is, locally on S, free of rank 1.)

Denote the isocrystal of $\mathcal{A}$ by N. From the polarization λ we get we a perfect symplectic form $\langle , \rangle$ on the Dieudonné module of $\mathcal{A}$ and hence also on N. Let Λ_0 be a W-lattice in N which is stable under F and V and under the action of $\mathbb{Z}_{p^2}$ and for which the dual lattice equals Λ_0 (via the identification induced by $\langle , \rangle$). The functor $\mathcal{M}^{HB}$ is a special case of [RZ], Definition 3.21. (In the notation of loc. cit., $O_B = \mathbb{Z}_{p^2}$ and $\mathcal{L} = \{p^i \Lambda_0; i \in \mathbb{Z}\}$. In loc. cit. there are imposed some additional conditions which are automatic here.) By Theorem 3.25 of loc. cit. the functor $\mathcal{M}^{HB}$ is representable by a formal scheme which we also call $\mathcal{M}^{HB}$. This formal scheme is formally locally of finite type over W and is formally smooth over $\mathbb{Z}_p$.

Following [KR2], §5, we define in this context the space of special endomorphisms

$$V' = \{j \in \mathrm{End}(N; F); j\iota_0(a) = \iota_0(a^\sigma)j \text{ and } j^* = j\},$$

where $*$ denotes the adjoint with respect to the alternating form $\langle , \rangle$. As shown in loc. cit., V' is a 4-dimensional vector space over $\mathbb{Q}_p$ with quadratic form

$$Q(j) = j^2.$$

For $j \in V'$ we define the special cycle $Z(j)$ associated to j to be the closed formal subscheme of $\mathcal{M}^{HB}$ consisting of all points (X, ϱ) such that $\varrho \circ j \circ \varrho^{-1}$ lifts to an endomorphism of X. Again, the fact that $Z(j)$ is a closed formal subscheme of $\mathcal{M}^{HB}$ follows from [RZ], Proposition 2.9. We fix $j_1 \in V'$ with $j_1^2 = \varepsilon_1 p$ for some unit $\varepsilon_1 \in \mathbb{Z}_p^\times$. (In [KR2], p. 188 the space V' is described more precisely and from this description one easily sees that such j_1 exist.) We define

$$V'[j_1] = \{j \in V' \mid j \perp j_1 \text{ with respect to the bilinear form associated to } Q\}.$$

Our next aim is to identify $Z(j_1)$ with the Drinfeld moduli scheme $\mathcal{M}$ introduced in §2. Let $\varepsilon \in \mathbb{Z}_{p^2}^\times$ be such that $\varepsilon \cdot \varepsilon^\sigma = \varepsilon_1$. Let $(X, \varrho) \in Z(j_1)$. We define an O_B-operation

ι on the points X by keeping the $\mathbb{Z}_{p^2}$-action ι_0 and by setting $\iota(\Pi) = \iota(\varepsilon^{-1})j_1$. Since $\mathcal{A}$ has height 4, for any point (X, ϱ) the p-divisible group X also has height 4. Since X is special, it has dimension 2. We must check that the condition given in (2) in the definition of $\mathcal{M}^{HB}$ is automatic for $\mathcal{M}$. But this is done in the proof of Proposition 3.3, Chapitre III of [BC]. (We may suppose that $\mathbb{X}$ is superspecial. Then the diagram on p. 138 of loc. cit. in case $S = \mathrm{Spec}(B)$, where B is here as in loc. cit. a $\mathbb{Z}_p^{\mathrm{nr}}$-algebra in which p is nilpotent, is the diagram that we need in this case (in loc. cit. the isomorphism $\tilde{\lambda}$ is called λ). Since the solution for $\tilde{\lambda}$ is unique in this case, we get the general case by glueing the solutions in formal neighborhoods of the geometric points.)

Lemma 4.1 *Let $j, j' \in V'$. Then $\iota_0(\delta)jj'$ has trace* 0.

Proof. We can choose a basis e_1, e_2, e_3, e_4 of the Dieudonné module M of $\mathcal{A}$ such that $e_1, e_2 \in M_0$ and $e_3, e_4 \in M_1$ and such that the matrix of j resp. the matrix of j' has the form

$$j = \begin{pmatrix} & & d & -b \\ & & -c & a \\ a & b & & \\ c & d & & \end{pmatrix}, \text{ resp. } j' = \begin{pmatrix} & & d' & -b' \\ & & -c' & a' \\ a' & b' & & \\ c' & d' & & \end{pmatrix}.$$

(Compare the proof of Proposition 4.5 below or [KR2], §5.) The diagonal entries of jj' are $(da' - bc'), (-cb' + ad'), (ad' - bc'), (-cb' + da')$. Since $\iota_0(\delta)$ acts on M_0 by multiplication by δ and on M_1 by multiplication by $-\delta$, the claim follows. □

Proposition 4.2 *Let $* = \mathrm{Int}(\delta j_1)$ where we identify B with $\mathbb{Q}_{p^2}[j_1]$. If $j \in V'[j_1]$ then j is $*$-special.*

Proof. Since for $j \in V'[j_1]$ we have $j\iota(\delta)j^{-1} = \iota(-\delta)$ and $jj_1j^{-1} = -j_1$, it follows that $j\iota(a)j^{-1} = \iota(a^*)$ for all $a \in B$. Using Lemma 4.1 the claim follows. □

From this proposition we conclude

Corollary 4.3 *Let $j \in V'[j_1]$. Identifying $Z(j_1)$ with $\mathcal{M}$ as above, the intersection $Z(j_1) \cap Z(j)$ equals the antispecial cycle $Z(j)$ in $\mathcal{M}$.* □

Remark 4.4 In [KR2], p. 243 it is erroneously asserted that conjugation by $j \in V'[j_1]$ induces the main involution on B.

Proposition 4.5 *Let $j \in V'$ be such that $Q(j) \neq 0$ and $Z(j) \neq \emptyset$. Then $Z(j)$ is a divisor in $\mathcal{M}^{HB}$.*

Proof. Let $x \in \mathcal{M}^{HB}$ be a closed point which belongs to $Z(j)$, and let R be the local ring $\mathcal{O}_{\mathcal{M}^{HB},x}$. Let J be the ideal of R coming from the ideal sheaf of $Z(j)$, and let $\mathfrak{m}$ be the maximal ideal of R. We must show that J is a nonzero principal ideal. Let $A := R/(\mathfrak{m}J)$, and let $\overline{A} := R/J$. We have $\overline{A} = A/I$, where $I = J/(\mathfrak{m}J)$. By Nakayama's lemma it is enough to show that I is a nonzero principal ideal. Since $I^2 = 0$, the ideal I carries a nilpotent *pd*-structure. Now A is separable and complete for the topology defined by the ideal (p). Therefore, by considering projective limits, it follows that we can apply Grothendieck-Messing theory for the pair A, $\overline{A}$. There is an A-valued and an $\overline{A}$-valued point of $\mathcal{M}^{HB}$ in the natural way. The latter also gives an $\overline{A}$-valued point of $Z(j)$. Let M be the value in A of the crystal of the p-divisible group belonging to the A-valued point of $\mathcal{M}^{HB}$, and let $\overline{M}$ be the value in $\overline{A}$ of the crystal of the p-divisible group belonging to the $\overline{A}$-valued point of $\mathcal{M}^{HB}$. These are free modules of rank 4 over A resp. $\overline{A}$, which are equipped with a perfect alternating form $\langle,\rangle$ and a $\mathbb{Z}_{p^2}$-action ι_0 such that $\iota_0(a)$ is selfadjoint. We have $\overline{M} = M \otimes_A \overline{A}$. From the $\mathbb{Z}_{p^2}$-actions we get $\mathbb{Z}/2$-gradings $M = M_0 \oplus M_1$ resp. $\overline{M} = \overline{M}_0 \oplus \overline{M}_1$.

Let $\overline{\mathcal{F}} \hookrightarrow \overline{M}$ be the Hodge filtration over $\overline{A}$ corresponding to the $\overline{A}$-valued point of $Z(j)$. (The submodule $\overline{\mathcal{F}}$ is free of rank 2 and stable under $\mathbb{Z}_{p^2}$ and under j.) By Grothendieck-Messing Theory, to a lifting of the corresponding p-divisible group over A corresponds a lifting of the Hodge filtration over A. Let $\mathcal{F} \hookrightarrow M$ be the Hodge filtration corresponding to the A-valued point of $\mathcal{M}^{HB}$. This lifts the Hodge filtration over $\overline{A}$.

Let $\{e_1, e_2, e_3, e_4\}$ be a basis of M such that $e_1, e_2 \in M_0$ and $e_3, e_4 \in M_1$. Then the images $\overline{e_i}$ of the e_i in $\overline{M}$ form a basis of $\overline{M}$. We may suppose that $\{\overline{e_2}, \overline{e_3}\}$ is a basis of $\overline{\mathcal{F}}$. Since $M_0 \perp M_1$ (which follows from the fact that $\iota_0(\delta)$ is selfadjoint), and since the determinant of the matrix of the bilinear form $\langle,\rangle$ is a unit, we may suppose that $\langle e_1, e_2\rangle = \langle e_3, e_4\rangle = 1$. As in [KR2], §5 it follows that with respect to the basis $e_1, ..., e_4$ the matrix of j has the form

$$j = \begin{pmatrix} & j_1 \\ j_0 & \end{pmatrix},$$

where $j_i \in \mathrm{Hom}(M_i, M_{i+1})$ and $j_1 = j_0^*$. Further for

$$j_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we have

$$j_1 = j_0^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Since $\mathcal{F}$ lifts $\overline{\mathcal{F}}$ we find a basis f_0, f_1 of $\mathcal{F}$ such that $f_0 = e_2 + m_1 e_1$ and $f_1 = e_3 + m_4 e_4$ for some $m_1, m_4 \in I$.

Now, let $\mathfrak{b} \subset I$ be an ideal in A and let $B = A/\mathfrak{b}$.

Claim: *The map* $\mathrm{Spf}\, B \to \mathcal{M}^{HB}$ *factors through* $Z(j)$ *if and only if* $bm_4 - cm_1 - d = 0$ *in* B.

By the same reasons as above we may apply Grothendieck-Messing theory to the pairs $B, \overline{A}$ and A, B. Let M_B be the value in B of the crystal of the p-divisible group belonging to the B-valued point of $\mathcal{M}^{HB}$. It equals $M \otimes_A B$. To the lifting $\mathrm{Spf}\, B \to \mathcal{M}^{HB}$ of $\mathrm{Spf}\, \overline{A} \to \mathcal{M}^{HB}$ corresponds the Hodge filtration $\mathcal{F}_B \hookrightarrow M_B$, where $\mathcal{F}_B = \mathcal{F} \otimes_A B$. (Note that the map $\mathrm{Spf}\, A \to \mathcal{M}^{HB}$ lifts $\mathrm{Spf}\, B \to \mathcal{M}^{HB}$.)

In M_B we have $j(f_0) = be_3+de_4+m_1(ae_3+ce_4)$. The image $j(f_0)$ lies in $\mathcal{F}_B$ if and only if $j(f_0) = xf_1$ in $\mathcal{M}_B$ for some $x \in B$. That means $be_3+de_4+m_1(ae_3+ce_4) = x(e_3+m_4e_4)$ in $\mathcal{M}_B$, hence $x = b + am_1$ and $xm_4 = d + cm_1$ in B, and hence

$$j(f_0) \in \mathcal{F} \Leftrightarrow bm_4 - cm_1 - d = 0 \text{ in } B.$$

Evaluating the corresponding condition for $j(f_1)$ one gets the same equation. This confirms the claim.

In case $\mathfrak{b} = I$ it follows in particular that $bm_4 - cm_1 - d = 0$ in $\overline{A}$. Hence in A we have an inclusion $(bm_4 - cm_1 - d) \subset I$. Therefore we can apply the claim in case $\mathfrak{b} = (bm_4 - cm_1 - d)$ and conclude that the map Spf $A/(bm_4 - cm_1 - d) \to \mathcal{M}^{HB}$ factors through $Z(j)$. On the other hand, by definition, I is the minimal ideal of A with the property that Spf $A/I \to \mathcal{M}^{HB}$ factors through $Z(j)$. It follows that $I = (bm_4 - cm_1 - d)$.

It remains to show that the equation for $Z(j)$ is nowhere trivial. If the equation is trivial in some local ring of $\mathcal{M}^{HB}$ then it follows that there is some (open) formal affine neighborhood in which the equation for $Z(j)$ is trivial. If the equation for $Z(j)$ is not trivial in some local ring of $\mathcal{M}^{HB}$ then it follows that there is some (open) formal affine neighborhood in which $Z(j)$ is given by one non trivial equation. It follows that the set of points of $\mathcal{M}^{HB}$ in whose local ring the equation for $Z(j)$ is trivial and the set of points of $\mathcal{M}^{HB}$ in whose local ring the equation for $Z(j)$ is not trivial are both open. By [KR2], Lemma 8.2 the latter set is not empty. Since $\mathcal{M}^{HB}$ is connected, it follows that the equation for $Z(j)$ is nowhere trivial. □

Lemma 4.6 *Suppose that $j \in V'[j_1]$. Then*

$$\mathcal{O}_{Z(j_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}^{HB}}} \mathcal{O}_{Z(j)} = \mathcal{O}_{Z(j_1)} \otimes_{\mathcal{O}_{\mathcal{M}^{HB}}} \mathcal{O}_{Z(j)}.$$

More precisely, the object of the derived category on the l.h.s. is represented by the r.h.s..

Proof. By Proposition 4.5 we know that $Z(j_1) = \mathcal{M}$ is a divisor in $\mathcal{M}^{HB}$. Hence, locally on $\mathcal{M}^{HB}$, there is an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{M}^{HB}} \xrightarrow{f} \mathcal{O}_{\mathcal{M}^{HB}} \longrightarrow \mathcal{O}_{\mathcal{M}} \longrightarrow 0,$$

where $f = 0$ is the equation of $Z(j_1)$. Using this resolution we see that $\mathcal{O}_{Z(j_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}^{HB}}} \mathcal{O}_{Z(j)}$ is represented by the complex $\mathcal{O}_{Z(j)} \xrightarrow{f} \mathcal{O}_{Z(j)}$. Its cohomology sheaves are $\mathcal{O}_{Z(j)} \otimes \mathcal{O}_{Z(j_1)}$ and $\mathcal{T}or_1^{\mathcal{O}_{\mathcal{M}^{HB}}}(\mathcal{O}_{Z(j_1)}, \mathcal{O}_{Z(j)})$.

Hence we only have to show that $\mathcal{T}or_1^{\mathcal{O}_{\mathcal{M}^{HB}}}(\mathcal{O}_{Z(j_1)}, \mathcal{O}_{Z(j)}) = 0$. So we have to show that in every local ring of $Z(j)$ the image of f is not a zero divisor. For this we consider a local ring of $\mathcal{M}^{HB}$. Let $g = 0$ be the equation for $Z(j)$ in this local ring. (The proof of Proposition 4.5 shows that $Z(j)$ is in fact given by one equation in any local ring of $\mathcal{M}^{HB}$.) Then we have to show that g and the image of f in the local ring have no common prime divisor. (Since $\mathcal{M}^{HB}$ is regular, its local rings are unique factorization domains.) Assuming the contrary it would follow that $\mathcal{M}$ and $Z(j)$ have a common component. But this contradicts the fact that their intersection is pure one dimensional by the results of section 2, which we can apply because of Corollary 4.3. □

Let $j_2, j_3 \in V'[j_1]$ be such that the $\mathbb{Z}_p$-span $\mathbf{j} = \mathbb{Z}_p j_2 + \mathbb{Z}_p j_3$ has rank 2 as a submodule of V' and such that Q induces a nondegenerate bilinear form on $\mathbf{j}$. We further suppose that the matrix of the bilinear form on $\mathbf{j}$ associated to Q with respect to the basis $j_2,\ j_3$ is equivalent to $\mathrm{diag}(\varepsilon_2 p^{\beta_2}, \varepsilon_3 p^{\beta_2})$ with $\varepsilon_i \in \mathbb{Z}_p^\times$ and $1 \leq \beta_2 \leq \beta_3$. In this situation we define the intersection product of $Z(j_1)$, $Z(j_2)$ and $Z(j_3)$ by the *Euler-Poincaré* characteristic of the derived tensor product,

$$(Z(j_1), Z(j_2), Z(j_3)) := \chi(\mathcal{M}^{HB}, \mathcal{O}_{Z(j_1)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(j_2)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(j_3)}).$$

This is well defined since $Z(j_1) \cap Z(j_2) \cap Z(j_3)$ is proper over $\mathrm{Spec}\, k$. Indeed, $Z(j_1) \cap Z(j_2) \cap Z(j_3) = (\mathcal{M} \cap Z(j_2)) \cap (\mathcal{M} \cap Z(j_3)))$. This is included in $Z(\Pi j_2) \cap Z(\Pi j_3)$ regarded as an intersection of special cycles inside $\mathcal{M}$. But this is proper over $\mathrm{Spec}\, k$ by [KR1], Proposition 3.6 and Corollary 2.14.

Proposition 4.7 *There is an equality of intersection multiplicities on $\mathcal{M}^{HB}$ resp. $\mathcal{M}$,*

$$(Z(j_1), Z(j_2), Z(j_3)) = ((\mathcal{M} \cap Z(j_2)), (\mathcal{M} \cap Z(j_3))),$$

where we regard the intersections $\mathcal{M} \cap Z(j_i)$ as antispecial cycles in $\mathcal{M}$, cf. Corollary 4.3.

The latter intersection multiplicity is given explicitly by Theorem 3.1 for $* = \mathrm{Int}(\delta j_1)$ after replacing j_1, j_2 in this theorem by j_2, j_3.

Proof. We have

$$\begin{aligned} &\chi(\mathcal{M}^{HB}, \mathcal{O}_{Z(j_1)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(j_2)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(j_3)}) \\ =&\chi(\mathcal{M}^{HB}, (\mathcal{O}_{Z(j_1)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(j_2)}) \otimes^{\mathbb{L}}_{\mathcal{O}_{Z(j_1)}} (\mathcal{O}_{Z(j_1)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(j_3)})). \end{aligned}$$

By Lemma 4.6, and since $Z(j_1) = \mathcal{M}$, the latter expression equals

$$\begin{aligned} &\chi(\mathcal{M}, (\mathcal{O}_{\mathcal{M}} \otimes \mathcal{O}_{Z(j_2)}) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} (\mathcal{O}_{\mathcal{M}} \otimes \mathcal{O}_{Z(j_3)})) = \chi(\mathcal{M}, \mathcal{O}_{\mathcal{M} \cap Z(j_2)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{M} \cap Z(j_3)}) \\ =&((\mathcal{M} \cap Z(j_2)), (\mathcal{M} \cap Z(j_3))). \end{aligned}$$

Using Proposition 4.2 and Corollary 4.3 it follows that this intersection multiplicity can be calculated as in Theorem 3.1 for $* = \mathrm{Int}(\delta j_1)$. □

5 Representation densities

We recall that, for $S \in \mathrm{Sym}_m(\mathbb{Z}_p)$ and $T \in \mathrm{Sym}_n(\mathbb{Z}_p)$ with $\det(S) \neq 0$ and $\det(T) \neq 0$, the representation density is defined as

$$\alpha_p(S, T) = \lim_{t \to \infty} p^{-tn(2m-n-1)/2} \mid \{x \in M_{m,n}(\mathbb{Z}/p^t\mathbb{Z});\ S[x] - T \in p^t \mathrm{Sym}_n(\mathbb{Z}_p)\} \mid .$$

Given S as above, let

$$S_r = \begin{pmatrix} S & & \\ & 1_r & \\ & & -1_r \end{pmatrix}.$$

Then there is a rational function $A_{S,T}(X) \in \mathbb{Q}(X)$ of X such that

$$\alpha_p(S_r, T) = A_{S,T}(p^{-r}).$$

Let

$$\alpha_p^{'}(S,T) = \frac{\partial}{\partial X}(A_{S,T}(X))|_{X=1}.$$

(Comp. [KR1], §7.)

Let j_1 be as in section 4, and let $j_2, j_3 \in V^{'}[j_1]$ also be as in section 4, i.e. such that the $\mathbb{Z}_p$-span $\mathbf{j} = \mathbb{Z}_p j_2 + \mathbb{Z}_p j_3$ is of rank 2 and nondegenerate for the bilinear form associated to the quadratic form Q, and such that the matrix of this bilinear form on $\mathbf{j}$ with respect to the basis $j_2,\ j_3$ is equivalent to $\mathrm{diag}(\varepsilon_2 p^{\beta_2}, \varepsilon_3 p^{\beta_2})$ with $\varepsilon_i \in \mathbb{Z}_p^\times$ and $1 \leq \beta_2 \leq \beta_3$. Let $S = \mathrm{diag}(1,-1,1,-\Delta)$. Then $T = \mathrm{diag}(\varepsilon_1 p, \varepsilon_2 p^{\beta_2}, \varepsilon_3 p^{\beta_2})$ is represented by the space $V^{'}$, hence T is not represented by S, see [Ku], Proposition 1.3.

Theorem 5.1 *Using the notation just introduced we have*

$$(Z(j_1), Z(j_2), Z(j_3)) = -\frac{p^4}{(p^2+1)(p^2-1)}\alpha_p^{'}(S,T).$$

Proof. For $\eta \in \mathbb{Z}_p^\times$ let $S(\eta) = \mathrm{diag}(1,-1,1,-\eta)$ so that $S = S(\Delta)$. Then it follows from [S], Lemma 3.5, that there is a polynomial $g_T(X) \in \mathbb{Z}[X]$ such that $\alpha_p(S(\eta)_r, T) = g_T(\chi(\eta)\cdot p^{-r-2})$, where χ denotes the quadratic residue character on $\mathbb{Z}_p^\times$. On the other hand, by [Ka] there exists a polynomial $f_T(X) \in \mathbb{Q}[X]$ with $\alpha_p(S(1)_r, T) = f_T(p^{-r})$. Hence for any $r \in \mathbb{N}$ we have $g_T(p^{-r-2}) = f_T(p^{-r})$ and hence $g_T(p^{-2}X) = f_T(X)$. Therefore $\alpha_p(S_r, T) = f_T(-p^{-r})$ and hence $A_{S,T}(X) = f_T(-X)$. Katsurada's polynomial $f_T(X)$ is given explicitly in [W]. Following this article, we can write

$$f_T(X) = \tilde{\gamma}_p(T;X)\tilde{F}_p(T;X),$$

where

$$\tilde{\gamma}_p(T;X) = (1-p^{-2}X)(1-p^{-2}X^2)$$

and where $\tilde{F}_p(T;X)$ is defined as follows. First we define some invariants of $T = \mathrm{diag}(\varepsilon_1 p, \varepsilon_2 p^{\beta_2}, \varepsilon_3 p^{\beta_2})$. Let

$$\tilde{\xi} = \begin{cases} \chi(-\varepsilon_1\varepsilon_2) & \text{if } \beta_2 \text{ is odd,} \\ 0 & \text{if } \beta_2 \text{ is even,} \end{cases}$$

and let

$$\sigma = \begin{cases} 2 & \text{if } \beta_2 \text{ is odd,} \\ 1 & \text{if } \beta_2 \text{ is even.} \end{cases}$$

Further, let

$$\eta = \begin{cases} +1 & \text{if } T \text{ is isotropic,} \\ -1 & \text{if } T \text{ is anisotropic.} \end{cases}$$

By [W], 2.11, we then have

$$\begin{aligned} \tilde{F}_p(T;X) = &\sum_{i=0}^{1} \sum_{j=0}^{(1+\beta_2-\sigma)/2-i} p^{i+j} X^{i+2j} \\ &+ \eta \sum_{i=0}^{1} \sum_{j=0}^{(1+\beta_2-\sigma)/2-i} p^{(1+\beta_2-\sigma)/2-j} X^{\beta_3+\sigma+i+2j} \\ &+ \tilde{\xi}^2 p^{(1+\beta_2-\sigma+2)/2} \sum_{i=0}^{1} \sum_{j=0}^{\beta_3-\beta_2+2\sigma-4} \tilde{\xi}^j X^{\beta_2-\sigma+2+i+j}. \end{aligned}$$

To distinguish whether T is isotropic or anisotropic we recall the following fact (see [W], p. 189). Let $i, j \in \{1,2,3\}$ with $i \neq j$ and $\beta_i \equiv \beta_j \mod 2$, and define $k \in \{1,2,3\}$ by $\{i,j,k\} = \{1,2,3\}$. Then T is isotropic if and only if $\chi(-\varepsilon_i\varepsilon_j) = 1$ or $\beta_k \equiv \beta_j \mod 2$. On the other hand, since T is represented by V' we have

$$-1 = (-1)^{1+\beta_2+\beta_3} \chi(-1)^{1+\beta_2+\beta_3+1\cdot\beta_2+1\cdot\beta_3+\beta_2\beta_3} \chi(\varepsilon_1)^{\beta_2+\beta_3} \chi(\varepsilon_2)^{1+\beta_3} \chi(\varepsilon_3)^{1+\beta_2}, \qquad (5.1)$$

see [Ku], (1.16).

Now we evaluate $\alpha_p'(S,T) = \frac{\partial}{\partial X}(\tilde{\gamma}_p(T;-X)\tilde{F}_p(T;-X))|_{X=1}$. We show that $\alpha_p'(S,T)$ equals $-(1+p^{-2})(1-p^{-2})$ times the expression given in Theorem 3.1 in case $* = \mathrm{Int}(\delta j_1)$ (comp. Proposition 4.7) after replacing ε_1 resp. ε_2 and α_1 resp. α_2 in Theorem 3.1 by ε_2 resp. ε_3 and β_2 resp. β_3. In the notation of Theorem 3.1 we have $\eta_* = -\Delta\varepsilon_1$. (Note that by Remark 3.2 (using $\beta \geq 1$) we do not need to exclude the case $p = 3$.) Using these substitutions we distinguish the same cases as in Theorem 3.1.

First case: *β_2 is odd and $\chi(-\Delta\varepsilon_1\varepsilon_2) = -1$, i.e. $\chi(-\varepsilon_1\varepsilon_2) = 1$.*

It follows immediately that $\tilde{\xi} = 1$ and $\sigma = 2$.

By (5.1) we get $-1 = (-1)^{\beta_3}\chi(-\varepsilon_1\varepsilon_2)^{1+\beta_3}$, hence β_3 is odd. From the criterion above we see that T is isotropic and hence $\eta = 1$. Now an easy calculation shows $\tilde{F}_p(T;-1) = 0$ and

$$\frac{\partial}{\partial X}\tilde{F}_p(T;-X)|_{X=1} = -\beta_2 - \beta_3 - 3 + p^{(\beta_2+1)/2} + 2\frac{p^{(\beta_2+1)/2} - 1}{p-1},$$

which yields the claim in this case.

Second case: *β_2 is odd and $\chi(-\Delta\varepsilon_1\varepsilon_2) = 1$, i.e. $\chi(-\varepsilon_1\varepsilon_2) = -1$.*

It follows immediately that $\tilde{\xi} = -1$ and $\sigma = 2$. We distinguish the subcases β_3 *even* and β_3 *odd.*

If β_3 is even we see from the criterion above that T is anisotropic and hence $\eta = -1$.

If β_3 is odd the criterion shows that T is isotropic and hence $\eta = 1$.

In both cases an easy calculation shows $\tilde{F}_p(T;-1) = 0$ and

$$\frac{\partial}{\partial X}\tilde{F}_p(T;-X)|_{X=1} = -\beta_2 - \beta_3 - 3 + (\beta_3 - \beta_2 + 1)p^{(\beta_2+1)/2} + 2\frac{p^{(\beta_2+1)/2} - 1}{p-1},$$

which yields the claim in this case.

Third case: *β_2 is even.*

It follows immediately that $\tilde{\xi} = 0$ and $\sigma = 1$. We distinguish the subcases β_3 *even* and β_3 *odd.*

If β_3 is even we get from (5.1) that $\chi(-\varepsilon_2\varepsilon_3) = 1$, and by the criterion above T is isotropic in this case and hence $\eta = 1$.

If β_3 is odd we get from (5.1) that $\chi(-\varepsilon_1\varepsilon_3) = -1$, and by the criterion above T is anisotropic in this case and hence $\eta = -1$.

In both cases an easy calculation shows $\tilde{F}_p(T;-1) = 0$ and

$$\frac{\partial}{\partial X}\tilde{F}_p(T;-X)|_{X=1} = -\beta_2 - \beta_3 - 3 + 2\frac{p^{\beta_2/2+1} - 1}{p-1},$$

which yields the claim in this case. □

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