

Intersections of arithmetic Hirzebruch-Zagier cycles

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Introduction

In this paper we will establish a close connection between the intersection multiplicities of three arithmetic Hirzebruch-Zagier cycles and the Fourier coefficients of the derivative of a certain Siegel-Eisenstein series at its center of symmetry. Our main result proves a conjecture of Kudla and Rapoport.

Kudla has proposed a general program which relates intersection multiplicities of special cycles on (arithmetic models of) Shimura varieties to Fourier coefficients (of derivatives) of Eisenstein series, see his paper [Ku1], his ICM-talk [Ku2], his Bourbaki-talk [Ku3] and his CDM-talk [Ku4]. Besides the paper [Ku1], which is based on the paper [GK] by Gross and Keating, other examples of results in this direction may be found in the papers [KR1] - [KR4] by Kudla and Rapoport, in the monograph [KRY] by Kudla, Rapoport and Yang, in [AR], which is mainly an exposition of [GK], in the papers [H1], [H2] and [H3] by Howard and in other papers. The present paper contributes to this program in a situation of degenerate intersections.

We now describe our results in detail.

We fix a prime number $p \neq 2$. Arithmetic Hirzebruch-Zagier cycles are defined in the paper [KR2] by Kudla and Rapoport as cycles over a certain moduli scheme \mathcal{M} of abelian schemes over $\mathbb{Z}_{(p)}$. The generic fibre of \mathcal{M} is the canonical model over \mathbb{Q} of a Hilbert-Blumenthal surface.

Let us make this more precise. Let V be a quadratic space over \mathbb{Q} with signature $(2, 2)$. Let $C(V)$ be the corresponding Clifford algebra, and let $C^+(V)$ be its even part. Then $C^+(V)$ is of the form $B_0 \otimes_{\mathbb{Q}} k$, where B_0 is an indefinite quaternion algebra over \mathbb{Q} , and k , the center of $C^+(V)$, is a real quadratic extension of \mathbb{Q} .

We suppose that there exists a self-dual $\mathbb{Z}_{(p)}$ -lattice $\Lambda \subset V$ (satisfying an additional technical condition, see section 1), and we fix such a lattice. Then p is unramified in k . We assume that p is inert in k , in particular, k is a field (see [KR2], section 11 for the case of a split prime). Let \mathcal{O}_C be the Clifford algebra of Λ and let \mathcal{O}_k be the ring of $\mathbb{Z}_{(p)}$ -integers in k . Finally, we consider the algebraic group G over \mathbb{Q} with

$$G(R) = \{g \in (C^+(V) \otimes_{\mathbb{Q}} R)^{\times}; \nu(g) \in R^{\times}\}$$

for any \mathbb{Q} -algebra R . Here ν denotes the spinor norm. We denote by \mathbb{A} the adèles of \mathbb{Q} , by \mathbb{A}_f the finite adèles of \mathbb{Q} and by \mathbb{A}_f^p the finite adèles of \mathbb{Q} with trivial p -adic component. Further $(\text{Sch}_{\text{ln}}/\mathbb{Z}_{(p)})$ denotes the category of locally noetherian schemes over $\text{Spec } \mathbb{Z}_{(p)}$.

If $K^p \subset G(\mathbb{A}_f^p)$ is a sufficiently small compact open subgroup, there is a moduli scheme \mathcal{M} which represents the following functor on $(\text{Sch}_{\text{ln}}/\mathbb{Z}_{(p)})$. It associates to a locally noetherian $\mathbb{Z}_{(p)}$ -scheme S the set of isomorphism classes of tuples $(A, \lambda, \iota, \overline{\eta}^p)$, where A is an abelian scheme over S up to prime to p isogeny, λ is a $\mathbb{Z}_{(p)}^{\times}$ -class of p -principal polarizations on A , further $\iota : \mathcal{O}_C \otimes \mathcal{O}_k \rightarrow \text{End}(A) \otimes \mathbb{Z}_{(p)}$ is a homomorphism (satisfying a compatibility condition with the Rosati involution induced by λ), and $\overline{\eta}^p$ is a K^p -level structure. See section 1 for the precise conditions. Then \mathcal{M} is quasi-projective and smooth of relative dimension 2 over $\text{Spec } \mathbb{Z}_{(p)}$.

To define arithmetic Hirzebruch-Zagier cycles, we recall that a *special endomorphism* of a tuple (A, λ, ι) over $S \in (\text{Sch}_{\text{In}}/\mathbb{Z}_{(p)})$ (as above) is an element $j \in \text{End}(A) \otimes \mathbb{Z}_{(p)}$ such that $j = j^*$ for the Rosati involution and satisfying a compatibility condition with ι , see section 1. If S is connected, then for any $(A, \lambda, \iota, \overline{\eta^p})$ over S as above the $\mathbb{Z}_{(p)}$ -module of special endomorphisms of (A, λ, ι) becomes a quadratic $\mathbb{Z}_{(p)}$ -module via the quadratic form Q given by $j^2 = Q(j) \cdot \text{id}$.

Now let $\omega \subset V(\mathbb{A}_f^p)$ be compact, open and stable under the K^p -action, and let $t \in \mathbb{Q}^\times$. We consider the functor on $(\text{Sch}_{\text{In}}/\mathbb{Z}_{(p)})$ which associates to $S \in (\text{Sch}_{\text{In}}/\mathbb{Z}_{(p)})$ the set of isomorphism classes of 5-tuples $(A, \lambda, \iota, \overline{\eta^p}, j)$, where $(A, \lambda, \iota, \overline{\eta^p})$ is as before, and where j is a special endomorphism of (A, λ, ι) such that $j^2 = t$ and satisfying a compatibility condition with $\overline{\eta^p}$ and ω . Again, if K^p is sufficiently small, this functor is representable by a scheme $Z(t, \omega)$ which maps by a finite and unramified morphism (forgetting j) to \mathcal{M} . An *arithmetic Hirzebruch-Zagier cycle* or *special cycle* is a scheme of the form $Z(t, \omega)$.

We now fix three special cycles $Z_1 = Z(t_1, \omega_1)$, $Z_2 = Z(t_2, \omega_2)$ and $Z_3 = Z(t_3, \omega_3)$. Let

$$Z = Z_1 \times_{\mathcal{M}} Z_2 \times_{\mathcal{M}} Z_3.$$

A nonempty special cycle is of dimension 2 while \mathcal{M} is of dimension 3. This suggests to intersect three special cycles. If $S \in (\text{Sch}_{\text{In}}/\mathbb{Z}_{(p)})$ is connected and $\xi = (A, \lambda, \iota, \overline{\eta^p}, j_1, j_2, j_3) \in Z(S)$, then following [KR2] we associate to ξ its *fundamental matrix* T_ξ whose entry at (a, b) is given by $\frac{1}{2}(Q(j_a + j_b) - Q(j_a) - Q(j_b))$. It is a symmetric 3×3 matrix with entries in \mathbb{Q} . The map $\xi \mapsto T_\xi$ is locally constant, and hence we can write

$$Z = \coprod_T Z_T,$$

where Z_T is the union of the connected components of Z which have fundamental matrix T . We assume from now on that Z is not empty. Let $T \in \text{Sym}_3(\mathbb{Q})$ such that Z_T is not empty. Note that then all entries of T lie in $\mathbb{Z}_{(p)}$ and that the diagonal entries of T are t_1, t_2, t_3 . For such a T we define the intersection multiplicity

$$\chi_T(Z_1, Z_2, Z_3) = \chi(Z_T, \mathcal{O}_{Z_1} \otimes^{\mathbb{L}} \mathcal{O}_{Z_2} \otimes^{\mathbb{L}} \mathcal{O}_{Z_3}).$$

Here, $\otimes^{\mathbb{L}}$ is the derived tensor product of $\mathcal{O}_{\mathcal{M}}$ -modules and χ is the Euler-Poincaré characteristic.

If T is singular, then the generic fibre of Z_T might be nonempty. Hence we additionally assume that T is nonsingular. Then Z_T lies over the supersingular locus of \mathcal{M} and its support is proper over \mathbb{F}_p so that $\chi_T(Z_1, Z_2, Z_3)$ is finite. One of the main results of [KR2] states that Z_T is a discrete set of points if and only if T (as matrix over $\mathbb{Z}_{(p)}$) is not divisible by p and that it is otherwise of dimension 1.

It is the aim of this paper to give an explicit expression for $\chi_T(Z_1, Z_2, Z_3)$ and to express it in terms of certain Eisenstein series. The case that Z_T is a discrete set of points was treated in [KR2], so we will focus on the case of a degenerate intersection, i.e., the case that T is divisible by p .

We now come to the Eisenstein series. Let $P \subset \text{Sp}_6$ be the Siegel parabolic subgroup. It is given as follows. $P = MN$, where

$$M = \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & {}_t a^{-1} \end{pmatrix} \mid a \in \text{GL}(3) \right\}$$

and

$$N = \left\{ n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \text{Sym}_3 \right\}.$$

Let $\tilde{K} = \tilde{K}_\infty \tilde{K}_f = \prod_v \tilde{K}_v$, where

$$\tilde{K}_v = \begin{cases} \mathrm{Sp}_6(\mathbb{Z}_l), & \text{if } v = l < \infty; \\ \left\{ \left(\begin{array}{cc} a & b \\ -b & a \end{array} \right) \mid a + ib \in \mathrm{U}_3(\mathbb{R}) \right\}, & \text{if } v = \infty. \end{cases}$$

Then we have the Iwasawa decomposition $\mathrm{Sp}_6(\mathbb{A}) = P(\mathbb{A})\tilde{K} = N(\mathbb{A})M(\mathbb{A})\tilde{K}$.

To $\omega = \omega_1 \times \omega_2 \times \omega_3$ we will associate in section 6 a section Φ of the induced representation $I_3(s, \chi_V)$ of $\mathrm{Sp}_6(\mathbb{A})$ (see [Ku1]). We consider the following Eisenstein series on $\mathrm{Sp}_6(\mathbb{A})$ (as in [KR2], see also [RW])

$$E(g, s, \Phi) = \sum_{\gamma \in P(\mathbb{Q}) \backslash \mathrm{Sp}_6(\mathbb{Q})} \Phi(\gamma g, s).$$

There is a Fourier decomposition (see [Ku1])

$$E(g, s, \Phi) = \sum_{U \in \mathrm{Sym}_3(\mathbb{Q})} E_U(g, s, \Phi).$$

This yields a Fourier decomposition of its derivative

$$E'(g, s, \Phi) = \sum_{U \in \mathrm{Sym}_3(\mathbb{Q})} E'_U(g, s, \Phi).$$

The main theorem of this paper is the following.

Theorem 0.1 *Let $h \in \mathrm{Sp}_6(\mathbb{R})$ and suppose that $\omega_1 \times \omega_2 \times \omega_3$ is locally centrally symmetric. Then for any nonsingular $T \in \mathrm{Sym}_3(\mathbb{Q})$ such that Z_T is nonempty there is the following relation between the T -th Fourier coefficient of the derivative in $s = 0$ of $E(g, s, \Phi)$ and the intersection multiplicity $\chi_T(Z_1, Z_2, Z_3)$:*

$$E'_T(h, 0, \Phi) = -\frac{1}{2} \log(p) \cdot \kappa \cdot \chi_T(Z_1, Z_2, Z_3) \cdot W_T^2(h),$$

where κ is some volume constant given in section 6.

To explain this notation, we write $h = n(b)m(a)k \in N(\mathbb{R})M(\mathbb{R})\tilde{K}_\infty = \mathrm{Sp}_6(\mathbb{R})$. Let $\tau = b + i \cdot a^t a \in \mathfrak{H}_3$. Then the Whittaker function $W_T^2(h)$ (see [KR2], [Ku1]) is of the form

$$W_T^2(h) = c(a, k) e^{2\pi i \cdot \mathrm{tr}(T\tau)},$$

where $c(a, k)$ depends on a and k . Conversely, any $\tau \in \mathfrak{H}_3$ can be written as $\tau = b + i \cdot a^t a$, where $\det(a) > 0$. Having chosen such an a , let $h = n(b)m(a)$. One obtains a well defined Siegel modular form

$$\phi(\tau) = -\left(\frac{1}{2} \log(p) \cdot \kappa \cdot \det(a)\right)^{-1} \cdot E'(h, 0, \Phi)$$

of weight 2 on \mathfrak{H}_3 (wrt. a suitable arithmetic subgroup of $\mathrm{Sp}_6(\mathbb{Q})$), and for its Fourier decomposition we obtain for T as above

$$\phi_T(\tau) = \chi_T(Z_1, Z_2, Z_3) \cdot e^{2\pi i \cdot \mathrm{tr}(T\tau)},$$

comp. also [KR2], chapter 12.

The theorem was proved in [KR2] in case that Z_T is a discrete set of points or, equivalently, that T is not divisible by p .

We now sketch the further content of this paper and some of the techniques used to prove Theorem 0.1. We fix some nonsingular $T \in \text{Sym}_3(\mathbb{Q})$ such that Z_T is nonempty. Since T is nonsingular, Z_T lies over the supersingular locus of the special fibre of \mathcal{M} ([KR2], Corollary 3.8). Let $\mathbb{F} = \overline{\mathbb{F}}_p$ be a fixed algebraic closure of \mathbb{F}_p , and let us fix $\xi \in Z_T(\mathbb{F})$. We write $\xi = (A, \lambda, \iota, \overline{\eta}^p, j_1, j_2, j_3)$ and denote by $A(p^\infty)$ be the corresponding p -divisible group. It is equipped with an action by $\mathcal{O}_C \otimes \mathcal{O}_k \otimes \mathbb{Z}_p \cong M_4(\mathbb{Z}_{p^2})$, which allows us to write $A(p^\infty) = \mathcal{A}^4$. Then \mathcal{A} is a principally polarized supersingular formal p -divisible group over \mathbb{F} of height 4 and dimension 2 which is equipped with a \mathbb{Z}_{p^2} -action satisfying some compatibility condition, see section 2.

Let $W = W(\mathbb{F})$ be the ring of Witt vectors of \mathbb{F} . For any W -scheme S we write $\overline{S} = S \times_{\text{Spec } W} \text{Spec } \mathbb{F}$. We consider the following functor \mathcal{M}^{HB} on the category Nilp of W -schemes S such that p is locally nilpotent in \mathcal{O}_S . It associates to a scheme $S \in \text{Nilp}$ the set of isomorphism classes of pairs (X, ϱ) , consisting of a principally polarized p -divisible group X over S which is equipped with an \mathbb{Z}_{p^2} -action and a quasi-isogeny of height zero $\varrho : \mathcal{A} \times_{\text{Spec } \mathbb{F}} \overline{S} \rightarrow X \times_S \overline{S}$, such that certain compatibility conditions are satisfied, see section 2.

By Theorem 3.25 of [RZ], the functor \mathcal{M}^{HB} is representable by a formal scheme over $\text{Spf } W$ which we also denote by \mathcal{M}^{HB} . Let $\widehat{\mathcal{M}}_W / \widehat{\mathcal{M}}_W^{ss}$ be the completion of $\mathcal{M}_W = \mathcal{M} \times_{\text{Spec } \mathbb{Z}_{(p)}} \text{Spec } W$ along its supersingular locus \mathcal{M}_W^{ss} . Then $\widehat{\mathcal{M}}_W / \widehat{\mathcal{M}}_W^{ss}$ and \mathcal{M}^{HB} are closely connected via the uniformization theorem (Theorem 6.23 of loc. cit.), see (2.1). The underlying reduced subscheme of \mathcal{M}^{HB} is a union of projective lines over \mathbb{F} which are in bijection with the vertices of the building $\mathcal{B} := \mathcal{B}(\text{PGL}_2(\mathbb{Q}_{p^2}))$. Two such projective lines $\mathbb{P}_{[\Lambda]}$ and $\mathbb{P}_{[\Lambda']}$ intersect in a point if and only if the vertices $[\Lambda]$ and $[\Lambda']$ are neighbours in the building.

As in [KR2], we define

$$V' = \{j \in \text{End}(A) \otimes \mathbb{Q}; j = j^* \text{ and } \iota(c \otimes a) \circ j = j \circ \iota(c \otimes a^\sigma)\}.$$

It is a quadratic space over \mathbb{Q} of dimension 4. As above, the quadratic form is given by $j^2 = Q(j) \cdot \text{id}$. We may regard V' as a \mathbb{Q} -subspace of $\text{End}^0(\mathcal{A}) = \text{End}(\mathcal{A}) \otimes \mathbb{Q}$. We will also define a quadratic \mathbb{Q}_p -space V'_p of special endomorphisms of \mathcal{A} which contains V' as a dense \mathbb{Q} -subspace, cf. section 2.

For any $j \in V'$ (in fact also for $j \in V'_p$) we define the *formal special cycle* $Z(j)$ to be the closed formal subscheme of \mathcal{M}^{HB} consisting of all points (X, ϱ) such that $\varrho \circ j \circ \varrho^{-1}$ lifts to an endomorphism of X .

Recall our fixed three special endomorphisms j_1, j_2, j_3 associated to ξ . We define the intersection multiplicity $(Z(j_1), Z(j_2), Z(j_3))$ of the associated formal special cycles again to be the Euler-Poincaré characteristic of the derived tensor product of the structure sheaves of the $Z(j_i)$. One shows (Proposition 2.2) that the global intersection multiplicity $\chi_T(Z_1, Z_2, Z_3)$ is a multiple of $(Z(j_1), Z(j_2), Z(j_3))$ by an explicit integer. Thus in order to calculate $\chi_T(Z_1, Z_2, Z_3)$, it is enough to calculate $(Z(j_1), Z(j_2), Z(j_3))$. To this end we first investigate the structure of formal special cycles. Let us for the moment fix $j \in V'$ such that $j^2 \neq 0$ and $a := \nu_p(j^2) \geq 0$, where ν_p is the p -adic valuation. First we show that $Z(j)$ is a relative divisor in \mathcal{M}^{HB} . We give a complete description of the special fibre of $Z(j)$. To state the result, we recall from [KR2] that j gives rise to a morphism β of the building whose square is a scalar. Then $\nu_p(\beta^2) = a - 1$. For $[\Lambda] \in \mathcal{B}$ denote by $d_{[\Lambda]}$ the distance of $[\Lambda]$ to \mathcal{B}^β , the fixed point locus of β . Then \mathcal{B}^β is one single edge if a is even, and it is a subbuilding of the form $\mathcal{B}(\text{PGL}_2(\mathbb{Q}_p))$ if a is odd.

Theorem 0.2 *The special fibre $Z(j)_p$ of $Z(j)$ is of the form*

$$Z(j)_p = \sum_{[\Lambda]; d_{[\Lambda]} \leq \frac{a-1}{2}} (1 + p + \dots + p^{\frac{a-1}{2} - d_{[\Lambda]}}) \mathbb{P}_{[\Lambda]} + p^{a/2} \cdot s.$$

Here, s is a divisor in the special fibre of \mathcal{M}^{HB} which is $\neq 0$ if and only if a is even. In this case s is reduced and irreducible, and it meets the supersingular locus only in the unique supersingular point of $Z(j/p^{a/2})$.

This theorem is proved using the theory of displays of Zink ([Z1], [Z2]).

We also associate to any formal special cycle $Z(j)$ (such that $j^2 \in \mathbb{Z}_p \setminus \{0\}$) a divisor $D(j)$, the "difference divisor", whose ideal is locally generated by the quotient of a generator of the ideal of $Z(j)$ and a generator of the ideal of $Z(j/p)$. We show that $D(j)$ is always a regular formal scheme. The proof of this uses the fact that for $j_1 \in V'_p$ such that $\nu_p(j_1^2) = 1$, the formal special cycle $Z(j_1)$ is isomorphic to the formal model of the Drinfeld upper half plane (for \mathbb{Q}_p and base changed with $\mathrm{Spf} W$), and the fact that for $j \perp j_1$ the structure of $Z(j) \cap Z(j_1)$ is known from [T]. This "testing by the Drinfeld space" is one of the basic techniques in our paper.

The next step is to investigate the structure of intersections of formal special cycles. First we show the statements of the following

- Theorem 0.3** 1. Let $j, j' \in V'$ be linearly independent. Suppose further that for any $x \in \mathcal{M}^{HB}(\mathbb{F})$ the equation in x (that is a generator of the ideal in $\mathcal{O}_{\mathcal{M}^{HB}, x}$) of at least one of the corresponding two formal special cycles is not divisible by p . Then $\mathcal{O}_{Z(j)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(j')}$ is represented in the derived category by $\mathcal{O}_{Z(j)} \otimes \mathcal{O}_{Z(j')}$.
2. The derived tensor product $\mathcal{O}_{Z(j_1)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(j_2)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(j_3)}$ depends only on the $\mathbb{Z}_{(p)}$ -span \mathbf{j} of j_1, j_2, j_3 in V' .
3. If T (as matrix over $\mathbb{Z}_{(p)}$) is not divisible by p , then $\mathcal{O}_{Z(j_1)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(j_2)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(j_3)}$ is represented in the derived category by $\mathcal{O}_{Z(j_1)} \otimes \mathcal{O}_{Z(j_2)} \otimes \mathcal{O}_{Z(j_3)}$.
4. Suppose that at least one of $\nu_p(j_1^2), \nu_p(j_2^2), \nu_p(j_3^2)$ is odd. Then

$$(Z(j_1), Z(j_2), Z(j_3)) = \sum_{l,m,n} (D(j_1/p^l), D(j_2/p^m), D(j_3/p^n)),$$

where the sum is taken over all possible triples (l, m, n) (i.e. setting $a_i = \nu_p(j_i^2)$, we have $l \leq [a_1/2]$, $m \leq [a_2/2]$, $n \leq [a_3/2]$, where $[]$ denotes Gauss brackets).

Point 1 is the key observation for the proof of point 2. Since $p \neq 2$, point 2 allows us to assume that j_1, j_2, j_3 are pairwise perpendicular to each other, i.e., T is a diagonal matrix $T = \mathrm{diag}(\varepsilon_1 p^{a_1}, \varepsilon_2 p^{a_2}, \varepsilon_3 p^{a_3})$, where $\varepsilon_i \in \mathbb{Z}_{(p)}^\times$ for all i and $a_1 \leq a_2 \leq a_3$. In this case the assumption of point 4 is fulfilled, which allows us by induction to restrict ourselves to the calculation of the intersection multiplicity $(D(j_1), D(j_2), D(j_3))$. Point 3 shows that (if T is not divisible by p) the length (over $\mathbb{Z}_{(p)}$) of the local ring of a point in Z_T resp. the length of the artinian W -scheme $Z(j_1) \cap Z(j_2) \cap Z(j_3)$ is the same as the intersection multiplicity $(Z(j_1), Z(j_2), Z(j_3))$ in our sense (recall that in the situation of point 3, Z_T is a discrete set of points and the length of its local rings is calculated in [KR2]).

Given two perpendicular special endomorphisms $y_1, y_2 \in V'$, we investigate the multiplicities of the several projective lines $\mathbb{P}_{[\Lambda]}$ in the intersections $D(j_1) \cap D(j_2)$ as a divisor in (say) $D(j_1)$ (Propositions 4.6, 4.7). Further we investigate (the existence of) horizontal components of $D(j_1) \cap D(j_2)$, that is of components which do not have support in the special fibre (Propositions 4.10, 4.11). Using this we calculate the intersection multiplicity $(D(j_1), D(j_2), D(j_3))$ (resp. in some cases $(D(j_1), Z(j_2), D(j_3))$). By induction, this leads to the following

Theorem 0.4 *Suppose that T is $\mathrm{GL}_3(\mathbb{Z}_p)$ -equivalent to $\mathrm{diag}(\varepsilon_1 p^{a_1}, \varepsilon_2 p^{a_2}, \varepsilon_3 p^{a_3})$, where $\varepsilon_i \in \mathbb{Z}_p^\times$ for all i and $a_1 \leq a_2 \leq a_3$. Then there is the following explicit expression for the intersection multiplicity $(Z(j_1), Z(j_2), Z(j_3))$.*

$$\begin{aligned} (Z(j_1), Z(j_2), Z(j_3)) &= - \sum_{i=0}^{a_1} \sum_{j=0}^{\frac{a_1+a_2-\sigma}{2}-i} p^{i+j} (-1)^i (i+2j) \\ &\quad - \eta \sum_{i=0}^{a_1} \sum_{j=0}^{\frac{a_1+a_2-\sigma}{2}-i} p^{\frac{a_1+a_2-\sigma}{2}-j} (-1)^{a_3+\sigma+i} (a_3+\sigma+i+2j) \\ &\quad - \tilde{\xi}^2 p^{\frac{a_1+a_2-\sigma}{2}+1} \sum_{i=0}^{a_1} \sum_{j=0}^{a_3-a_2+2\sigma-4} \tilde{\xi}^j (-1)^{a_2-\sigma+i+j} (a_2-\sigma+2+i+j). \end{aligned}$$

See section 5 for the definition of the invariants $\eta \in \{\pm 1\}$, $\sigma \in \{1, 2\}$ and $\tilde{\xi} \in \{\pm 1, 0\}$ of T . As mentioned before, this theorem is proved by induction on $a_1 + a_2 + a_3$. The induction start is given by the treatment of the cases $a_1 = 0$ in [KR2] (together with point 3 of Theorem 0.3) and $a_1 = 1$ in [T].

We may express $(Z(j_1), Z(j_2), Z(j_3))$ in terms of some representation densities. To state the result, recall that for $S \in \mathrm{Sym}_m(\mathbb{Z}_p)$ and $U \in \mathrm{Sym}_n(\mathbb{Z}_p)$ with $\det(S) \neq 0$ and $\det(U) \neq 0$, the representation density is defined as

$$\alpha_p(S, U) = \lim_{t \rightarrow \infty} p^{-tn(2m-n-1)/2} |\{x \in M_{m,n}(\mathbb{Z}/p^t\mathbb{Z}); S[x] - U \in p^t \mathrm{Sym}_n(\mathbb{Z}_p)\}|.$$

Theorem 0.5 *Let $S = \mathrm{diag}(1, -1, 1, -\Delta)$, where $\Delta \in \mathbb{Z}_p^\times$ is not a square. There is the following relation between intersection multiplicities and representation densities:*

$$(Z(j_1), Z(j_2), Z(j_3)) = - \frac{p^4}{(p^2+1)(p^2-1)} \alpha'_p(S, T).$$

(See section 6 for an explanation of the derivative $\alpha'_p(S, T)$.) This theorem was proved before in case $a_1 = 0$ in [KR2] and in case $a_1 = 1$ in [T]. Using this theorem and the connection of $E'_T(h, 0, \Phi)$ and $\alpha'_p(S, T)$ (see the proof of Theorem 6.2), Theorem 0.1 follows.

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1 Arithmetic Hirzebruch-Zagier cycles

In this section we review the notion of arithmetic Hirzebruch-Zagier cycles and introduce the intersection problem we want to consider. We closely follow the paper [KR2] by Kudla and Rapoport to which we refer for more details.

Let V be a vector space over \mathbb{Q} of dimension 4 which is equipped with a quadratic form q such that the signature of V with respect to q is $(2, 2)$. Denote by $C^+(V)$ the even part of the Clifford algebra $C(V)$. Let β be the main involution of $C(V)$. Denote the center of $C^+(V)$ by k . It is a real quadratic extension of \mathbb{Q} . We consider the algebraic group G over \mathbb{Q} with

$$G(R) = \{g \in (C^+(V) \otimes_{\mathbb{Q}} R)^{\times}; \nu(g) = g \cdot g^{\beta} \in R^{\times}\}$$

for any \mathbb{Q} -algebra R . Let K be a compact open subgroup of $G(\mathbb{A}_f)$. One associates to G and K a Shimura variety $Sh(G, D)_K$ with \mathbb{C} -valued points

$$Sh(G, D)_K(\mathbb{C}) = G(\mathbb{Q}) \backslash [D \times G(\mathbb{A}_f)/K],$$

where D is the space of oriented negative 2-planes in $V(\mathbb{R})$. If K is sufficiently small, then there is a canonical model M_K over \mathbb{Q} of $Sh(G, D)_K$ which represents a moduli functor over \mathbb{Q} whose points consist of polarized abelian schemes with endomorphism structure and K -level structure. See [KR2] for the precise statement and more details.

Now let $p \neq 2$ be a prime number. We are interested in a model of M_K over $\mathbb{Z}_{(p)}$.

Let $v \in V$ such that $q(v) > 0$. Then $C^+(V)$ can be written in the form $B_0 \otimes_{\mathbb{Q}} k$, where B_0 is the fixed algebra of $\text{Ad}(v)$ in $C^+(V)$. Further B_0 is an indefinite quaternion algebra over \mathbb{Q} . We fix an element $\tau \in B_0^{\times}$ such that $\tau^{\beta} = -\tau$ and $\tau^2 < 0$. Then

$$x \mapsto x^* = \tau x^{\beta} \tau^{-1} \tag{1.1}$$

is a positive involution of $C(V)$. We assume that there exists a self-dual $\mathbb{Z}_{(p)}$ -lattice $\Lambda \subset V(\mathbb{Q})$. Then p is unramified in k . We further assume that $\Lambda = \tau \Lambda \tau^{-1}$, and we fix such a lattice Λ . Denote by \mathcal{O}_C the Clifford algebra of Λ , and denote by \mathcal{O}_k the ring of $\mathbb{Z}_{(p)}$ -integers in k . Then \mathcal{O}_C is invariant under $*$. We assume that p is inert in k , in particular k is a field. We denote by U the underlying \mathbb{Q} -vector space of $C(V)$. It is equipped with a nondegenerate alternating form given by $\langle x, y \rangle = \text{tr}^0(y^{\beta} \tau x)$, where tr^0 denotes the reduced trace on $C(V)$. Further U inherits an action i by $C(V) \otimes_{\mathbb{Q}} k$ via $i(c \otimes a)x = cxa$. Finally we fix a compact open subgroup $K^p \subset G(\mathbb{A}_f^p)$.

We consider the moduli problem over $\mathbb{Z}_{(p)}$ which associates to a locally noetherian scheme S over $\mathbb{Z}_{(p)}$ the set of isomorphism classes of 4-tuples $(A, \lambda, \iota, \bar{\eta}^p)$, where

- (1) A is an abelian scheme over S , up to prime to p isogeny,
- (2) $\lambda : A \rightarrow \hat{A}$ is a $\mathbb{Z}_{(p)}^{\times}$ -class of p -principal polarizations on A ,
- (3) $\iota : \mathcal{O}_C \otimes \mathcal{O}_k \rightarrow \text{End}(A) \otimes \mathbb{Z}_{(p)}$ is a homomorphism satisfying

$$\iota(c \otimes a)^* = \iota(c^* \otimes a)$$

for the Rosati involution with respect to λ on $\text{End}^0(A)$ resp. the involution $*$ on $C(V)$ introduced above,

- (4) $\bar{\eta}^p$ is a K^p -equivalence class of $\mathcal{O}_C \otimes \mathcal{O}_k$ -equivariant isomorphisms

$$\left(\prod_{l \neq p} T_l(A) \right) \otimes \mathbb{Q} \xrightarrow{\sim} U(\mathbb{A}_f^p).$$

We require that the elements of $\overline{\eta^p}$ preserve the symplectic forms on $\left(\prod_{l \neq p} T_l(A)\right) \otimes \mathbb{Q}$ (coming from λ), resp. on $U(\mathbb{A}_f^p)$ up to a scalar in $(\mathbb{A}_f^p)^\times$.

We further require that the action of $\iota(c \otimes a)$ on $\text{Lie}(A)$ satisfies the determinant condition, see [KR2] resp. [Ko]. This implies in particular that the relative dimension of A over S is 8.

If K^p is sufficiently small, this moduli problem is represented by a quasi-projective scheme \mathcal{M} over $\text{Spec } \mathbb{Z}_{(p)}$ which is smooth of relative dimension 2 over $\text{Spec } \mathbb{Z}_{(p)}$. If further $K = K_p \cdot K^p$, where $K_p \subset G(\mathbb{Q}_p)$ is the stabilizer of $\Lambda \otimes \mathbb{Z}_p$, then

$$M_K \cong \mathcal{M} \times_{\text{Spec } \mathbb{Z}_{(p)}} \text{Spec } \mathbb{Q}.$$

From now on, we suppose that K has this form.

Definition 1.1 Let (A, λ, ι) (over a locally noetherian $\mathbb{Z}_{(p)}$ -scheme S) as above. A *special endomorphism* of (A, λ, ι) is an element $j \in \text{End}(A) \otimes \mathbb{Z}_{(p)}$ such that $j = j^*$ for the Rosati involution and $\iota(c \otimes a) \circ j = j \circ \iota(c \otimes a^\sigma)$ for all $c \in \mathcal{O}_C$ and $a \in \mathcal{O}_k$. (Here $\langle \sigma \rangle = \text{Gal}(k/\mathbb{Q})$).

Suppose $S \in (\text{Sch}_{\text{ln}}/\mathbb{Z}_{(p)})$ (the category of locally noetherian schemes over $\text{Spec } \mathbb{Z}_{(p)}$) is connected. Then for any $(A, \lambda, \iota, \overline{\eta^p})$ over S as above the $\mathbb{Z}_{(p)}$ -module of special endomorphisms of (A, λ, ι) becomes a quadratic $\mathbb{Z}_{(p)}$ -module via the quadratic form Q given by $j^2 = Q(j) \cdot \text{id}$.

Now we come to the notion of *special cycles* or *arithmetic Hirzebruch-Zagier cycles*. Let $\omega \subset V(\mathbb{A}_f^p)$ be compact, open and stable under the K^p -action, and let $t \in \mathbb{Q}^\times$. We consider the functor on $(\text{Sch}_{\text{ln}}/\mathbb{Z}_{(p)})$ which associates to $S \in (\text{Sch}_{\text{ln}}/\mathbb{Z}_{(p)})$ the set of isomorphism classes of 5-tuples $(A, \lambda, \iota, \overline{\eta^p}, j)$, where $(A, \lambda, \iota, \overline{\eta^p})$ is as before and j is a special endomorphism of (A, λ, ι) such that

1. $Q(j) = t$
2. For $\eta \in \overline{\eta^p}$ we have $\eta j \eta^{-1} \in \omega$.

The second condition means that for any (equivalent: for one) $\eta \in \overline{\eta^p}$ the endomorphism $\eta j \eta^{-1}$ of $U(\mathbb{A}_f^p)$ is given by right multiplication by an element of ω . Again, if K^p is small enough (which we always assume), this functor is representable by a scheme $Z(t, \omega)$ which maps by a finite and unramified morphism (forgetting j) to \mathcal{M} .

Definition 1.2 A *special cycle* or *arithmetic Hirzebruch-Zagier cycle* over \mathcal{M} is a scheme of the form $Z(t, \omega)$ as described above.

We now fix three special cycles $Z_1 = Z(t_1, \omega_1)$, $Z_2 = Z(t_2, \omega_2)$ and $Z_3 = Z(t_3, \omega_3)$. Let

$$Z = Z_1 \times_{\mathcal{M}} Z_2 \times_{\mathcal{M}} Z_3.$$

If $S \in (\text{Sch}_{\text{ln}}/\mathbb{Z}_{(p)})$ is connected and $\xi = (A, \lambda, \iota, \overline{\eta^p}, j_1, j_2, j_3) \in Z(S)$, then we associate to ξ its *fundamental matrix* T_ξ . By definition, its entry at (a, b) is given by $\frac{1}{2}(Q(j_a + j_b) - Q(j_a) - Q(j_b))$. Thus it is a symmetric 3×3 matrix with entries in \mathbb{Q} . Since the map $\xi \mapsto T_\xi$ is locally constant, we can write

$$Z = \coprod_T Z_T,$$

where Z_T is the union of the connected components of Z which have fundamental matrix T . We assume that Z is not empty, and we fix T such that Z_T is not empty. Note that then all entries of T lie in $\mathbb{Z}_{(p)}$. We define the intersection multiplicity

$$\chi_T(Z_1, Z_2, Z_3) = \chi(Z_T, \mathcal{O}_{Z_1} \otimes^{\mathbb{L}} \mathcal{O}_{Z_2} \otimes^{\mathbb{L}} \mathcal{O}_{Z_3})$$

Here, $\otimes^{\mathbb{L}}$ is the derived tensor product of $\mathcal{O}_{\mathcal{M}}$ -modules and χ is the Euler-Poincaré characteristic (defined analogous to [KR1], section 4: Recall that this means the following. Let $\pi : \mathcal{M} \rightarrow \text{Spec } \mathbb{Z}_{(p)}$. Then for a sheaf of modules \mathcal{F} on \mathcal{M} one defines $\chi(\mathcal{F}) = \sum_i (-1)^i \text{lg}_{\mathbb{Z}_{(p)}}(R^i \pi_* \mathcal{F})$. This is finite if \mathcal{F} is coherent and has support in the special fibre \mathcal{M}_p of \mathcal{M} and this support is proper over \mathbb{F}_p . Further, χ is additive in short exact sequences. For a (bounded) complex of sheaves of modules \mathcal{F}^\bullet on \mathcal{M} one defines $\chi(\mathcal{F}^\bullet) = \sum_i (-1)^i \chi(\mathcal{F}^i)$.) The first argument Z_T of χ indicates that we only consider the value of χ which comes from the part of $\mathcal{O}_{Z_1} \otimes^{\mathbb{L}} \mathcal{O}_{Z_2} \otimes^{\mathbb{L}} \mathcal{O}_{Z_3}$ which has support in Z_T .

If T is singular, then the generic fibre of Z_T might be nonempty. Hence we additionally assume that our fixed T is nonsingular. Then (as follows from [KR2], Proposition 3.8 and sections 5 and 8), the support of Z_T lies in the supersingular locus of \mathcal{M} and is proper over \mathbb{F}_p . Using this one sees that then $\chi_T(Z_1, Z_2, Z_3)$ is finite. It is the aim of this paper to give an explicit expression for $\chi_T(Z_1, Z_2, Z_3)$ in terms of certain Eisenstein series.

2 Formal special cycles and the local intersection problem

In this section the intersection problem is reformulated in terms of an intersection problem of formal special cycles on a formal moduli space of p -divisible groups. We will solve our intersection problem in the subsequent sections in this reformulation.

Since our fixed T is assumed to be nonsingular, Z_T lies over the supersingular locus of the special fibre of \mathcal{M} ([KR2], Corollary 3.8). Let $\mathbb{F} = \overline{\mathbb{F}_p}$ be a fixed algebraic closure of \mathbb{F}_p , and let us fix $\xi \in Z_T(\mathbb{F})$. We write $\xi = (A, \lambda, \iota, \overline{\eta^p}, j_1, j_2, j_3)$. As in [KR2], we define

$$V' = \{j \in \text{End}^0(A); j = j^* \text{ and } \iota(c \otimes a) \circ j = j \circ \iota(c \otimes a^\sigma) \quad \forall c \otimes a \in \mathcal{O}_C \otimes \mathcal{O}_k\}.$$

Note that V' depends only on the isogeny class of (A, λ, ι) . Further, as shown in [KR2], its dimension as \mathbb{Q} -vector space is 4, and it is equipped with the quadratic form Q given by $j^2 = Q(j) \cdot \text{id}$.

Let $A(p^\infty)$ be the p -divisible group corresponding to A . It is equipped with an action by $\mathcal{O}_C \otimes \mathcal{O}_k \otimes \mathbb{Z}_p \cong M_4(\mathbb{Z}_{p^2})$ which allows us to write $A(p^\infty) = \mathcal{A}^4$. Then \mathcal{A} is a supersingular formal p -divisible group over \mathbb{F} of height 4 and dimension 2 which is equipped with an action

$$\iota : \mathbb{Z}_{p^2} \rightarrow \text{End}(\mathcal{A}),$$

such that \mathcal{A} is special with respect to ι (see below for a definition of the term *special*). Further \mathcal{A} is equipped with a principal polarization

$$\lambda : \mathcal{A} \xrightarrow{\sim} \hat{\mathcal{A}},$$

such that for the Rosati involution $\iota(a)^* = \iota(a)$ for all $a \in \mathbb{Z}_{p^2}$.

Let $W = W(\mathbb{F})$ be the ring of Witt vectors of \mathbb{F} and denote its Frobenius also by σ . For any W -scheme S we write $\overline{S} = S \times_{\text{Spec } W} \text{Spec } \mathbb{F}$. We consider the following functor \mathcal{M}^{HB} on the category Nilp of W -schemes S such that p is locally nilpotent in \mathcal{O}_S . It associates to a scheme $S \in \text{Nilp}$ the set of isomorphism classes of the following data.

- (1) A p -divisible group X over S , with an action

$$\iota : \mathbb{Z}_{p^2} \rightarrow \text{End}(X),$$

such that X is special with respect to this \mathbb{Z}_{p^2} -action.

(2) A quasi-isogeny of height zero

$$\varrho : \mathcal{A} \times_{\mathrm{Spec} \mathbb{F}} \overline{S} \rightarrow X \times_S \overline{S},$$

which commutes with the action of \mathbb{Z}_{p^2} such that the following condition holds. Let $\lambda_{\overline{S}} : \mathcal{A}_{\overline{S}} \rightarrow \hat{\mathcal{A}}_{\overline{S}}$ be the map induced by λ . Then we require the existence of an isomorphism $\tilde{\lambda} : X \rightarrow \hat{X}$ such that for the induced map $\tilde{\lambda}_{\overline{S}} : X_{\overline{S}} \rightarrow \hat{X}_{\overline{S}}$ we have the relation $\lambda_{\overline{S}} = \hat{\varrho} \circ \tilde{\lambda}_{\overline{S}} \circ \varrho$.

Here, a p -divisible group X over S with \mathbb{Z}_{p^2} -action is said to be *special* if the induced $\mathbb{Z}_{p^2} \otimes \mathcal{O}_S$ -module $\mathrm{Lie} X$ is, locally on S , free of rank 1.

Let \mathcal{M}^\bullet be the same functor on Nilp as \mathcal{M}^{HB} but without the condition on the height of the quasi-isogeny, and such that the relation $\lambda_{\overline{S}} = \hat{\varrho} \circ \tilde{\lambda}_{\overline{S}} \circ \varrho$ is required to hold only up to a constant in \mathbb{Q}_p^\times . Then by Theorem 3.25 of [RZ], the functor \mathcal{M}^\bullet is representable by a formal scheme which we also denote by \mathcal{M}^\bullet . From this it follows that \mathcal{M}^{HB} also also representable by a formal scheme which we also denote by \mathcal{M}^{HB} , and $\mathcal{M}^\bullet = \mathcal{M}^{HB} \times \mathbb{Z}$. The formal schemes \mathcal{M}^\bullet and \mathcal{M}^{HB} are formally locally of finite type over $\mathrm{Spf} W$ and are formally smooth over $\mathrm{Spf} W$.

Denote by $\widehat{\mathcal{M}_W / \mathcal{M}_W^{ss}}$ the completion of $\mathcal{M}_W = \mathcal{M} \times_{\mathrm{Spec} \mathbb{Z}_{(p)}} \mathrm{Spec} W$ along its supersingular locus \mathcal{M}_W^{ss} . Then by Theorem 6.23 of loc. cit., we have (using that our K^p is sufficiently small) an isomorphism of formal schemes over $\mathrm{Spf} W$

$$\widehat{\mathcal{M}_W / \mathcal{M}_W^{ss}} \cong G'(\mathbb{Q}) \setminus (\mathcal{M}^{HB} \times \mathbb{Z} \times G(\mathbb{A}_f^p) / K^p), \quad (2.1)$$

where G' is the inner form of G described in [KR2], section 4, i.e., G' is defined as G for $B = C^+(V) = B_0 \otimes_{\mathbb{Q}} k$, but for the quaternion algebra B' over k which is ramified at the two archimedean primes and is isomorphic to B at all finite primes. Then $G'(\mathbb{Q})$ can be identified with the group of quasi-isogenies of the abelian variety A , which respect ι and (up to scalar) λ . Further $g \in G'(\mathbb{Q})$ acts on \mathcal{M}^\bullet (and therefore on $\mathcal{M}^{HB} \times \mathbb{Z}$) by sending (X, ϱ) to $(X, \varrho g^{-1})$. See [RZ] for more details.

Denote the isocrystal of \mathcal{A} by N . It is equipped with σ - resp. σ^{-1} -linear operators F and V such that $FV = VF = p$. From the polarization λ we get a perfect symplectic form \langle, \rangle on the Dieudonné module of \mathcal{A} and hence also on N . As in [KR2], section 5, we define in this context the space of special endomorphisms of \mathcal{A}

$$V'_p = \{j \in \mathrm{End}(N, F); j\iota(a) = \iota(a^\sigma)j \quad \forall a \in \mathbb{Z}_{p^2} \quad \text{and} \quad j^* = j\},$$

where $*$ denotes the adjoint with respect to the alternating form \langle, \rangle . We may and will regard V' as the \mathbb{Q} -subspace of V'_p consisting of the special endomorphisms which come from the abelian variety A . Then V' is dense in V'_p . As shown in loc. cit., V'_p is a 4-dimensional vector space over \mathbb{Q}_p with a quadratic form, which we also denote by Q and which is again given by

$$j^2 = Q(j) \cdot \mathrm{id}.$$

Instead of $Q(j)$ we will also write j^2 . Note that V' is positive definite ([KR2], Proposition 3.5), that $V'(\mathbb{Q}_p) \cong V'_p$ and that $V'(\mathbb{A}_f^p) \cong V(\mathbb{A}_f^p)$ via some (fixed) $\eta \in \overline{\eta}^p$ (where $\overline{\eta}^p$ belongs to ξ). We will always identify $V'(\mathbb{A}_f^p)$ and $V(\mathbb{A}_f^p)$. In particular we may regard V' as being contained in $V(\mathbb{A}_f^p)$.

Given special endomorphisms $y_1, \dots, y_n \in V'_p$, we again define their *fundamental matrix* to be the symmetric $n \times n$ -matrix whose entry at (a, b) is given by $\frac{1}{2}(Q(y_a + y_b) - Q(y_a) - Q(y_b))$.

Definition 2.1 Let $j \in V'_p$ and regard it as an element of $\mathrm{End}^0(\mathcal{A})$. Then the *formal special cycle* $Z(j)$ associated to j is the closed formal subscheme of \mathcal{M}^{HB} consisting of all points (X, ϱ) such that $\varrho \circ j \circ \varrho^{-1}$ lifts to an endomorphism of X .

The fact that $Z(j)$ is a closed formal subscheme of \mathcal{M}^{HB} follows from [RZ], Proposition 2.9.

Our next aim is a description of the underlying reduced scheme \mathcal{M}^{HBred} of \mathcal{M}^{HB} . Let X be the set of W -lattices L in the isocrystal N such that for the dual lattice L^\perp we have $L^\perp = cL$ (with respect to \langle, \rangle) for some constant c , and which are stable under F , under V and under the \mathbb{Z}_{p^2} -action. The latter gives a grading $L = L_0 \oplus L_1 \subset N = N_0 \oplus N_1$ (see also [KR2], section 4). We call the index $i \in \mathbb{Z}/2$ critical for L if $F^2 L_i = pL_i$. By [KR2], Lemma 4.2, for each such lattice at least one index is critical. We call L superspecial if both indices are critical. We may assume that ξ is superspecial, which means that the corresponding Dieudonné module $M \subset N$ of \mathcal{A} is superspecial. (It follows for example from [KR2], Lemma 8.12 or from the beginning of the next section that there is always a superspecial point in Z_T .)

The \mathbb{F} -valued points of \mathcal{M}^{HB} correspond to those lattices in X which have the same volume as M (by the condition that the quasi-isogenies have height 0). We denote the set of such lattices by X^{HB} .

Let now $\eta_i^0 = M_i^{p^{-1}F^2}$ for $i \in \mathbb{Z}/2$. If $L \in X^{HB}$ has critical index 0, then let $\eta_0^L = L_0^{p^{-1}F^2}$. It has the same volume as η_0^0 . We assign to L the point on the projective line $\mathbb{P}(L_0/pL_0) = \mathbb{P}(\eta_0^L \otimes_{\mathbb{Z}_{p^2}} \mathbb{F})$ corresponding to the line $FL_1/pL_0 \subset L_0/pL_0 = \eta_0^L \otimes_{\mathbb{Z}_{p^2}} \mathbb{F}$.

If $L \in X^{HB}$ has critical index 1, then let $\eta_1^L = L_1^{p^{-1}F^2}$. It has the same volume as η_1^0 . We assign to L the point on $\mathbb{P}(L_1/pL_1) = \mathbb{P}(\eta_1^L \otimes_{\mathbb{Z}_{p^2}} \mathbb{F})$ corresponding to the line $FL_0/pL_1 \subset L_1/pL_1 = \eta_1^L \otimes_{\mathbb{Z}_{p^2}} \mathbb{F}$. If e_1, e_2 is a basis of η_0^0 , then $F\eta_1^L$ has the same volume as the \mathbb{Z}_{p^2} -lattice in $\eta_0^0 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ spanned by e_1, pe_2 .

In both cases, any point on $\mathbb{P}(L_i/pL_i)$ is recovered by some lattice in X^{HB} , see also [KR2], Lemma 4.3. If we identify $\eta_0^0 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with $\mathbb{Q}_{p^2}^2$, we see that the projective lines occurring in the above construction for the lattices in X^{HB} are in bijection with the vertices of the building $\mathcal{B} := \mathcal{B}(\mathrm{PGL}_2(\mathbb{Q}_{p^2}))$.

For each superspecial lattice we get two points, each lying on a different projective line. If we glue for any superspecial lattice the corresponding two projective lines along these (\mathbb{F}_{p^2} -rational) points, then we see that (at least on the level of \mathbb{F} -valued points) \mathcal{M}^{HBred} is a union of projective lines dual to \mathcal{B} : For each vertex $[\Lambda] \in \mathcal{B}$ we have a projective line $\mathbb{P}_{[\Lambda]}$ (of the form $\mathbb{P}(\eta_1^L \otimes_{\mathbb{Z}_{p^2}} \mathbb{F})$ or $\mathbb{P}(\eta_0^L \otimes_{\mathbb{Z}_{p^2}} \mathbb{F})$ for a suitable L as above) over \mathbb{F} and with a \mathbb{F}_{p^2} -rational structure. Further for each edge, the projective lines corresponding to the vertices of the edge intersect in a \mathbb{F}_{p^2} -rational point. All \mathbb{F}_{p^2} -rational points are such intersection points and these correspond precisely to the superspecial points (comp. [KR2], Lemma 4.3).

To see that this description is also true on the level of schemes, we show that the \mathbb{F} -valued points on each projective line are in fact the \mathbb{F} -valued points of a closed subscheme of \mathcal{M}^{HBred} which equals the projective line as a scheme. To this end one chooses three pairwise perpendicular special endomorphisms $y_1, y_2, y_3 \in V'_p$ such that $y_1^2 = p$ and such that the scheme $(Z(y_1) \cap Z(y_2) \cap Z(y_3))^{red}$ is (at least on the level of points) precisely the projective line. (It is easy to see that such y_1, y_2, y_3 exist using the description of V' given in the beginning of the next section and Lemma 3.2 below or using [KR2], Lemma 8.12.) By the identification of $Z(y_1)$ and the Drinfeld upper half plane (whose underlying reduced subscheme is known to be a union of projective lines) given in [T], section 4, it follows that this equality also holds on the level of schemes. (See also [KR2], (4.10).)

We now come back to our intersection problem. Let S_1, S_2, S_3 be closed subschemes of \mathcal{M}^{HB} . Then we define the (possibly infinite) intersection multiplicity

$$(S_1, S_2, S_3) := \chi(\mathcal{M}^{HB}, \mathcal{O}_{S_1} \otimes^{\mathbb{L}} \mathcal{O}_{S_2} \otimes^{\mathbb{L}} \mathcal{O}_{S_3}),$$

Here, $\otimes^{\mathbb{L}}$ is the derived tensor product of $\mathcal{O}_{\mathcal{M}^{HB}}$ -modules and χ is the Euler-Poincaré char-

acteristic. Again, this is finite if $S_1 \cap S_2 \cap S_3$ has support in the supersingular locus and this support is proper over \mathbb{F} .

Recall our fixed $\xi \in Z_T(\mathbb{F})$. Writing again $\xi = (A, \lambda, \iota, \eta, j_1, j_2, j_3)$, the special endomorphisms j_1, j_2, j_3 induce special endomorphisms of \mathcal{A} which we also call j_1, j_2, j_3 .

Our next aim is to express the intersection multiplicity $\chi_T(Z_1, Z_2, Z_3)$ in terms of the intersection multiplicity $(Z(j_1), Z(j_2), Z(j_3))$, the calculation of which we will call the local intersection problem. Denote by Z' the center of G' and denote by $Z'(\mathbb{Q})^0$ the set of elements $z \in Z'(\mathbb{Q})$ such that $\nu_p(\det(z)) = 0$ (where ν_p denotes the p -adic valuation). Let $\underline{x} = (j_1, j_2, j_3) \in V'^3$. Further let

$$I(\underline{x}, \omega_1 \times \omega_2 \times \omega_3) = \{gK^p \in G(\mathbb{A}_f^p)/K^p ; g^{-1}\underline{x}g \in \omega_1 \times \omega_2 \times \omega_3\}.$$

Proposition 2.2 *The intersection multiplicity $\chi_T(Z_1, Z_2, Z_3)$ can be expressed in the following way,*

$$\chi_T(Z_1, Z_2, Z_3) = 2 \cdot (Z(j_1), Z(j_2), Z(j_3)) \cdot |Z'(\mathbb{Q})^0 \setminus I(\underline{x}, \omega_1 \times \omega_2 \times \omega_3)|.$$

The proof is analogous to the proof of the corresponding Theorem 8.5 in [KR1]. As in loc. cit., it is based on the following incidence relations.

Let $Z' = Z(t', \omega')$ be a special cycle. For a special endomorphism $j \in V'_p$ define the closed subscheme $Z^\bullet(j)$ of \mathcal{M}^\bullet by the obvious analogon to $Z(j)$. Further denote by V'_t the set of elements j in V' with $j^2 = t'$. Denote by \widehat{Z}'_W the module over $\widehat{\mathcal{M}_W}/\widehat{\mathcal{M}_W}^{ss}$ obtained from Z' by completing $Z' \times_{\text{Spec } \mathbb{Z}_{(p)}} \text{Spec } W$ along \mathcal{M}_W^{ss} . Then the uniformization morphism (2.1) induces an inclusion

$$\widehat{Z}'_W \hookrightarrow G'(\mathbb{Q}) \setminus (V'_t \times \mathcal{M}^\bullet \times G(\mathbb{A}_f^p)/K^p),$$

and a point $(j, (X, \varrho), gK^p)$ lies in the image if and only if $g^{-1}jg \in \omega'$ and $(X, \varrho) \in Z^\bullet(j)$. For several special cycles Z'_i we get an analogous description for the fibre product of the \widehat{Z}'_{iW} . See the discussion in [KR1], section 8 for more details.

The proposition shows that it is enough to handle the local intersection problem and this is what we will do in the subsequent sections.

3 On the structure of formal special cycles

Our first object of study in this section is the underlying reduced subscheme of a formal special cycle. Here the results follow from [KR2], sections 5 and 8. We denote by ν_p the p -adic valuation.

Let $j \in V'_p$ such that $j^2 \neq 0$ and $\nu_p(j^2) \geq 0$. Let $Y = N_0^{p^{-1}F^2}$. Then $\beta := F^{-1}j | Y$ is an σ -linear endomorphism and induces an endomorphism of \mathcal{B} . If $\nu_p(j^2) = 0$, then the underlying reduced subscheme of $Z(j)$ is a single superspecial point. It is the superspecial point corresponding the unique edge \mathcal{B}^β in \mathcal{B} which is fixed by β . If $\nu_p(j^2) \geq 1$, then the underlying reduced subscheme of $Z(j)$ is a connected union of projective lines. The corresponding locus in the building is given by the set

$$\mathcal{T}(\beta) = \{x \in \mathcal{B} ; \text{dist}(x, \mathcal{B}^\beta) \leq \frac{1}{2}\nu_p(\det \beta)\}.$$

Here, \mathcal{B}^β is the fixed point set of β in \mathcal{B} , and dist denotes the distance in the building. The fixed point set \mathcal{B}^β is the midpoint of an edge if $\nu_p(j^2)$ is even and it is a subbuilding of the form $\mathcal{B}(\text{PGL}_2(\mathbb{Q}_p))$ if $\nu_p(j^2)$ is odd. See [KR2], sections 5 and 8 for more details.

Definition 3.1 1. For a special endomorphism $j \in V'_p$ such that $j^2 \neq 0$ denote by $Core(j)$ the set of superspecial points in \mathcal{M}^{HB} such that the corresponding midpoints of edges in \mathcal{B} belong to \mathcal{B}^β , where $\beta := F^{-1}j \mid Y$.

2. If j is a special endomorphism such that $\nu_p(j^2)$ is odd and if we write $j = p^b j_0$, where $\nu_p(j_0^2) = 1$, then the center of $Z(j)$, denoted by $Cent(j)$, is the set of all $\mathbb{P}_{[\Lambda]}$ which are contained in $Z(j_0)$.

In the sequel, we say that a special endomorphism $j \in V'_p$ is even, resp. odd, if $\nu_p(j^2)$ is even resp. odd. Thus, if j is even, then $Core(j)$ consists of a single superspecial point. If j is odd, then $Core(j)$ is an infinite set.

In the following lemma we denote by χ the quadratic residue character of \mathbb{Z}_p^\times .

Lemma 3.2 Let $y_1, y_2 \in V'_p$ such that $y_1 \perp y_2$ and $y_1^2, y_2^2 \neq 0$. Suppose that $a_i = \nu_p(y_i^2) \geq 0$ and write $y_i^2 = \varepsilon_i p^{a_i}$.

1. If y_1 and y_2 are both odd and $\chi(-\varepsilon_1 \varepsilon_2) = 1$, then $Cent(y_1) \cap Cent(y_2)$ consists of precisely one projective line.
2. If y_1 and y_2 are both odd and $\chi(-\varepsilon_1 \varepsilon_2) = -1$, then $Cent(y_1) \cap Cent(y_2)$ consists of an apartment of projective lines, i.e. each $\mathbb{P}_{[\Lambda]} \in Cent(y_1) \cap Cent(y_2)$ is intersected by precisely two others.
3. If y_1 is odd and y_2 is even, then the point in $Core(y_2)$ is the intersection point of two projective lines belonging to $Cent(y_1)$.
4. If $y_3 \in V'_p$ is a third special endomorphism with $y_3^2 = \varepsilon_3 p^{a_3}$ (where $a_3 := \nu_p(y_3^2) \geq 0$) and $y_3 \perp y_1, y_2$, then $Z(y_1) \cap Z(y_2) \cap Z(y_3)$ is not empty, and if y_1 and y_2 are odd and y_3 is even, then $\chi(-\varepsilon_1 \varepsilon_2) = -1$. Further, if y_1, y_2, y_3 are all odd, then $Cent(y_1) \cap Cent(y_2) \cap Cent(y_3)$ consists of precisely one projective line.

Proof. The first three points follow by combining [T], Proposition 4.2, together with loc.cit Proposition 2.4 and [KR1], Corollary 2.6, or can be proved directly like [KR1], Corollary 2.6. The first assertion of point (iv) is clear, the second follows from the first three points. The last assertion follows from [KR2], Proposition 8.13 or alternatively from [KR1], Proposition 2.12. \square

Proposition 3.3 Let $j \in V'_p$ such that $j^2 \neq 0$ and $\nu_p(j^2) \geq 0$. Then $Z(j)$ is a relative divisor over $\text{Spf } W$.

Proof. The fact that $Z(j)$ is a divisor is proved in [T], Proposition 4.5. The fact that the local equation of $Z(j)$ is nowhere divisible by p follows from [KR3] Lemma 3.6 and the fact that not the whole supersingular locus of \mathcal{M}^{HB} (which is connected) belongs to $Z(j)$ (see above). \square

Let $L \in X^{HB}$ be a superspecial lattice. The space $N_L \subset V'_p$ of special endomorphisms, which map L into itself is a quadratic \mathbb{Z}_p -modul. (It is the set of all special endomorphisms $j \in V'_p$ such that the p -divisible group corresponding to L belongs to $Z(j)$.) We write $L = L_0 \oplus L_1$, and then (by the compatibility condition with ι) for $y \in N_L$ we can write

$$y = \begin{pmatrix} & y_1 \\ y_0 & \end{pmatrix}; \quad y_i \in \text{Hom}(L_i, L_{i+1})$$

A standard basis of L is a basis e_1, e_2, e_3, e_4 of L such that e_1, e_2 is a basis of L_0 and such that $e_3 = Fe_1$ and $e_4 = p^{-1}Fe_2$ and such that for the symplectic form coming from the principal

polarization we have $\langle e_1, e_2 \rangle = 1$. After choosing a standard basis of L (see [KR2], Lemma 5.1 for the existence of a standard basis) we have by [KR2], Corollary 5.2

$$N_L \cong \left\{ x = \begin{pmatrix} a & b \\ c & pa^\sigma \end{pmatrix}; a, b, c \in \mathbb{Z}_{p^2}, b^\sigma = -b, c^\sigma = -c \right\},$$

and the quadratic form is given by

$$Q(x) = paa^\sigma - bc.$$

In this isomorphism, x is the matrix of y_0 in the standard basis. We write $\mathbb{Z}_{p^2} = \mathbb{Z}_p[\delta]/(\delta^2 - \Delta)$, where $\Delta \in \mathbb{Z}_p^\times$ is not a square in \mathbb{Z}_p .

A \mathbb{Z}_p -basis of N_L is given by

$$s_1 = \begin{pmatrix} & -\delta \\ \delta^{-1} & \end{pmatrix}, \quad s_2 = \begin{pmatrix} & \delta \\ \delta^{-1} & \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & \\ & p \end{pmatrix}, \quad s_4 = \begin{pmatrix} \delta & \\ & -\delta p \end{pmatrix}.$$

The matrix of the bilinear form induced by Q with respect to this basis is:

$$S' = \text{diag}(1, -1, p, -\Delta p).$$

Lemma 3.4 *Let $y_1, y_2 \in V'_p$ such that $y_1^2, y_2^2 \neq 0$ and let $x \in (Z(y_1) \cap Z(y_2))(\mathbb{F})$. Then there exists a special endomorphism $y \in V'_p$ such that $y \perp y_1, y_2$ and such that $\nu_p(y^2) = 1$ and $x \in Z(y)(\mathbb{F})$.*

Proof. We may assume that y_1 and y_2 are linearly independent.

Suppose that $x \in \mathbb{P}_{[\Lambda]}$. Let $z \in \mathbb{P}_{[\Lambda]}$ be superspecial such that $z \notin \text{Core}(y_1) \cap \text{Core}(y_2)$. It is enough to find $y \in V'_p$ such that $\nu_p(y^2) = 1$ and $y \perp y_1, y_2$ and such that $\mathbb{P}_{[\Lambda]} \subset Z(y)$. We consider the basis s_1, \dots, s_4 described above for the quadratic \mathbb{Z}_p -module of special endomorphisms in V'_p which are endomorphisms of the p -divisible group belonging to z . Write $y_1 = \sum a_i s_i$ and $y_2 = \sum b_i s_i$, where $a_i, b_i \in \mathbb{Q}_p$. If $a_4 = b_4 = 0$, then we can choose $y = s_4$. Hence we may assume that $a_4 \neq 0$. Let $\tilde{y}_2 = y_2 - (b_4/a_4)y_1 = \sum c_i s_i$. Then $\tilde{y}_2 \neq 0$ and $c_4 = 0$. We consider the special endomorphisms

$$r_1 = -a_4 c_3 \cdot s_1 + \frac{1}{p} a_4 c_1 \cdot s_3 + \frac{-1}{\Delta p} (a_1 c_3 - a_3 c_1) \cdot s_4$$

and

$$r_2 = a_4 c_3 \cdot s_2 + \frac{1}{p} a_4 c_2 \cdot s_3 + \frac{-1}{\Delta p} (a_2 c_3 - a_3 c_2) \cdot s_4.$$

One checks that r_1 and r_2 are both perpendicular to y_1 and to \tilde{y}_2 and hence also to y_2 .

Suppose that the coefficients of s_3 and s_4 in r_1 and in r_2 do simultaneously vanish. Since $a_4 \neq 0$ it follows that $c_1 = c_2 = 0$. Hence $c_3 \neq 0$ and hence $a_1 = a_2 = 0$. Hence one of the special endomorphisms $(p+1)s_1 + s_2$ and $(p-1)s_1 + s_2$ fulfills the requirements of y (for both we have $\nu_p(y^2) = 1$ and $y \perp y_1, y_2$, and $\mathbb{P}_{[\Lambda]}$ belongs to the formal special cycle of one of them).

Suppose now that we have chosen i such that the coefficients of s_3 and s_4 in r_i do not both vanish. We choose the integer m minimal such that $y = p^m r_i$ lies in the \mathbb{Z}_p -span of s_1, \dots, s_4 . Then $p^m a_4 c_3 \equiv 0 \pmod{p}$ since otherwise $y^2 \in \mathbb{Z}_p^\times$, hence $\{z\} = \text{Core}(y)$, hence $z \in \text{Core}(y_1) \cap \text{Core}(y_2)$ which contradicts our assumption on z . Hence the coefficient of s_3 or of s_4 in y is a unit and hence $\nu_p(y^2) = 1$ and y fulfills all requirements above. (Note that both projective lines passing through z belong to $Z(y)$ (see also Lemma 3.2), in particular $\mathbb{P}_{[\Lambda]} \subset Z(y)$.) \square

For $j \in V'_p$ as in Proposition 3.3 we define $D(j) = Z(j) - Z(j/p)$. This is meant in the following sense. If $Z(j)$ is (locally) given by the equation $f = 0$ and $Z(j/p)$ is (also locally) given by $g = 0$, then $D(j)$ is the relative effective divisor locally given by $f \cdot g^{-1} = 0$. Further, if $\nu_p(j^2) = 0$ or $= 1$, then $D(j) = Z(j)$.

Lemma 3.5 *Let $y, y_1 \in V'_p$ such that $y_1 \perp y$ and $y^2 \in \mathbb{Z}_p \setminus \{0\}$ and $\nu_p(y_1^2) = 1$. Let $y^2 = \varepsilon p^a$ and $y_1^2 = \varepsilon_1 p$, where $\varepsilon, \varepsilon_1 \in \mathbb{Z}_p^\times$.*

1. *If $a \geq 2$, then $D(y) \cap Z(y_1)$ is a reduced union of projective lines. The same holds if $a = 1$ and the image of $-\varepsilon\varepsilon_1$ in \mathbb{F}_p is not a square.*
2. *If $a = 0$, then $Z(y) \cap Z(y_1)$ is a reduced, irreducible horizontal divisor in $Z(y_1)$ (i.e. has no component with support in the special fibre of $Z(y_1)$) which intersects the special fibre of $Z(y_1)$ only in the point of $\text{Core}(y)$. Further, the intersection multiplicity of this horizontal component with each of the two projective lines in the special fibre of $Z(y_1)$ passing through the point of $\text{Core}(y)$ equals 1.*
3. *If $a = 0$ and $y_0 \in V'_p$ satisfies $y_0 \perp y, y_1$ and $\nu_p(y_0^2) = 1$, then $(Z(y_0), Z(y_1), Z(y)) = 2$.*
4. *If $a = 0$ and $y_0 \in V'_p$ satisfies $y_0 \perp y, y_1$ and $\nu_p(y_0^2) = 0$, then $(Z(y_0), Z(y_1), Z(y)) = 1$.*

Proof. All claims follow from [T]. More precisely, in the situation of 1., by the equations for antispecial cycles in given in [T], section 2, and the connection of antispecial cycles and formal special cycles in our sense discussed in [T], section 4, $D(y) \cap Z(y_1)$ is indeed the reduced union of the projective lines $\mathbb{P}_{[\Lambda]}$ which belong to the special fibres of $Z(y)$ and $Z(y_1)$.

If $a = 0$, then the assertion of 2. follows by comparing the intersection $Z(y) \cap Z(y_1)$ with the intersection $Z(py) \cap Z(y_1)$. More precisely, again by combining as above the results of sections 2 and 4 of [T], $Z(py) \cap Z(y_1)$ consists of two (reduced) projective lines in the special fibre and a horizontal component which has all the properties which are claimed for the horizontal component of $Z(y) \cap Z(y_1)$ in the statement of the lemma. Since no projective lines in the special fibre belong to $Z(y)$, it follows that the horizontal components of $Z(y) \cap Z(y_1)$ and of $Z(py) \cap Z(y_1)$ are equal.

Using this, in case $\nu_p(y_0^2) = 1$, the claim that

$$(Z(y_0), Z(y_1), Z(y)) = ((Z(y_0) \cap Z(y_1)), (Z(y) \cap Z(y_1))) = 2$$

(intersection multiplicity in $Z(y_1)$, see [T], Proposition 4.7 for the first equality) follows from the fact that (by the same reasoning as above) $Z(y_0) \cap Z(y_1)$ is the reduced union of the projective lines $\mathbb{P}_{[\Lambda]}$ which belong to the special fibres of $Z(y_0)$ and $Z(y_1)$.

In case $\nu_p(y_0^2) = 0$ one has to prove that (as intersection multiplicity in $Z(y_1)$) one has $((Z(y_0) \cap Z(y_1)), (Z(y) \cap Z(y_1))) = 1$. In other words, this means that the horizontal components of $(Z(py_0) \cap Z(y_1))$ and of $(Z(py) \cap Z(y_1))$ intersect in $Z(y_1)$ with multiplicity 1. This in turn follows from the comparison of intersections of (horizontal components of) special cycles in the sense of [KR1] and antispecial cycles resp. formal special cycles in our sense given in [T], sections 3, 4. \square

Proposition 3.6 *Let $j \in V'_p$ be as in Proposition 3.3, i.e. $j^2 \neq 0$ and $\nu_p(j^2) \geq 0$. Then $D(j)$ is a regular formal scheme.*

Proof. If $\nu_p(j^2) = 0$, then we find $j_0, j_1 \in V'$ such that $\text{Core}(j) \subset (Z(j_0) \cap Z(j_1))(\mathbb{F})$ and all three endomorphisms are perpendicular to each other and such that $\nu_p(j_0^2) = 0$ and $\nu_p(j_1^2) = 1$. Then by Lemma 3.5 the length (over W) of the local ring (which is a local Artin ring) of

$Z(j) \cap Z(j_0) \cap Z(j_1)$ is 1, hence this ring is regular. But then $Z(j)$ is also regular in the point of $\text{Core}(j)$.

If $\nu_p(j^2) = 1$, then $Z(j)$ is isomorphic to the Drinfeld upper half plane, see [T], section 4 for the precise statement, which is regular.

Suppose now that $\nu_p(j^2) \geq 2$. Let $x \in Z(j)(\mathbb{F})$. We have to show that the local ring $\mathcal{O}_{D(j),x}$ is regular.

First case $x \notin Z(j/p)(\mathbb{F})$.

We choose $j_1 \in V'_p$ such that $j_1 \perp j$ and $\nu_p(j_1^2) = 1$ and such that $x \in Z(j_1)(\mathbb{F})$, see Lemma 3.4. Then $Z(j_1)$ is isomorphic to the Drinfeld upper half plane (see [T], section 4). By Lemma 3.5 we know that locally around x , the intersection $Z(j_1) \cap D(j)$ is isomorphic to a projective line over \mathbb{F} , hence this intersection is regular in x . Hence $D(j)$ is regular in x as well.

Second case $x \in Z(j/p)(\mathbb{F})$.

If x is not superspecial, then we choose j_1 as in case 1 and the reasoning is also the same as above. Suppose now that x is superspecial, $\{x\} = \mathbb{P}_{[\Lambda_0]} \cap \mathbb{P}_{[\Lambda_1]}$, and $\mathbb{P}_{[\Lambda_0]} \subset Z(j/p)$. We choose j_1 linearly independent of j such that $\nu_p(j_1^2) = 1$ and $\mathbb{P}_{[\Lambda_0]} \subset Z(j_1)$ but $\mathbb{P}_{[\Lambda_1]} \not\subset Z(j_1)$. We consider the \mathbb{Z}_p -submodule \mathbf{j} of V'_p spanned by j/p and j_1 . Then there is $j_2 \in \mathbf{j}$ such that $j_1 \perp j_2$ and $\{j_1, j_2\}$ is a basis of \mathbf{j} . (This can be seen as follows: Consider an arbitrary quadratic \mathbb{Z}_p -module (M, q) which is free of finite rank. Then for any element $m \in M$ such that $\nu_p(q(m))$ is minimal, there is an orthogonal basis of M which contains m as an element. Thus, if we apply this to \mathbf{j} , and arguing by contradiction, we see that if our claim were wrong there would exist some $j' \in \mathbf{j}$ such that $\nu_p(j'^2) = 0$. But this contradicts the fact that $\mathbb{P}_{[\Lambda_0]} \subset Z(j'')$ for all $j'' \in \mathbf{j}$). We write $j/p = aj_1 + bj_2$. Then $D(j) \cap Z(j_1) = D(pbj_2) \cap Z(j_1)$ which is regular in x by the same reasoning as above. Again it follows that $D(j)$ is regular in x . \square

Our next aim is a description of the special fibre of a formal special cycle $Z(j)$, where $j \in V'$ and $j^2 \neq 0$. In the sequel we use the following notations. For a vertex $[\Lambda] \in \mathcal{B}$ denote by $d_{[\Lambda]}$ the distance of $[\Lambda]$ to \mathcal{B}^β , where $\beta := F^{-1}j \mid Y$ as above. For any formal scheme X over $\text{Spf } W$ we denote by X_p the special fibre of X .

Theorem 3.7 *Let $j \in V'$ such that $j^2 \neq 0$ and $a := \nu_p(j^2) \geq 0$. Then the divisor $Z(j)_p$ in \mathcal{M}^{HB}_p is of the form*

$$Z(j)_p = \sum_{[\Lambda]; d_{[\Lambda]} \leq \frac{a-1}{2}} (1 + p + \dots + p^{\frac{a-1}{2} - d_{[\Lambda]}}) \mathbb{P}_{[\Lambda]} + p^{a/2} \cdot s.$$

Here, s is a divisor in \mathcal{M}^{HB}_p which is $\neq 0$ if and only if j is even. In this case s meets the supersingular locus only in the point of $\text{Core}(j)$.

We will see later (Lemma 4.9) that (in case that j is even) the intersection multiplicity of s with each of the two projective lines to which the point in $\text{Core}(j)$ belongs is 1. In particular, s is reduced and irreducible.

Proof. We know by Proposition 3.3 that $Z(j)_p$ is a divisor in \mathcal{M}^{HB}_p . It follows from the description of the underlying reduced scheme of $Z(j)$ given above that the part of $Z(j)_p$ having support in the supersingular locus is of the form

$$Z(j)_p^{ss} = \sum_{[\Lambda]; d_{[\Lambda]} \leq \frac{a-1}{2}} m_{[\Lambda]} \mathbb{P}_{[\Lambda]}$$

(without knowing the values of the multiplicities $m_{[\Lambda]}$ of the irreducible components $\mathbb{P}_{[\Lambda]}$). Further it follows from [KR2], section 10 that the rest of $Z(j)_p$ is of the form $p^{a/2} \cdot s$ where s is

as above. (In the corresponding statement Corollary 10.3 in loc. cit. it is erroneously asserted that the multiplicity of s is p^a instead of $p^{a/2}$.)

Thus it remains to find the values of $m_{[\Lambda]}$. It suffices to show that

- (i) If $d_{[\Lambda]} = \frac{a-1}{2}$, then $m_{[\Lambda]} = 1$.
- (ii) If $\mathbb{P}_{[\Lambda]}$ intersects $\mathbb{P}_{[\Lambda']}$ and $d_{[\Lambda]} < d_{[\Lambda']} \leq \frac{a-1}{2}$, then $m_{[\Lambda]} + m_{[\Lambda']} = 2 \cdot (1 + p + \dots + p^{\frac{a-1}{2}-d_{[\Lambda']}}) + p^{\frac{a-1}{2}-d_{[\Lambda]}}$.

For any point $x \in Z(j)_p(\mathbb{F})$ we use the following notation. Let I_x be the ideal of $Z(j)_p$ in the local ring $\mathcal{O}_{\mathcal{M}_p^{HB}, x}$ and let \mathfrak{m}_x be the maximal ideal of $\mathcal{O}_{\mathcal{M}_p^{HB}, x}$. Denote by n_x the maximal integer n with the property that $I_x \subset \mathfrak{m}_x^n$. Denote by X (instead of X_x) the p -divisible group over \mathbb{F} belonging to the point x .

Suppose now we are in the situation of (i) and let $x \in \mathbb{P}_{[\Lambda]}$ be a superspecial point which does not belong to another $\mathbb{P}_{[\Lambda']} \subset Z(j)_p$. Let $f = 0$ be the equation of $\mathbb{P}_{[\Lambda]}$ in the local ring $\mathcal{O}_{\mathcal{M}_p, x}$. It follows that $I_x = (f^{m_{[\Lambda]}})$. Since f belongs to a regular parameter system of \mathfrak{m} , it follows that $n_x = m_{[\Lambda]}$.

Suppose now that we are in the situation of (ii) and let x be the intersection point of $\mathbb{P}_{[\Lambda]}$ and $\mathbb{P}_{[\Lambda']}$. Then it follows as above that $n_x = m_{[\Lambda]} + m_{[\Lambda']}$. Further one sees easily that j as an endomorphism of X is divisible by $p^{\frac{a-1}{2}-d_{[\Lambda]}}$ but not by $p^{\frac{a-1}{2}-d_{[\Lambda]}+1}$. (Note that $d_{[\Lambda]} = \frac{a-1}{2}$ means that $\mathbb{P}_{[\Lambda]}$ belongs to $Z(j)$ but not to $Z(j/p)$.)

Using the notation just introduced it follows that it is enough to show the following:

- (1) Suppose x is a superspecial point belonging to precisely one $\mathbb{P}_{[\Lambda]} \subset Z(j)_p$. Then $n_x = 1$.
- (2) Suppose we are in the situation of (ii) and x is as above the intersection point of $\mathbb{P}_{[\Lambda]}$ and $\mathbb{P}_{[\Lambda']}$. Suppose further that j is as an endomorphism of X divisible by p^r but not by p^{r+1} . Then $n_x = 2 \cdot (1 + p + \dots + p^{r-1}) + p^r$.

For this we are going to use the theory of displays (see [Z2]) in the manner of [KR3], section 8. Denote by (M, F, V) the Dieudonné module to the p -divisible group X corresponding to the point x in (1) or in (2). Then the display for X is given by the data (M, VM, F, V^{-1}) . We find a basis f_1, \dots, f_4 of M for which F and V have matrices

$$F = \begin{pmatrix} & p & & \\ & & 1 & \\ 1 & & & \\ & p & & \end{pmatrix} \sigma, \quad V = \begin{pmatrix} & p & & \\ & & 1 & \\ 1 & & & \\ & p & & \end{pmatrix} \sigma^{-1},$$

and such that for the alternating form coming from the polarization we have $\langle f_1, f_2 \rangle = \langle f_3, f_4 \rangle = 1$, see [KR2], section 5. Let $e_1 = f_1$, $e_2 = f_4$, $e_3 = f_2$, $e_4 = f_3$. Denote by T the W -span of e_1, e_2 and by L the W -span of e_3, e_4 . Then

$$M = L \oplus T, \quad VM = L \oplus pT.$$

Define the matrix (α_{ij}) by

$$Fe_j = \sum_i \alpha_{ij} e_i \text{ for } j = 1, 2,$$

$$V^{-1}e_j = \sum_i \alpha_{ij} e_i \text{ for } j = 3, 4.$$

Hence,

$$(\alpha_{ij}) = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}.$$

It follows (see [Z1], p. 48) that the universal deformation of X over $\mathbb{F}[[t_1, t_2, t_3, t_4]]$ corresponds to the display $(L \oplus T) \otimes W(\mathbb{F}[[t_1, t_2, t_3, t_4]])$ with matrix $(\alpha_{ij})^{\text{univ}}$ (wrt. the basis e_1, \dots, e_4 and with entries in $W(\mathbb{F}[[t_1, t_2, t_3, t_4]])$) given by

$$(\alpha_{ij})^{\text{univ}} = \begin{pmatrix} 1 & [t_1] & [t_2] \\ & 1 & [t_3] & [t_4] \\ & & 1 & \\ & & & 1 \end{pmatrix} \cdot \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} = \begin{pmatrix} [t_2] & [t_1] & & 1 \\ [t_4] & [t_3] & 1 & \\ & & 1 & \\ 1 & & & \end{pmatrix}.$$

Here the $[t_i]$ denote the Teichmüller representatives of the t_i .

Now let $A' = W[[t_1, t_2, t_3, t_4]]$, let $R' = \mathbb{F}[[t_1, t_2, t_3, t_4]]$. Denote by t resp. u the image of t_1 resp. of t_4 in $A'/(t_2, t_3)$ so that $A'/(t_2, t_3) = W[[t, u]] =: A$. Let $R = \mathbb{F}[[t, u]]$. We extend the Frobenius σ on W to A' resp. A putting $\sigma(t_i) = t_i^p$ resp. $\sigma(t) = t^p$ and $\sigma(u) = u^p$. Further for any $n \in \mathbb{N}$ denote by \mathfrak{a}_n resp. \mathfrak{r}_n the ideal in A resp. in R generated by the monomials $t^i u^{n-i}$, $i = 0, \dots, n$, and let $A_n = A/\mathfrak{a}_n$ and $R_n = R/\mathfrak{r}_n$. Then A' is a frame for R' resp. A is a frame for R resp. A_n is a frame for R_n .

For an $A' - R'$ -window $(M', M'_1, \Phi', \Phi'_1)$ let $M_1'^\sigma = A' \otimes_{A', \sigma} M'_1$ and denote by $\Psi' : M_1'^\sigma \rightarrow M'$ the linearization of Φ'_1 . It is an isomorphism of A' -modules. Denote by $\alpha' : M'_1 \rightarrow M_1'^\sigma$ the composition of the inclusion map $M'_1 \hookrightarrow M'$ followed by Ψ'^{-1} . In this way, the category of pairs (M'_1, α') consisting of a free A' -module of finite rank and an A' -linear injective homomorphism $\alpha' : M'_1 \rightarrow M_1'^\sigma$ such that $\text{Coker } \alpha'$ is a free R' -module becomes equivalent to the category of formal p -divisible groups over R' . (Note that the so called nilpotence condition is fulfilled automatically here, comp. also [KR3], section 8.) A corresponding description holds for the category of formal p -divisible groups over R resp. over R_n . We consider the $A' - R'$ window $(M', M'_1, \Phi', \Phi'_1)$ given by

$$M' = M \otimes A', \quad M'_1 = VM \otimes A', \quad \Phi' = \begin{pmatrix} t_2 & t_1 & p \\ t_4 & t_3 & p \\ & 1 & \\ 1 & & \end{pmatrix} \sigma, \quad \Phi'_1 = \frac{1}{p} \cdot \Phi',$$

the matrix of Φ' being described in the basis e_1, e_2, e_3, e_4 . The corresponding display is the universal display described above (easy to see using the procedure described on p.2 of [Z2]). Hence $(M', M'_1, \Phi', \Phi'_1)$ is the universal window. The corresponding matrix of α' wrt. the basis pe_1, pe_2, e_3, e_4 of M'_1 resp. the basis $p(1 \otimes e_1), p(1 \otimes e_2), 1 \otimes e_3, 1 \otimes e_4$ of $M_1'^\sigma$ is given by

$$\alpha' = \begin{pmatrix} & & & 1 \\ & & 1 & \\ p & -t_3 & -t_4 & \\ p & -t_1 & -t_2 & \end{pmatrix}.$$

The p -divisible group X is equipped with a principal polarization λ and with a \mathbb{Z}_{p^2} -action ι .

Lemma 3.8 *The ideal in A' resp. R' describing the universal deformation of (X, λ, ι) is (in both rings) given by (t_2, t_3) . The universal object (M_1, α) over A (corresponding to the completion of $\mathcal{O}_{\mathcal{M}_p^{\text{HB}}, x}$) is given by the image of (M'_1, α') over A (i.e. is obtained from (M'_1, α') by*

tensoring over A' with A). In the basis pf_1, f_2, f_3, pf_4 of M_1 resp. the basis $p(1 \otimes f_1), 1 \otimes f_2, 1 \otimes f_3, p(1 \otimes f_4)$ of M_1^σ the map α is then given by the matrix

$$\alpha = \begin{pmatrix} & & 1 & \\ p & -t & -u & p \\ & & & \\ & & 1 & \end{pmatrix} = \begin{pmatrix} & U \\ T & \end{pmatrix}.$$

Here

$$U = \begin{pmatrix} 1 & \\ -u & p \end{pmatrix} \text{ resp. } T = \begin{pmatrix} p & -t \\ & 1 \end{pmatrix}.$$

Proof. The principal polarization λ of X corresponds to the perfect symplectic pairing on M given by $\langle e_1, e_3 \rangle = 1, \langle e_2, e_4 \rangle = -1$ and $\langle e_i, e_j \rangle = 0$ for all other pairs $i \leq j$ (see above). One checks easily that it lifts to a perfect symplectic bilinear form \langle, \rangle of the display over $W(R'/(t_2, t_3))$ induced (by base change) by the universal one above, such that \langle, \rangle satisfies the condition $\langle V^{-1}(x), V^{-1}(y) \rangle^V = \langle x, y \rangle$ (where V is the Verschiebung of $W(R'/(t_2, t_3))$). Hence λ lifts over $R'/(t_2, t_3)$, comp [G], p. 231. Further the \mathbb{Z}_{p^2} -operation ι also lifts over $R'/(t_2, t_3)$, since on $M'_1 \otimes A'/(t_2, t_3)$ we have a $\mathbb{Z}/2$ -grading lifting the $\mathbb{Z}/2$ -grading on VM and such that the map induced by α' is homogenous of degree 1. More precisely, the $\mathbb{Z}/2$ -grading is given by $(M'_1 \otimes A'/(t_2, t_3))^0 = \text{span of } \{pe_1, e_3\}$ and $(M'_1 \otimes A'/(t_2, t_3))^1 = \text{span of } \{e_2, pe_4\}$. Hence (X, λ, ι) lifts over $R'/(t_2, t_3)$. Since the completion of $\mathcal{O}_{\mathcal{M}_p^{HB}, x}$ is isomorphic to $R'/(t_2, t_3)$, it follows that the ideal (t_2, t_3) in R' is the ideal describing the deformation of (X, λ, ι) . Hence the universal object (M_1, α) over A corresponding to the completion of $\mathcal{O}_{\mathcal{M}_p^{HB}, x}$ is given by the image of (M'_1, α') over A . It follows that in the basis pf_1, f_2, f_3, pf_4 of M_1 resp. the basis $p(1 \otimes f_1), 1 \otimes f_2, 1 \otimes f_3, p(1 \otimes f_4)$ of M_1^σ the map α is then given by the matrix

$$\alpha = \begin{pmatrix} & & 1 & \\ p & -t & -u & p \\ & & & \\ & & 1 & \end{pmatrix} = \begin{pmatrix} & U \\ T & \end{pmatrix}$$

as asserted. □

Next we want to determine the maximal n such that j lifts over R_n . The $\mathbb{Z}/2$ -grading of the Dieudonné module M of X is given by $M^0 = \text{span of } \{f_1, f_2\}$ and $M^1 = \text{span of } \{f_3, f_4\}$. The matrix of j in the basis f_1, \dots, f_4 is of the form

$$j = \begin{pmatrix} & pa^\sigma & -b \\ a & b & -c & a \\ c & pa^\sigma & & \end{pmatrix}; \quad a, b, c \in \mathbb{Z}_{p^2}, \quad b^\sigma = -b, \quad c^\sigma = -c,$$

see [KR2], section 5. Write $j = p^r \bar{j}$, where

$$\bar{j} = \begin{pmatrix} & pa_0^\sigma & -b_0 \\ a_0 & b_0 & -c_0 & a_0 \\ c_0 & pa_0^\sigma & & \end{pmatrix}; \quad a_0, b_0, c_0 \in \mathbb{Z}_{p^2} \text{ are not all divisible by } p.$$

Lemma 3.9 *If $b_0 \equiv 0 \pmod{p}$ and $c_0 \equiv 0 \pmod{p}$, then x is the intersection point of $\mathbb{P}_{[\Lambda]}$ with another projective line $\mathbb{P}_{[\Lambda']} \subset Z(j)$ and $d_{[\Lambda']} = d_{[\Lambda]} = 0$.*

Proof. Since in this case $\nu_p(\bar{j}^2) = 1$ we can identify $Z(\bar{j})$ with the Drinfeld upper half plane (see [T], chapter 4) and therefore the statement just means that under this identification x is a superspecial point in the Drinfeld upper half plane (in the sense of [KR1]). Denoting the Dieudonné module of x with its $\mathbb{Z}/2$ -grading as above by $M = M^0 \oplus M^1$, we have to show that $\bar{j}M^0 = VM^0$ and $\bar{j}M^1 = VM^1$. Now $\bar{j}M^0$ is the W -span of $a_0f_3 + c_0f_4$ and $b_0f_3 + pa_0f_4$. Since a_0 is a unit in W and b_0 and c_0 are divisible by p this is the same as the W -span of f_3 and $b_0f_3 + pa_0f_4$, and this is the same as the W -span of f_3 and pf_4 which is VM^0 . The equality $\bar{j}M^1 = VM^1$ is proved in the same way. \square

It follows that neither in the situation of (1) nor in the situation of (2) the assumption made in the lemma is fulfilled. Write now $(M_1(n), \alpha(n))$ for $M_1 \otimes A_n$ and the map $\alpha(n) : M_1(n) \rightarrow M_1(n)^\sigma$ induced by α . Write $j(1)$ for the endomorphism of $M_1(1)$ induced by j . In the basis pf_1, f_2, f_3, pf_4 it is given by the matrix

$$j(1) = \begin{pmatrix} & a^\sigma & -b \\ pa & b & -c \\ c & a^\sigma & pa \end{pmatrix} = \begin{pmatrix} & j_1(1) \\ j_0(1) & \end{pmatrix}.$$

Here

$$j_0(1) = \begin{pmatrix} pa & b \\ c & a^\sigma \end{pmatrix} \text{ resp. } j_1(1) = \begin{pmatrix} a^\sigma & -b \\ -c & pa \end{pmatrix}.$$

In order to lift j over R_n we need an endomorphism $j(n)$ of $M_1(n)$ lifting $j(1)$ such that the following diagram commutes:

$$\begin{array}{ccc} M_1(n) & \xrightarrow{\alpha(n)} & M_1(n)^\sigma \\ j(n) \downarrow & & \downarrow \sigma(j(n)) \\ M_1(n) & \xrightarrow{\alpha(n)} & M_1(n)^\sigma. \end{array}$$

In other words we are looking for liftings $j_0(n)$ of $j_0(1)$ and $j_1(n)$ of $j_1(1)$ over A_n such that

$$Uj_0(n) = \sigma(j_1(n))T \quad \text{and} \quad Tj_1(n) = \sigma(j_0(n))U. \quad (*)$$

Suppose $n = p^l$, where $l \geq 1$, and suppose we have found liftings $j_0(p^{l-1})$ and $j_1(p^{l-1})$ satisfying (*). For any choice of liftings $j_0(p^l)$ and $j_1(p^l)$ of $j_0(p^{l-1})$ and $j_1(p^{l-1})$ the matrices $\sigma(j_i(p^l))$ are equal to $\sigma(j_i(p^{l-1}))$ interpreted as matrix over A_{p^l} . Hence there are liftings $j_0(p^l)$ and $j_1(p^l)$ satisfying (*) if and only if

$$U^{-1}\sigma(j_1(p^{l-1}))T \quad \text{and} \quad T^{-1}\sigma(j_0(p^{l-1}))U$$

are integral and in this case

$$j_0(p^l) = U^{-1}\sigma(j_1(p^{l-1}))T \quad \text{and} \quad j_1(p^l) = T^{-1}\sigma(j_0(p^{l-1}))U.$$

Define now inductively matrices $X(l)$ and $Y(l)$ over $A_{p^l} \otimes_{\mathbb{Z}} \mathbb{Q}$ as follows: $X(0) = j_0(1)$ and $Y(0) = j_1(1)$ and

$$X(l+1) = U^{-1}\sigma(Y(l))T \quad \text{and} \quad Y(l+1) = T^{-1}\sigma(X(l))U.$$

(Again $\sigma(Y(l))$ and $\sigma(X(l))$ are well defined over $A_{p^{l+1}} \otimes_{\mathbb{Z}} \mathbb{Q}$.)

The following lemma can easily be proved by induction.

Lemma 3.10 *Let $l \geq 1$. The matrix $X(l)$ is of the form*

$$X(l) = \frac{1}{p^l} \left(\begin{array}{c} 0 \\ 0 \end{array} \begin{array}{c} 0 \\ (tu)^{1+p+\dots+p^{l-2}} \cdot ((-1)^l \sigma^{l+1}(a) \cdot (tu)^{p^{l-1}} + b \cdot u^{p^{l-1}} - c \cdot t^{p^{l-1}}) \end{array} \right) + p \cdot A(l),$$

where $A(l)$ has entries in A_{p^l} . Similarly, $Y(l)$ is of the form

$$Y(l) = \frac{1}{p^l} \left(\begin{array}{c} (tu)^{1+p+\dots+p^{l-2}} \cdot ((-1)^l \sigma^{l+1}(a) \cdot (tu)^{p^{l-1}} + b \cdot u^{p^{l-1}} - c \cdot t^{p^{l-1}}) \\ 0 \end{array} \right) + p \cdot B(l),$$

where $B(l)$ has entries in A_{p^l} . \square

Lemma 3.10 shows together with the discussion above that j lifts over R_{p^r} but not over $R_{p^{r+1}}$. More precisely, if we are in the situation of (1) or (2), then b and c are not both divisible by p^{r+1} (Lemma 3.9) and hence we see from Lemma 3.10 (applied in case $l = r + 1$) that j lifts over $R_{2 \cdot (1+p+\dots+p^{r-1})+p^r}$ but not over $R_{2 \cdot (1+p+\dots+p^{r-1})+p^{r+1}}$. Hence $n_x = 2 \cdot (1+p+\dots+p^{r-1})+p^r$ as claimed in (1) resp. in (2). \square

From Theorem 3.7 we immediately obtain the following corollary (using the same notations and assumptions).

Corollary 3.11 *If $a = 0$, then $D(j)_p$ is of the form $D(j)_p = s$. If $a > 0$, then $D(j)_p$ is of the form*

$$D(j)_p = \sum_{[\Lambda]; d_{[\Lambda]} \leq \frac{a-1}{2}} p^{\frac{a-1}{2} - d_{[\Lambda]}} \mathbb{P}_{[\Lambda]} + p^{a/2-1} (p-1) \cdot s.$$

\square

Lemma 3.12 *Let $j \in V'$ such that $j^2 \in \mathbb{Z}_{(p)} \setminus \{0\}$, suppose that let $x \in Z(j)(\mathbb{F})$ and let $D \subset \mathcal{M}^{HB}$ be a divisor which is regular in x , such that locally around x we have $D_p \subset Z(j)_p$. Let $R = \mathcal{O}_{D,x}$ and let (f) be the ideal of $Z(j) \cap D$ in R (i.e., $\mathcal{O}_{Z(j) \cap D, x} = R/(f)$). Then the ideal of $Z(pj) \cap D$ in R is of the form $(h \cdot p \cdot f)$, where h is coprime to p in R .*

Proof. We may assume that $f \neq 0$.

The canonical map

$$\varrho_0 : \mathrm{Spf} R/(f) \rightarrow D$$

factors through $Z(j) \cap D$.

Claim: *The canonical map*

$$\varrho_1 : \mathrm{Spf} R/(pf) \rightarrow D$$

factors through $Z(pj) \cap D$.

By our assumption $D_p \subset Z(j)_p$ (locally around x) we know that f is divisible by p . We have $R/(f) = (R/(pf))/I$, where $I = (f)/(pf)$. Since f is divisible by p , the ideal I carries a nilpotent pd -structure. Hence we may apply Grothendieck-Messing theory for the pair $R/(pf)$, $R/(f)$. We denote by M the value of the crystal of the $R/(pf)$ -valued point ϱ_1 in $R/(pf)$, and by \overline{M} the value of the crystal of the $R/(f)$ -valued point ϱ_0 in $R/(f)$. Then $\overline{M} = M \otimes R/(f)$. Denote by $\mathcal{F} \hookrightarrow M$ the Hodge filtration of ϱ_1 and by $\overline{\mathcal{F}} \hookrightarrow \overline{M}$ the Hodge filtration of ϱ_0 . Then the Hodge filtration of ϱ_1 lifts the Hodge filtration of ϱ_0 . We have $j\overline{\mathcal{F}} \subset \overline{\mathcal{F}}$, and we have to show that $pj\mathcal{F} \subset \mathcal{F}$. As in the proof of Proposition 4.5 in [T] we find a basis e_1, e_2, e_3, e_4 of M , such that, if \overline{e}_i denote the images of the e_i in \overline{M} , then a basis of $\overline{\mathcal{F}}$ is given by $\overline{e}_2, \overline{e}_3$, and such that, for suitable $m_1, m_4 \in I$, a basis of \mathcal{F} is given by $f_0 = e_2 + m_1 e_1$ and $f_1 = e_3 + m_4 e_4$. Since $j\overline{\mathcal{F}} \subset \overline{\mathcal{F}}$ it follows that $jf_i \in \langle e_2, e_3 \rangle + IM$ for $i = 1, 2$. Hence $pj f_i \in \langle p e_2, p e_3 \rangle + pIM =$

$\langle pe_2, pe_3 \rangle = \langle pf_0, pf_1 \rangle \subset \mathcal{F}$. This proves the claim.

Let $\mathcal{O}_{Z(pj) \cap D, x} = R/(g)$. Since D is regular in x we know that R is a UFD. We write $g = h \cdot s \cdot f$, where h is prime to p and $V(s)$ has support in the special fibre of D . We must show that $s = p$. We already know that s is divisible by p . Suppose that q is prime divisor of p in R such that g divisible by qpf . Let $C = R/(qpf)$, let $\overline{C} = R/(qf)$, and let $\overline{\overline{C}} = R/(f)$. This is a chain of nilpotent pd -extensions.

We consider the canonical C -, resp. \overline{C} -, resp. $\overline{\overline{C}}$ -valued points

$$\varrho_C : \mathrm{Spf} C \rightarrow D, \quad \varrho_{\overline{C}} : \mathrm{Spf} \overline{C} \rightarrow D, \quad \varrho_{\overline{\overline{C}}} : \mathrm{Spf} \overline{\overline{C}} \rightarrow D.$$

Claim: $\varrho_{\overline{C}}$ factors through $Z(j)$.

We denote by M the value of the crystal of the ϱ_C -valued point in C , analogously we define \overline{M} and $\overline{\overline{M}}$. Further let $\mathcal{F} \hookrightarrow M$ by the Hodge filtration of ϱ_C and analogously we define $\overline{\mathcal{F}} \hookrightarrow \overline{M}$ and $\overline{\overline{\mathcal{F}}} \hookrightarrow \overline{\overline{M}}$. As in the proof of Proposition 4.5 in [T] we find a basis e_1, e_2, e_3, e_4 of M , such that, if \overline{e}_i denote the images of the e_i in \overline{M} , then a basis of $\overline{\mathcal{F}}$ is given by $\overline{e}_2, \overline{e}_3$, and such that for suitable $m_1, m_4 \in (qf)/(qpf)$ a basis of \mathcal{F} is given by $f_0 = e_2 + m_1 e_1$ and $f_1 = e_3 + m_4 e_4$. The endomorphism j induces an endomorphism of the C -module M , and we have $pj\mathcal{F} \subset \mathcal{F}$. To show the above claim we must show that $j\overline{\mathcal{F}} \subset \overline{\mathcal{F}}$. Since $pj f_0 \in \mathcal{F}$ and by the explicit form of f_0 and f_1 we can write $j f_0 = u + x e_4$ for a suitable $u \in \mathcal{F}$ and some $x \in C$ with the property that $px = 0$. Let $\hat{x} \in R$ be a lift of x . Then $p\hat{x} \in (qpf)$, hence $\hat{x} = yqf$ for a suitable $y \in R$. hence $x \in (qf)/(qpf)$, hence $j\overline{e}_2 \in \overline{\mathcal{F}}$. In the same way one shows that $j\overline{e}_3 \in \overline{\mathcal{F}}$. From this the claim follows.

The claim implies the assertion, since it implies that f is divisible by qf , a contradiction. \square

We reformulate the lemma in a more geometrical manner in the following proposition.

Proposition 3.13 *In the situation of Lemma 3.12 and under the assumption that $f \neq 0$, the divisor $D(pj) \cap D$ in D is locally around x of the form*

$$D(pj) \cap D = D_p + h,$$

where h is an effective divisor which is horizontal, i.e. it has no component with support in D_p . \square

4 On the structure of intersections of formal special cycles

By the proof of [T], Proposition 4.5, for any $j \in V'_p$, and any $x \in \mathcal{M}^{HB}(\mathbb{F})$ the ideal of $Z(j)$ in $\mathcal{O}_{\mathcal{M}^{HB}, x}$ is generated by one element (we do not use the assumptions $j^2 \neq 0$ and $\nu_p(j^2) \geq 0$ made in Proposition 3.3).

Lemma 4.1 *Let $j, j' \in V'$ be linearly independent. Suppose further that for any $x \in \mathcal{M}^{HB}(\mathbb{F})$ the equation in x (that is a generator of the ideal in $\mathcal{O}_{\mathcal{M}^{HB}, x}$) of at least one of the corresponding two formal special cycles is not divisible by p . Then*

$$\mathcal{O}_{Z(j)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(j')} = \mathcal{O}_{Z(j)} \otimes \mathcal{O}_{Z(j')}.$$

More precisely, the object on the left hand side is represented in the derived category by the object on the right hand side. The same formula holds if $Z(j)$ or $Z(j')$ or both are replaced by $D(j)$ resp. $D(j')$.

Proof. Let $x \in \mathcal{M}^{HB}(\mathbb{F})$ and let $R = \mathcal{O}_{\mathcal{M}^{HB},x}$. Let the ideals of $Z(j)$ and $Z(j')$ in R be generated by f and f' . Suppose that f is not divisible by p . We consider the exact sequence

$$0 \longrightarrow R \xrightarrow{f} R \longrightarrow R/(f) \longrightarrow 0.$$

Tensoring this with $R/(f')$ we see that

$$\mathcal{TOR}_1(\mathcal{O}_{Z(j)}, \mathcal{O}_{Z(j')})_x = \ker(R/(f') \xrightarrow{f} R/(f')).$$

To show that this vanishes we have to show that f and f' have no common divisor in the regular ring R . Let $g = \gcd(f, f')$. We have to show that g is a unit. It follows from our assumption in the lemma that g is coprime to p . We may assume that $x \in (Z(j) \cap Z(j'))(\mathbb{F})$. Let now T_x be the matrix of the bilinear form associated to Q for the basis j, j' of $\mathbf{j} = \mathbb{Z}_{(p)}$ -span of j, j' in V' and let $t = j^2$ and $t' = j'^2$. Since $j, j' \in V'$, we find special cycles $Z(t, \omega)$ and $Z(t', \omega')$ and an \mathbb{F} -valued point y of $Z(t, \omega) \times_{\mathcal{M}} Z(t', \omega')$ such that the corresponding p -divisible group resp. its corresponding special endomorphisms are the p -divisible group of x resp. j, j' (see also the incidence relation under Proposition 2.2). It follows that for the completions $\hat{\mathcal{O}}_{Z(t, \omega) \times_{\mathcal{M}} Z(t', \omega'), y} \otimes_{\mathbb{Z}_p} W \cong \hat{R}/(f, f')$ (we denote the images of f, f' and g in \hat{R} by the same letters). Suppose that g is not a unit. Then $R/(g) \otimes_W \mathbb{Q}$ has dimension 1. Thus $\mathcal{O}_{Z(t, \omega) \times_{\mathcal{M}} Z(t', \omega'), y} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$ has dimension 1. It follows that there is some \mathbb{C} -valued point ξ of the generic fibre of $Z(t, \omega) \times_{\mathcal{M}} Z(t', \omega')$ such that the corresponding 2×2 fundamental matrix in ξ (defined as above) equals T_x . But then the proof of Proposition 1.4 of [KR2] shows that T_x is positive definite. Then again by the same proposition, the generic fibre of the part of $Z(t, \omega) \times_{\mathcal{M}} Z(t', \omega')$ where the fundamental matrix equals T_x has dimension 0 which is a contradiction to the fact that $\dim(\mathcal{O}_{Z(t, \omega) \times_{\mathcal{M}} Z(t', \omega'), y} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}) = 1$. The proof in case that $Z(j)$ or $Z(j')$ or both are replaced by $D(j)$ resp. $D(j')$ is the same. \square

Proposition 4.2 *Let $y_1, y_2, y_3 \in V'$ such that the corresponding fundamental matrix is non-singular and has entries in $\mathbb{Z}_{(p)}$. Then the derived tensor product $\mathcal{O}_{Z(y_1)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(y_2)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(y_3)}$ depends only on the $\mathbb{Z}_{(p)}$ -span \mathbf{y} of y_1, y_2, y_3 in V' .*

Proof. First we observe that for any basis y'_1, y'_2, y'_3 of \mathbf{y} the derived tensor product $\mathcal{O}_{Z(y'_1)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(y'_2)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(y'_3)}$ is invariant under any permutation of the y'_i .

Claim $\mathcal{O}_{Z(y'_1)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(y'_2)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(y'_3)}$ is invariant if one y_i is replaced by $\varepsilon y_i + z y_l$, where $\varepsilon \in \mathbb{Z}_{(p)}^\times$, $z \in \mathbb{Z}_{(p)}$ and $l \neq i$.

We want to apply the lemma setting $j = y_i, j' = y_l$. We must show that in no \mathbb{F} -valued point of \mathcal{M}^{HB} the equation of $Z(y_i)$ and $Z(y_l)$ is divisible by p . Suppose that there is some \mathbb{F} -valued point of \mathcal{M}^{HB} in which both equations are divisible by p . Then the same is true for any linear combination of y_i and y_l . Further it follows from Proposition 3.3 that $y_i^2 = 0$ and $y_l^2 = 0$ and hence again the same is true for any linear combination of y_i and y_l . Thus it follows (by diagonalizing) that the fundamental matrix of y_i and y_l is the zero matrix. But then it follows that the fundamental matrix of y_1, y_2, y_3 is singular which contradicts our assumption. Thus we may apply the lemma, by which $\mathcal{O}_{Z(y'_i)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(y'_l)} = \mathcal{O}_{Z(y'_i)} \otimes \mathcal{O}_{Z(y'_l)} = \mathcal{O}_{Z(y'_i) \cap Z(y'_l)}$ which only depends on the $\mathbb{Z}_{(p)}$ -span of y_i, y_l in V' . From this the claim follows.

Since we can transform the basis y_1, y_2, y_3 by a suitable sequence of permutations and operations as in the claim into any other basis of \mathbf{y} , the claim of the proposition follows. \square

In particular, the proposition shows that for the calculation of $(Z(j_1), Z(j_2), Z(j_3))$ we may assume our three fixed special endomorphisms j_1, j_2, j_3 are perpendicular to each other, i.e. we

may assume that T is a diagonal matrix.

Suppose for the moment that our fixed T is not divisible by p . Then, as shown in [KR2], section 6, the intersection $Z(j_1) \cap Z(j_2) \cap Z(j_3)$ is the formal spectrum of a local Artin ring whose length (over W) is calculated in loc. cit. We want to show that this length equals the intersection multiplicity in the sense defined above. To this end we show the following

Proposition 4.3 *If T (as matrix over $\mathbb{Z}_{(p)}$) is not divisible by p , then*

$$\mathcal{O}_{Z(j_1)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(j_2)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(j_3)} = \mathcal{O}_{Z(j_1)} \otimes \mathcal{O}_{Z(j_2)} \otimes \mathcal{O}_{Z(j_3)}$$

(meaning that the right hand side represents the left hand side in the derived category). In particular,

$$(Z(j_1), Z(j_2), Z(j_3)) = \text{lg}_W(\mathcal{O}_{Z(j_1) \cap Z(j_2) \cap Z(j_3), x}),$$

where x is the unique \mathbb{F} -valued point of $Z(j_1) \cap Z(j_2) \cap Z(j_3)$.

Proof. By Proposition 4.2 we may assume that j_1, j_2, j_3 are pairwise perpendicular to each other. We further assume $\nu_p(j_1^2) \leq \nu_p(j_2^2) \leq \nu_p(j_3^2)$. Then $\nu_p(j_1^2) = 0$. By Lemma 4.1 we have $\mathcal{O}_{Z(j_1)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(j_2)} = \mathcal{O}_{Z(j_1)} \otimes \mathcal{O}_{Z(j_2)}$ in the above sense. Let the ideal of $Z(j_i)$ in $\mathcal{O}_{\mathcal{M}^{HB}, x}$ be generated by f_i and let $R = \mathcal{O}_{\mathcal{M}^{HB}, x}/(f_1)$. Denote the images of f_2 resp. f_3 in R by \bar{f}_2 resp. \bar{f}_3 . The same reasoning as in the proof of Lemma 4.1 shows that the claim follows if we know that \bar{f}_2 and \bar{f}_3 are coprime in R (which is a UFD by Proposition 3.6). But this in turn follows from the fact that $\dim(R/(\bar{f}_2, \bar{f}_3)) = 0$. \square

Our next aim is to prove the following multilinearity property of the intersection product $(Z(j_1), Z(j_2), Z(j_3))$.

Proposition 4.4 *Suppose that at least one of j_1, j_2, j_3 is odd. Then*

$$(Z(j_1), Z(j_2), Z(j_3)) = \sum_{l, m, n} (D(j_1/p^l), D(j_2/p^m), D(j_3/p^n)),$$

where the sum is taken over all possible triples (l, m, n) (i.e. setting $a_i = \nu_p(j_i^2)$, we have $l \leq [a_1/2]$, $m \leq [a_2/2]$, $n \leq [a_3/2]$, where $[]$ denotes Gauss brackets).

Note that in case that j_1, j_2, j_3 are pairwise perpendicular to each other, the hypothesis of the proposition is fulfilled as we see from [Ku1], 1.16, see also (5.1). Note also that using the fact that $Z(j_1) \cap Z(j_2) \cap Z(j_3)$ has support in the supersingular locus (follows from [KR2], Proposition 3.8) and the fact that this support is proper over \mathbb{F} (follows for example from the beginning of section 3 and Lemma 3.2 or from [KR2], section 8) one sees that all of these intersection multiplicities are finite. The same holds for the other intersection multiplicities in the subsequent proof.)

Proof. Suppose j_1 is odd and $\nu_p(j_1^2) = 2r + 1$. Denote by \mathcal{I} the ideal sheaf of $D := D(j_1/p^r) (= Z(j_1/p^r))$ in $\mathcal{O}_{\mathcal{M}^{HB}}$. Denote by \mathcal{J} the ideal sheaf of $\Delta := Z(j_1) - D(j_1/p^r)$ (the notation in the latter expression is meant in the same sense as the notation $D(j) = Z(j) - Z(j/p)$). Our first aim is to show that

$$(Z(j_1), Z(j_2), Z(j_3)) = (\Delta, Z(j_2), Z(j_3)) + (D, Z(j_2), Z(j_3)).$$

We consider the canonical short exact sequences

$$0 \longrightarrow \mathcal{J}/(\mathcal{J} \cdot \mathcal{I}) \longrightarrow \mathcal{O}_{\mathcal{M}^{HB}}/(\mathcal{J} \cdot \mathcal{I}) \longrightarrow \mathcal{O}_{\mathcal{M}^{HB}}/\mathcal{J} \longrightarrow 0$$

and

$$0 \longrightarrow (\mathcal{J} + \mathcal{I})/\mathcal{I} \longrightarrow \mathcal{O}_{\mathcal{M}^{HB}}/\mathcal{I} \longrightarrow \mathcal{O}_{\mathcal{M}^{HB}}/(\mathcal{J} + \mathcal{I}) \longrightarrow 0.$$

Lemma 4.5 *The inclusion $\mathcal{J} \cdot \mathcal{I} \hookrightarrow \mathcal{J} \cap \mathcal{I}$ is an equality.*

Proof. We can check this locally. Let $x \in \mathcal{M}^{HB}(\mathbb{F})$ and let $R = \mathcal{O}_{\mathcal{M}^{HB}, x}$. For any $l \leq r$ let (f_l) be the ideal of $D(j/p^l)$ in R . By Proposition 3.6, each f_l is a prime element in R . It follows from Corollary 3.11 that the f_l are pairwise distinct (more precisely, Corollary 3.11 shows that even the special fibres of the several $\mathrm{Spf} R/(f_l)$ are pairwise distinct). Since \mathcal{I}_x equals (f_r) and \mathcal{J}_x equals $(\prod_{l \neq r} f_l)$, the claim of the lemma follows. \square

By the first exact sequence have

$$(Z(j_1), Z(j_2), Z(j_3)) = (\Delta, Z(j_2), Z(j_3)) + \chi(\mathcal{J}/(\mathcal{J} \cdot \mathcal{I}) \otimes^{\mathbb{L}} \mathcal{O}_{Z(j_2)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(j_3)}).$$

Now using the lemma we see that $(\mathcal{J} + \mathcal{I})/\mathcal{I} = \mathcal{J}/(\mathcal{J} \cap \mathcal{I}) = \mathcal{J}/(\mathcal{J} \cdot \mathcal{I})$. This shows together with the second exact sequence that

$$\chi(\mathcal{J}/(\mathcal{J} \cdot \mathcal{I}) \otimes^{\mathbb{L}} \mathcal{O}_{Z(j_2)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(j_3)}) = (D, Z(j_2), Z(j_3)) - (D \cap \Delta, Z(j_2), Z(j_3)).$$

Thus in order to show that

$$(Z(j_1), Z(j_2), Z(j_3)) = (\Delta, Z(j_2), Z(j_3)) + (D, Z(j_2), Z(j_3))$$

it remains to show the

Claim $(D \cap \Delta, Z(j_2), Z(j_3)) = 0$.

Using Lemma 4.1 we see that

$$\chi(\mathcal{O}_{D \cap \Delta} \otimes^{\mathbb{L}} \mathcal{O}_{Z(j_2)} \otimes^{\mathbb{L}} \mathcal{O}_{Z(j_3)}) = \chi(\mathcal{O}_{D \cap \Delta} \otimes^{\mathbb{L}} \mathcal{O}_{Z(j_2) \cap Z(j_3)}).$$

First we observe that by the proof of the above lemma, $\Delta \cap D$ is as a divisor in D . The same reasoning as in the proof of Proposition 4.6 below (second case in the proof) shows that $\Delta \cap D$ is as a divisor in D of the form $rD_p + h$, where h is a divisor of the form $\sum_x h_x$ where the sum runs over a discrete set of \mathbb{F} -valued points of D and h_x is a horizontal divisor (meaning that its equation is coprime to p) meeting the underlying reduced subscheme of D only in x . (More precisely, D plays the role of $D(y_1)$ in the situation of the proof of Proposition 4.6 and (for any $l < r$), $D(y_1/p^l)$ plays the role of $D(y_2)$ in that proof. Note that $\Delta = \sum_{l < r} D(y_1/p^l)$.) Now we regard rD_p and h as (formal) closed subschemes of \mathcal{M}^{HB} . Suppose that $h = 0$. Then $(D \cap \Delta, Z(j_2), Z(j_3)) = (rD_p, Z(j_2) \cap Z(j_3))$. This in turn equals 0 as follows from the exact sequence

$$0 \longrightarrow \mathcal{O}_D \xrightarrow{p^r} \mathcal{O}_D \longrightarrow \mathcal{O}_D/(p^r) \longrightarrow 0.$$

Thus the claim is proved if we can show that $h = 0$. Suppose that there is $x \in (D \cap \Delta)(\mathbb{F})$ such that $h_x \neq 0$. Using Lemma 3.4 we choose a special endomorphism j such that $\nu_p(j^2) = 1$ and $x \in Z(j)(\mathbb{F})$ and $j \perp j_1$. We write $j_1^2 = \varepsilon_1 p^{2r+1}$ and $j^2 = \varepsilon p$. It is easy to see that we can choose j such that the image of $-\varepsilon \varepsilon_1$ in \mathbb{F}_p is a square. Then it follows from the results of [T] (in the same way as the proof of Lemma 3.5) that $Z(j) \cap \Delta$ has support in the special fibre, that this support is proper over \mathbb{F} and that $Z(j) \cap D$ has precisely two horizontal components which meet the special fibre in the same projective line. Further their intersection points with this projective line are both neither intersection points of two projective lines which belong to $Z(j)$ nor intersection points of two projective lines which belong to D . Further the intersection multiplicity of each of these horizontal components with the projective line which they meet is 1. We consider the intersection multiplicity $(D, \Delta, Z(j))$. It can be written as $((Z(j) \cap D), (Z(j) \cap \Delta))$ (intersection multiplicity in $Z(j)$, compare also [T], section 4). Let $(Z(j) \cap D)_v$ resp. $(Z(j) \cap D)_h$ denote the vertical resp. horizontal part of $(Z(j) \cap D)$. Now

Lemma 5.2 below shows that $((Z(j) \cap D)_v, (Z(j) \cap \Delta)) = 0$ (compare the technique of calculating intersection multiplicities used in the propositions of section 5). Further $((Z(j) \cap D)_h, (Z(j) \cap \Delta)) = 2r$ and thus $((Z(j) \cap D), (Z(j) \cap \Delta)) = 0 + 2r = 2r$ (see also below for the additivity of intersection multiplicities in $Z(j)$ used here). On the other hand, $(D, \Delta, Z(j))$ can be written as $((D \cap \Delta), (D \cap Z(j)))$ (intersection multiplicity in D , again defined to be the Euler-Poincaré characteristic of the derived tensor product of the corresponding structure sheaves). Repeating the reasoning before the claim with $Z(j_1)$ replaced by $\Delta \cap D$ and D replaced by rD_p and Δ replaced by h , we see the following identity of intersection multiplicities in D

$$2r = \chi(\mathcal{O}_{D \cap \Delta} \otimes^{\mathbb{L}} \mathcal{O}_{Z(j) \cap D}) = (h, Z(j) \cap D) + (rD_p, Z(j) \cap D) - (h \cap rD_p, Z(j) \cap D)$$

Now $(h \cap rD_p, Z(j) \cap D) = 0$, since the structure sheaf of $h \cap rD_p$ is a skyscraper sheaf (see also [KR1], Lemma 4.1). (Analogously one sees the additivity of intersection multiplicities in $Z(j)$ used above.) As above we see that $(rD_p, (Z(j) \cap D)_v) = 0$ (applying the above exact sequence, we use that the support of $D \cap (Z(j) \cap D)_v = (Z(j) \cap D)_v$ lies in the special fibre and is proper over \mathbb{F} .) Thus $(rD_p, (Z(j) \cap D)) = 2r$. It follows that $(h, Z(j) \cap D) = 0$ and hence $h_x = 0$. Since x was arbitrary, it follows that $h = 0$. This confirms the claim.

Now repeating this reasoning we see that

$$(Z(j_1), Z(j_2), Z(j_3)) = \sum_l (D(j_1/p^l), Z(j_2), Z(j_3))$$

(note that in the remaining steps the reasoning in the last part of the proof becomes simpler since (in contrast to the above case $l = r$) for $l < r$ there are no horizontal components in $D(j/p^l) \cap Z(j)$ if j is as above.) The remaining multilinearity in the other variables follows from the multilinearity of intersections of divisors (such that the support of their intersection lies in the special fibre and is proper over \mathbb{F}) in $D(j_1/p^l)$, see the beginning of section 5. \square

The proposition and its proof show that (if at least one of j_1, j_2, j_3 is odd) it is enough to calculate the intersection multiplicity $(D(j_i), D(j_l), D(j_k))$ or (if at least one of j_i, j_l is odd) $(D(j_i), Z(j_k), D(j_l))$, where $\{i, k, l\} = \{1, 2, 3\}$. To calculate such intersections, we use the following observation (already made and used in the proof of the last proposition).

By Lemma 4.1, we have $\mathcal{O}_{D(j_i)} \otimes^{\mathbb{L}} \mathcal{O}_{D(j_l)} = \mathcal{O}_{D(j_i)} \otimes \mathcal{O}_{D(j_l)} = \mathcal{O}_{D(j_i) \cap D(j_l)}$. (Here, each $D(j_l)$ or $D(j_i)$ or both may be replaced by $Z(j_l)$ resp. $Z(j_i)$.) Thus

$$\chi(\mathcal{O}_{D(j_i)} \otimes^{\mathbb{L}} \mathcal{O}_{D(j_l)} \otimes^{\mathbb{L}} \mathcal{O}_{D(j_k)}) = \chi(\mathcal{O}_{D(j_i) \cap D(j_l)} \otimes^{\mathbb{L}} \mathcal{O}_{D(j_k)}) = \chi(\mathcal{O}_{D(j_i) \cap D(j_l)} \otimes_{\mathcal{O}_{D(j_i)}}^{\mathbb{L}} \mathcal{O}_{D(j_i) \cap D(j_k)}).$$

The same is true if $D(j_l)$ or $D(j_k)$ is replaced by $Z(j_l)$ resp. $Z(j_k)$. If the three special endomorphisms are pairwise perpendicular to each other (which we may assume), then by 1.16 of [Kul] (see also (5.1)), at least one of the three special endomorphisms is odd, so we may also assume that j_i is odd. This suggests to investigate intersections of the form $D(y_1) \cap D(y_2)$ where we assume that $y_1, y_2 \in V'$ such that $y_1^2, y_2^2 \in \mathbb{Z}_{(p)} \setminus \{0\}$ and $y_1 \perp y_2$ (and we also may assume that y_1 is odd). Note that by the proof of Lemma 4.1 (which shows that $D(y_1)$ and $D(y_2)$ have no common component), $D(y_1) \cap D(y_2)$ is a divisor in $D(y_1)$.

Let $[\Lambda] \in \mathcal{B}$. Let $y_1, y_2 \in V'$ such that $y_1^2, y_2^2 \in \mathbb{Z}_{(p)} \setminus \{0\}$ and $y_1 \perp y_2$ and $\mathbb{P}_{[\Lambda]} \subset D(y_1) \cap D(y_2)$. The divisor $D(y_i)_p$ in \mathcal{M}^{HB}_p contains $\mathbb{P}_{[\Lambda]}$ with multiplicity p^{r_i} for a suitable integer r_i given by Corollary 3.11. Now $\mathbb{P}_{[\Lambda]}$ is a closed irreducible reduced subscheme of codimension 1 in (the regular) $D(y_i)$, hence a prime divisor. The subsequent two propositions will tell us the multiplicity of $\mathbb{P}_{[\Lambda]}$ in the divisor $D(y_1) \cap D(y_2)$ in $D(y_1)$.

Proposition 4.6 *In the situation just described, suppose further that $r_1 \neq r_2$. Then the divisor $D(y_1) \cap D(y_2)$ in $D(y_1)$ contains $\mathbb{P}_{[\Lambda]}$ with multiplicity $p^{\min\{r_1, r_2\}}$.*

Proof. We choose $x \in \mathbb{P}_{[\Lambda]}(\mathbb{F})$ not superspecial. We choose $y \in V'_p$ such that $y \perp y_1, y_2$ and such that $x \in Z(y)(\mathbb{F})$ and $\nu_p(y^2) = 1$ (Lemma 3.4). Let $R = \mathcal{O}_{\mathcal{M}^{HB}, x}$ and let $(f_1), (f_2), (d)$ be the ideals of $D(y_1), D(y_2), Z(y)$ in R . It follows from Lemma 3.5 and from [T], chapter 2 that the ideal of $\mathbb{P}_{[\Lambda]}$ in x is given by $(p, d) = (f_1, d) = (f_2, d)$.

Claim 1 *We have the following identity of ideals in R : $(d^{p^{r_1}}, f_1) = (d^{p^{r_1}}, p) = (f_1, p)$.*

For any $z \in R$ we denote by \bar{z} the image of z in $\bar{R} := R/(p)$ which is a UFD. First we observe that $(d^{p^{r_1}}, p) = (f_1, p)$, since the equation of $\mathbb{P}_{[\Lambda]}$ in \bar{R} is given by $\bar{d} = 0$. After perhaps multiplying f_1 by a unit we can therefore write

$$f_1 = d^{p^{r_1}} + p\rho,$$

for some $\rho \in R$. By the above description of the ideal of $\mathbb{P}_{[\Lambda]}$ in R we can also write

$$f_1 = \varepsilon p + \sigma d$$

for some $\varepsilon \in R^\times$ and some $\sigma \in R$. From these equations we get $\bar{f}_1 = \bar{d}^{p^{r_1}} = \bar{\sigma}\bar{d}$, hence $\bar{\sigma} = \bar{d}^{p^{r_1}-1}$, hence $\sigma = d^{p^{r_1}-1} + p\sigma'$ for some $\sigma' \in R$. Hence we have $f_1 = \varepsilon p + d^{p^{r_1}} + pd\sigma' = p(\varepsilon + d\sigma') + d^{p^{r_1}}$. Since $\varepsilon + d\sigma'$ is a unit it follows that $p \in (f_1, d^{p^{r_1}})$. Hence $(f_1, d^{p^{r_1}}) = (f_1, d^{p^{r_1}}, p) = (f_1, p)$. This confirms the claim.

Now we distinguish the cases $r_2 < r_1$ and $r_1 < r_2$.

First case $r_2 < r_1$. Since the ideal of $\mathbb{P}_{[\Lambda]}$ in the local ring of $D(y_1)$ in x equals (d) , it is enough to show that $(f_1, f_2) \subset (d^{p^{r_2}}, f_1)$ and that $(f_1, f_2) \not\subset (d^{p^{r_2+1}}, f_1)$. This is the content of claims 2 and 3.

Claim 2 $(f_1, f_2) \subset (d^{p^{r_2}}, f_1)$.

We have $(f_2) \subset (d^{p^{r_2}}, p)$, hence $(f_1, f_2) \subset (d^{p^{r_2}}, f_1, p)$. The latter ideal equals by claim 1 the ideal $(d^{p^{r_2}}, d^{p^{r_1}}, f_1) = (d^{p^{r_2}}, f_1)$ since $r_2 < r_1$. Hence $(f_1, f_2) \subset (d^{p^{r_2}}, f_1)$ as claimed.

Claim 3 $(f_1, f_2) \not\subset (d^{p^{r_2+1}}, f_1)$.

Suppose $(f_1, f_2) \subset (d^{p^{r_2+1}}, f_1)$. Then $(f_1, f_2, p) \subset (d^{p^{r_2+1}}, f_1, p)$. Since $r_2 < r_1$ we have $(f_1, f_2, p) = (d^{p^{r_2}}, p)$ and $(d^{p^{r_2+1}}, f_1, p) = (d^{p^{r_2+1}}, p)$. Thus we get $(d^{p^{r_2}}, p) \subset (d^{p^{r_2+1}}, p)$, a contradiction which confirms the claim.

Combining claims 2 and 3 ends the proof in case $r_2 < r_1$.

Second case $r_2 > r_1$. By the first case, the vertical part (the part with support in the special fibre) of $D(y_1) \cap D(y_2)$ is in R given by the ideal $(f_2, d^{p^{r_1}})$. Now $(f_2, d^{p^{r_1}}) = (f_2, d^{p^{r_1}}, f_1) = (d^{p^{r_1}}, f_1)$. (The latter equality results from the fact that by claim 1 both ideals describe the ideal of the special fibre of $D(y_1)$ in R .) \square

Proposition 4.7 *Suppose we are in the situation described before Proposition 4.6. Suppose further that $r_1 = r_2 =: r$. Let $a_i = \nu_p(y_i^2)$ and write $y_i^2 = \varepsilon_i p^{a_i}$ and suppose that a_1 and a_2 are not both even and that $a_1 \leq a_2$. Suppose further that in case $a_1 = a_2$ we have $\chi(-\varepsilon_1 \varepsilon_2) = -1$ (here, χ is the quadratic residue character of \mathbb{Z}_p^\times). Then a_1 is odd and the divisor $D(y_1) \cap D(y_2)$ in $D(y_1)$ contains $\mathbb{P}_{[\Lambda]}$ with multiplicity $(\frac{a_1+1}{2} - r) \cdot p^r$. If a_2 is also odd, then the divisor $D(y_1) \cap D(y_2)$ in $D(y_2)$ also contains $\mathbb{P}_{[\Lambda]}$ with multiplicity $(\frac{a_1+1}{2} - r) \cdot p^r$.*

Before we prove this, we show the following lemma.

Lemma 4.8 *Suppose that $y_1, y_2 \in V'$ are both odd and that $y_1 \perp y_2$. Let $a_i = \nu_p(y_i^2)$ and write $y_i^2 = \varepsilon_i p^{a_i}$ with $a_1 \leq a_2$ and suppose $\chi(-\varepsilon_1 \varepsilon_2) = -1$. (Here, χ denotes the quadratic residue character of \mathbb{Z}_p^\times). Then the generic fibre of $Z(y_1) \cap Z(y_2)$ is empty (i.e., $Z(y_1) \cap Z(y_2)$ has support in the special fibre).*

Proof of Lemma 4.8. Since $y_1, y_2 \in V'$, it is enough to show that there are no special cycles $Z = Z(\varepsilon_1 p^{a_1}, \omega)$ and $Z' = Z(\varepsilon_2 p^{a_2}, \omega')$ such that there is some point in the generic fibre of $Z \times_{\mathcal{M}} Z'$ with fundamental matrix $\text{diag}(\varepsilon_1 p^{a_1}, \varepsilon_2 p^{a_2})$. (See also the incidence relation below Proposition 2.2.) Recall our fixed lattice $\Lambda \subset V$ in section 1. By the description of the generic fibre of special cycles given in [KR2], section 2, it is enough to show the following

Claim 1 *There are no elements $\lambda_1, \lambda_2 \in \omega_p := \Lambda \otimes \mathbb{Z}_p$ such that $\lambda_1 \perp \lambda_2$ and $q(\lambda_i) = \varepsilon_i p^{a_i}$.*

Here, q denotes the extension to ω_p of the quadratic form q on Λ . We will write \langle, \rangle for the induced bilinear form on ω_p . We also write λ^2 for $q(\lambda)$, and we write \langle, \rangle for the corresponding bilinear form, i.e. $\langle x, y \rangle = \frac{1}{2}((x+y)^2 - x^2 - y^2)$.

Suppose there are λ_1, λ_2 as in the claim. We may suppose that both of λ_1, λ_2 are not divisible by p .

If x_1, \dots, x_4 is a orthogonal basis of ω_p and if $q_i = x_i^2$, then, since Λ is self-dual, we have $q_i \in \mathbb{Z}_p^\times$ for all i . Further we have $\chi(q_1 q_2 q_3 q_4) = -1$ (comp. [KR2], 7.14).

We choose $z_1 \in \omega_p$ such that $z_1^2 \in \mathbb{Z}_p^\times$ and $\langle z_1, \lambda_1 \rangle \in \mathbb{Z}_p^\times$ (such z_1 exists, since λ_1 is not divisible by p). Let $\sigma_1 = z_1^2$ and let $z_2 = \lambda_1 - \frac{\langle z_1, \lambda_1 \rangle}{\sigma_1} z_1$. Then $0 \neq z_2 \perp z_1$ and $z_2^2 =: \sigma_2 \in \mathbb{Z}_p^\times$. Let $b_1 = \frac{\langle z_1, \lambda_1 \rangle}{\sigma_1}$. Then $b_1^2 \sigma_1 + \sigma_2 \equiv 0 \pmod{p}$, hence $\chi(-\sigma_1 \sigma_2) = 1$. We extend z_1, z_2 to an orthogonal basis z_1, \dots, z_4 of ω_p . Let $\sigma_i = z_i^2$. Then we conclude that $\chi(-\sigma_3 \sigma_4) = -1$. We write $\lambda_2 = c_1 z_1 + c_2 z_2 + c_3 z_3 + c_4 z_4$. We already know that $\lambda_1 = b_1 z_1 + z_2$. Since $\lambda_1 \perp \lambda_2$ we have $b_1 c_1 \sigma_1 + c_2 \sigma_2 = 0$.

Claim 2 $c_2 \in \mathbb{Z}_p^\times$.

Suppose $c_2 \equiv 0 \pmod{p}$. Since b_1 and σ_1 are units it follows that $c_1 \equiv 0 \pmod{p}$. Hence $c_3^2 \sigma_3 + c_4^2 \sigma_4 \equiv 0 \pmod{p}$. If c_3 or c_4 is divisible by p , it follows that both are divisible by p . But then λ_2 is also divisible by p , a contradiction. Hence c_3 and c_4 are units. Hence $\chi(\sigma_3) = \chi(c_3^2 \sigma_3) = \chi(-c_4^2 \sigma_4) = \chi(-\sigma_4)$, and hence $\chi(-\sigma_3 \sigma_4) = 1$. This contradiction confirms the claim that c_2 is a unit.

By claim 2 we may assume that $c_2 = 1$. From this we get $c_1 = -\frac{\sigma_2}{b_1 \sigma_1}$. Next we observe that $\nu_p((c_1 z_1 + z_2)^2) = \nu_p(\frac{\sigma_2^2}{b_1^2 \sigma_1^2} \sigma_1 + \sigma_2) = \nu_p(b_1^2 \sigma_1 + \sigma_2) = \nu_p(\lambda_1^2) = a_1$. Since $a_1 \leq a_2$ it follows that $c_3^2 \sigma_3 + c_4^2 \sigma_4 \equiv 0 \pmod{p^{a_1}}$. Since $\chi(-\sigma_3 \sigma_4) = -1$ and a_1 is odd it follows that $c_3, c_4 \equiv 0 \pmod{p^{\frac{a_1+1}{2}}}$. Hence $c_3^2 \sigma_3 + c_4^2 \sigma_4 \equiv 0 \pmod{p^{a_1+1}}$.

We introduce the following notation. For $x = \eta p^h \in \mathbb{Z}_p$ with $\eta \in \mathbb{Z}_p^\times$ we write $\chi(x) = \chi(\eta)$.

It follows that $\chi(\varepsilon_2) = \chi(\lambda_2^2) = \chi(c_1^2 \sigma_1 + \sigma_2)$. Hence

$$\begin{aligned} \chi(-\varepsilon_1 \varepsilon_2) &= \chi(-(b_1^2 \sigma_1 + \sigma_2)(c_1^2 \sigma_1 + \sigma_2)) = \chi(-(b_1^2 \sigma_1 + \sigma_2)(\frac{\sigma_2^2}{b_1^2 \sigma_1^2} \sigma_1 + \sigma_2)) \\ &= \chi(-(b_1^2 \sigma_1 + \sigma_2)(\sigma_2^2 \sigma_1 + \sigma_2 b_1^2 \sigma_1^2)) = \chi(-\sigma_1 \sigma_2 (b_1^2 \sigma_1 + \sigma_2)(\sigma_2 + b_1^2 \sigma_1)) = \chi(-\sigma_1 \sigma_2) = 1. \end{aligned}$$

This contradicts the assumption of the lemma. \square

Proof of Proposition 4.7. Since $y_1 \perp y_2$, the intersection $\text{Core}(y_1) \cap \text{Core}(y_2)$ is not empty. If y_1 was even, then $\text{Core}(y_1)$ would consist of a single point and y_2 would be odd, hence $a_2 > a_1$. Now using the Description of the special fibres of $D(y_1)$ and $D(y_2)$ given by Corollary 3.11, it is easy to see that then the case $r_1 = r_2$ cannot occur. Hence a_1 is odd.

To prove the claim on the multiplicity of $\mathbb{P}_{[\Lambda]}$ in $D(y_1) \cap D(y_2)$ we use induction on r , starting with the case $r = 0$.

We write \langle, \rangle for the bilinear form corresponding to Q on V' , i.e. $\langle x, y \rangle = \frac{1}{2}(Q(x+y) - Q(x) - Q(y))$. We choose an \mathbb{F} -valued point $x \in \mathbb{P}_{[\Lambda]}$ which is superspecial and such that if x is the intersection point $\mathbb{P}_{[\Lambda]}$ and $\mathbb{P}_{[\Lambda']}$, then $\mathbb{P}_{[\Lambda']} \not\subset Z(y_i), i = 1, 2$. Next we choose $y_0 \in V'$ such that $\nu_p(y_0^2) = 0$ and $x \in Z(y_0)(\mathbb{F})$ and $\langle y_0, y_1 \rangle \neq 0$. (Such y_0 exists since otherwise $\mathbb{P}_{[\Lambda']} \subset Z(y_1)$.)

We may suppose that y_2 is linearly independent of y_0 and y_1 . (Otherwise one chooses $y_3 \in V'$ such that $y_3 \perp y_0, y_1, y_2$ and $x \in Z(y_3)(\mathbb{F})$ and replaces y_0 by $y_0 + py_3$.) Let $z_1 = y_0$ and $z_2 = y_1 - \frac{\langle y_0, y_1 \rangle}{\langle y_0, y_0 \rangle} y_0$, let $b_1 = \frac{\langle y_0, y_1 \rangle}{\langle y_0, y_0 \rangle}$. We extend z_1, z_2 to an orthogonal basis z_1, \dots, z_4 of the quadratic $\mathbb{Z}_{(p)}$ -module N_x of special endomorphisms $y \in V'$ such that $x \in Z(y)(\mathbb{F})$. Let $\sigma_i = z_i^2$. Then σ_1 and σ_2 are units and (using the notation introduced in the proof of Lemma 4.8) we have $\chi(-\sigma_3\sigma_4) = -1$ (see also section 3).

We write $y_1 = b_1 z_1 + z_2$ and $y_2 = c_1 z_1 + c_2 z_2 + c_3 z_3 + c_4 z_4$. Then c_1 and c_2 are units since otherwise both would be divisible by p (since a_2 is divisible by p) and then we would have $\mathbb{P}_{[\Lambda']} \subset Z(y_2)$. Hence we may assume that $c_2 = 1$. Hence $b_1 c_1 \sigma_1 + \sigma_2 = 0$. Arguing as in Lemma 4.8 we see that $\nu_p(c_1^2 \sigma_1 + \sigma_2) = a_1$ and that $c_3^2 \sigma_3 + c_4^2 \sigma_4$ is divisible by p^{a_1} . Using $\chi(-\sigma_3\sigma_4) = -1$ we see that c_3 and c_4 are divisible by $p^{\frac{a_1-1}{2}}$. Then $c_3 z_3 + c_4 z_4 = p^{\frac{a_1-1}{2}} y_4$, where $y_4 \in N_x$ is an odd special endomorphism. If $a_1 < a_2$, then it follows immediately that $\nu_p(y_4^2) = 1$. Again reasoning as in Lemma 4.8 we see that $\chi(-(b_1^2 \sigma_1 + \sigma_2)(c_1^2 \sigma_1 + \sigma_2)) = 1$. Thus also in case $a_1 = a_2$ (in which case by assumption $\chi(-\varepsilon_1 \varepsilon_2) = -1$ and therefore $\chi(\varepsilon_2) \neq \chi(c_1^2 \sigma_1 + \sigma_2)$) it follows that $\nu_p(y_4^2) = 1$. It follows that the fundamental matrix of y_0, y_1, y_2 can be diagonalized to the matrix

$$\begin{pmatrix} \eta_0 & & \\ & \eta_1 & \\ & & \eta_2 p^{a_1} \end{pmatrix},$$

where $\eta_i \in \mathbb{Z}_{(p)}^\times$. Thus by [KR2], Proposition 6.2, it follows that $(Z(y_0), Z(y_1), Z(y_2)) = \frac{a_1+1}{2}$.

Next we choose $y \in V'_p$ such that $x \in Z(y)(\mathbb{F})$ and $y \perp y_0, y_1$ and $\nu_p(y^2) = 1$ (Lemma 3.4). Then the fundamental matrix of y, y_0, y_1 can be diagonalized to the matrix

$$\begin{pmatrix} \eta_0 & & \\ & \eta_1 & \\ & & y^2 \end{pmatrix}.$$

Thus the intersection multiplicity $(Z(y_0), Z(y_1), Z(y))$ equals 1 (Lemma 3.5). Since $\mathbb{P}_{[\Lambda]} \subset Z(y)$ it follows that $(Z(y_1) \cap Z(y_0), \mathbb{P}_{[\Lambda]}) = (D(y_1) \cap Z(y_0), \mathbb{P}_{[\Lambda]}) = 1$ (intersection multiplicity in $D(y_1)$). Hence it follows (writing $(Z(y_0), Z(y_1), Z(y_2)) = ((Z(y_1) \cap Z(y_0), (Z(y_1) \cap Z(y_2))) = ((D(y_1) \cap Z(y_0), (D(y_1) \cap D(y_2)))$ as intersection multiplicity in $D(y_1)$) that the multiplicity of $\mathbb{P}_{[\Lambda]}$ in $D(y_1) \cap D(y_2)$ as divisor in $D(y_1)$ is $\leq \frac{a_1+1}{2}$ and that it is $< \frac{a_1+1}{2}$ if and only if there is a horizontal component of $D(y_1) \cap D(y_2)$ passing through x . The same reasoning shows that for a_2 odd the multiplicity of $\mathbb{P}_{[\Lambda]}$ in $D(y_1) \cap D(y_2)$ as divisor in $D(y_2)$ is $\leq \frac{a_1+1}{2}$ and that it is $< \frac{a_1+1}{2}$ if and only if there is a horizontal component of $D(y_1) \cap D(y_2)$ passing through x . Thus in case $a_1 = a_2$ this ends the induction start by Lemma 4.8, which guarantees that there is no such horizontal component, and we also see that it suffices to show the claim in case $a_1 \neq a_2$ for the multiplicity of $\mathbb{P}_{[\Lambda]}$ in $D(y_1) \cap D(y_2)$ as divisor in $D(y_1)$.

Next it is easy to see that there is a special endomorphism y_3 of the form $c_1 z_1 + z_2 + \rho$, where ρ is in the $\mathbb{Z}_{(p)}$ -span of z_3 and z_4 in V' , such that $y_3^2 = \varepsilon_3 p^{a_1}$ (where $\varepsilon_3 \in \mathbb{Z}_{(p)}^\times$) and $\chi(-\varepsilon_1 \varepsilon_3) = -1$. We fix such a y_3 .

Thus $y_2 - y_3 = p^{\frac{a_1-1}{2}} y_4$, where y_4 is some element in the $\mathbb{Z}_{(p)}$ -span of z_3 and z_4 in V' .

As just shown, we already know that the multiplicity of $\mathbb{P}_{[\Lambda]}$ in $D(y_1) \cap D(y_3)$ (or $D(y_1) \cap Z(y_3)$) as divisor in $D(y_1)$ is $\frac{a_1+1}{2}$. Further, $D(y_1) \cap Z(y_4)$ (as divisor in $D(y_1)$) contains $\mathbb{P}_{[\Lambda]}$ with multiplicity at least 1. Now Proposition 3.13 shows that $D(y_1) \cap Z(p^{\frac{a_1-1}{2}} y_4)$ (as divisor in $D(y_1)$) contains $\mathbb{P}_{[\Lambda]}$ with multiplicity at least $1 + \frac{a_1-1}{2} = \frac{a_1+1}{2}$. Hence we see that $D(y_1) \cap Z(y_2) = D(y_1) \cap Z(y_3 + p^{\frac{a_1-1}{2}} y_4)$ also contains $\mathbb{P}_{[\Lambda]}$ with multiplicity at least $\frac{a_1+1}{2}$. Since locally around x we $D(y_2) = Z(y_2)$ and since we already know that this multiplicity is at

most $\frac{a_1+1}{2}$, this ends the start of the induction.

Now we come to the induction step from $r - 1$ to r .

If y_1 and y_2 would be replaced by y_1/p^r and y_2/p^r , then we would be in the situation of the induction start. Now we choose an \mathbb{F} -valued point x of $\mathbb{P}_{[\Lambda]}$ for y_1/p^r and y_2/p^r as we did in the induction start, and also, as before, x is the intersection point of $\mathbb{P}_{[\Lambda]}$ and $\mathbb{P}_{[\Lambda']}$. We also choose y_0 for y_1/p^r and y_2/p^r as before. Then the fundamental matrix of y_0, y_1, y_2 can be diagonalized to the matrix

$$\begin{pmatrix} \eta_0 & & \\ & \eta_1 p^{2r} & \\ & & \eta_2 p^{a_1} \end{pmatrix},$$

where $\eta_i \in \mathbb{Z}_{(p)}^\times$. Further it follows that the fundamental matrix of $y_0, y_1/p, y_2$ can be diagonalized to the matrix $\text{diag}(\eta_0, \eta_1 p^{2r-2}, \eta_2 p^{a_1})$ and that the fundamental matrix of $y_0, y_1/p, y_2/p$ can be diagonalized to the matrix $\text{diag}(\eta_0, \eta_1 p^{2r-2}, \eta_2 p^{a_1-2})$. Analogously one sees that the fundamental matrix of $y_0, y_1, y_2/p$ can be diagonalized to a matrix of the form $\text{diag}(\eta_0, \tilde{\eta}_1 p^{2r-2}, \tilde{\eta}_2 p^{a_1})$, where $\tilde{\eta}_i \in \mathbb{Z}_{(p)}^\times$. Using $D(y_1) = Z(y_1) - Z(y_1/p)$ and $D(y_2) = Z(y_2) - Z(y_2/p)$ and [KR2], Proposition 6.2, one checks that

$$(Z(y_0), D(y_1), D(y_2)) = \left(\frac{a_1+1}{2} - r + 1\right)p^{r-1} + \left(\frac{a_1+1}{2} - r\right)p^r. \quad (4.1)$$

Using the induction hypothesis (applied to y_1, y_2 and the projective line $\mathbb{P}_{[\Lambda']}$, note that $\mathbb{P}_{[\Lambda']}$ has multiplicity p^{r-1} in $D(y_1)_p$ and in $D(y_2)_p$) we see that the multiplicity of $\mathbb{P}_{[\Lambda']}$ in $D(y_1) \cap D(y_2)$ as divisor in $D(y_1)$ (and also as divisor in $D(y_2)$ if a_2 is odd) is $(\frac{a_1+1}{2} - (r-1))p^{r-1}$.

As before, we choose $y \in V'_p$ such that $y \perp y_0, y_1$ and such that $\nu_p(y^2) = 1$ and $x \in Z(y)(\mathbb{F})$. Then the fundamental matrix of y, y_0, y_1 can be diagonalized to the matrix

$$\begin{pmatrix} \eta_0 & & \\ & y^2 & \\ & & p^{2r} \eta_1 \end{pmatrix}.$$

Thus the intersection multiplicity $(Z(y_0), D(y_1), Z(y))$ equals 2 (use Lemma 3.5).

Since $\mathbb{P}_{[\Lambda]}, \mathbb{P}_{[\Lambda]}' \subset Z(y) \cap D(y_1)$ it follows that $(Z(y_0) \cap D(y_1), \mathbb{P}_{[\Lambda]}) = (Z(y_0) \cap D(y_1), \mathbb{P}_{[\Lambda]'}) = 1$ (intersection multiplicities in $D(y_1)$). Using (4.1) and the above expression for the multiplicity of $\mathbb{P}_{[\Lambda]'}$ in $D(y_1) \cap D(y_2)$, we see that the multiplicity of $\mathbb{P}_{[\Lambda]}$ in $D(y_1) \cap D(y_2)$ as divisor in $D(y_1)$ is $\leq (\frac{a_1+1}{2} - r)p^r$, with equality if and only if there is no horizontal component of $D(y_1) \cap D(y_2)$ passing through x . Thus, again by Lemma 4.8, we are done in case $a_1 = a_2$. The same reasoning shows in case that a_2 is odd the corresponding statement in $D(y_2)$. Thus for the rest we may restrict ourselves to the case that we consider $D(y_1) \cap D(y_2)$ as divisor in $D(y_1)$.

The remaining reasoning (for the case $a_1 \neq a_2$) is analogous to the reasoning in the induction start: One chooses $y_3 \in V'$ such that $y_2 - y_3$ is a linear combination of z_3 and z_4 (in particular $y_3 \perp y_1$) and such that $x \in Z(y_3)(\mathbb{F})$ and $\nu_p(y_3^2) = a_1$ and writing $y_3^2 = \varepsilon_3 p^{a_1}$ we have $\chi(-\varepsilon_1 \varepsilon_3) = -1$. Then $y_2 - y_3 = p^{\frac{a_1-1}{2}} y_4$ for some y_4 such that $x \in Z(y_4)(\mathbb{F})$. Then as just shown, $D(y_1) \cap D(y_3)$ as divisor in $D(y_1)$ contains $\mathbb{P}_{[\Lambda]}$ with multiplicity $(\frac{a_1+1}{2} - r)p^r$. Thus (using Proposition 4.6) we see that $D(y_1) \cap Z(y_3)$ as divisor in $D(y_1)$ contains $\mathbb{P}_{[\Lambda]}$ with multiplicity $1 + \dots + p^{r-1} + (\frac{a_1+1}{2} - r)p^r$. Now Proposition 4.6 and claim 1 in its proof show that $Z(p^r y_4) \cap D(y_1)$ as divisor in $D(y_1)$ contains $\mathbb{P}_{[\Lambda]}$ with multiplicity at least $1 + \dots + p^r$. Now using Proposition 3.13 we see that $Z(p^{\frac{a_1-1}{2}} y_4) \cap D(y_1)$ as divisor in $D(y_1)$ contains $\mathbb{P}_{[\Lambda]}$ with multiplicity at least $1 + \dots + p^r + (\frac{a_1-1}{2} - r)p^r = 1 + \dots + p^{r-1} + (\frac{a_1+1}{2} - r)p^r$.

Thus $D(y_1) \cap Z(y_2) = D(y_1) \cap Z(y_3 + p^{\frac{a_1-1}{2}} y_4)$ also contains $\mathbb{P}_{[\Lambda]}$ with multiplicity at least $1 + \dots + p^{r-1} + (\frac{a_1+1}{2} - r)p^r$. Thus we see (again using Proposition 4.6) that the multiplicity of $\mathbb{P}_{[\Lambda]}$ in $D(y_1) \cap D(y_2)$ (as divisor in $D(y_1)$) is at least $(\frac{a_1+1}{2} - r)p^r$. Since we already know that this multiplicity is at most $(\frac{a_1+1}{2} - r)p^r$, this is the precise multiplicity. \square

Propositions 4.6 and 4.7 give us the necessary informations about the vertical part (i.e. the part having support in the special fibre) of the divisors $D(j_i) \cap D(j_l)$ in say $D(j_i)$. Next we investigate the horizontal components (the components which do not have support in the special fibre). We consider two special endomorphisms $y_1, y_2 \in V'$ such that $y_1 \perp y_2$, and we suppose that y_1 is odd. By the results of [KR2] and [T] we may assume that $\nu_p(y_i) \geq 2$. (In case that the three special endomorphisms j_i are pairwise perpendicular to each other and at least one of the exponents $\nu_p(j_i^2)$ is ≤ 1 , the intersection multiplicity $(Z(j_1), Z(j_2), Z(j_3))$ is known by [KR2], resp. [T].) We start with the following lemma, which is given a geometric interpretation in the subsequent proposition.

Lemma 4.9 *Let $y_1, y_2 \in V'$ such that $y_1 \perp y_2$ and suppose that y_1 is odd and that y_2 is even. Write $a_i = \nu_p(y_i^2)$. Suppose that $a_1 > a_2 \geq 2$. Let x be the unique \mathbb{F} -valued point in $\text{Core}(y_2)$. Let $R = \mathcal{O}_{\mathcal{M}^{\text{HB}}, x}$ and let f_1 resp. f_2 be generators of the ideals of $D(y_1)$ resp. $D(y_2)$ in R . Let $d \in R$ such that the image of d in $R/(f_1)$ is a generator of the ideal of the underlying reduced subscheme of $D(y_1)$. By Proposition 4.6, we can write $f_2 = d^\alpha h + \rho f_1$, for some $h, \rho \in R$ such that the image of h in $R/(f_1)$ is coprime to p , and where $\alpha = p^{a_2/2-1}$. Further let $t \in R$ such that the image of t in $R/(p)$ is a generator of the ideal of the divisor s in Corollary 3.11 (applied in case $j = y_2$). Let $\beta = p^{a_2/2-1}(p-1)$. Then the ideals $I_1 := (f_1, h, d)$ and $I_2 := (t^\beta, d, p)$ are equal and the length (over W) of R/I_1 is 2β .*

Proof. We choose a special endomorphism y such that $y \perp y_1, y_2$ such that $\nu_p(y) = 1$ and $x \in Z(y)(\mathbb{F})$ (Lemma 3.4). By Lemma 3.5 (and using Lemma 3.2), we may assume that the ideal of $Z(y)$ in R is (d) . Further by Lemma 3.5 we know that $p \in (f_1, d)$ and $f_1 \in (d, p)$. Thus $I_1 = (f_1, h, d, p)$ and $I_2 = (f_1, t^\beta, d, p)$. For any $r \in R$ we denote by \bar{r} its image in $R/(p)$ (which is a UFD). Further let \bar{I}_i be the image of I_i in $R/(p)$. Then, since $p \in I_1 \cap I_2$, it is enough to show that $\bar{I}_1 = \bar{I}_2$. We write

$$f_2 = d^\alpha t^\beta + \sigma p$$

for some $\sigma \in R$. Then

$$\bar{f}_2 = \bar{d}^\alpha \bar{h} + \bar{\rho} \bar{f}_1 = \bar{d}^\alpha \bar{t}^\beta$$

and

$$\bar{f}_1 = \bar{d}^\gamma,$$

where $\gamma = p^{(a_1-1)/2}$. Combining both equations we get

$$\bar{t}^\beta = \bar{h} + \bar{\rho} \bar{d}^{\gamma-\alpha}.$$

Note that $\gamma - \alpha > 0$. Thus we see that $\bar{t}^\beta \in \bar{I}_1$ and $\bar{h} \in \bar{I}_2$. This confirms that $\bar{I}_1 = \bar{I}_2$ and hence $I_1 = I_2$.

Further $\text{lg}_W(R/I_1) = \text{lg}_W(R/I_2) = \beta \cdot \text{lg}_W(R/(p, d, t))$. By Lemma 3.5,1 (together with Lemma 3.2), the equation for $Z(y_1/p^{(a_1-1)/2})$ in $R/(d)$ is given by $p = 0$, hence we see that $\text{lg}_W(R/(p, d, t)) = (Z(y_1/p^{(a_1-1)/2}), Z(y_2/p^{a_2/2}), Z(y))$. The latter equals 2, as follows again from Lemma 3.5. Thus $\text{lg}_W(R/I_1) = 2\beta$ as claimed. \square

Proposition 4.10 *In the situation of Lemma 4.9 there is a horizontal part h of the divisor $D(y_1) \cap D(y_2)$ in $D(y_1)$ which passes through the unique point in $\text{Core}(y_2)$. Its intersection multiplicity (in $D(y_1)$) with each of the two projective lines which contain the point in $\text{Core}(y_2)$ is equal to $\beta = p^{a_2/2-1}(p-1)$.*

Proof. This follows immediately from Lemma 4.9. \square

Proposition 4.11 *Let $y_1, y_2 \in V'$ such that $y_1 \perp y_2$ and suppose that y_1 is odd. Let $a_i = \nu_p(y_i^2)$ and write $y_i^2 = \varepsilon_i p^{a_i}$. Suppose that $a_1, a_2 \geq 2$ and that in case $a_1 = a_2$ we have $\chi(-\varepsilon_1 \varepsilon_2) = -1$. Then in case that $a_1 > a_2$ and that a_2 is even, the horizontal component of $D(y_1) \cap D(y_2)$ passing through the point in $\text{Core}(y_2)$ (see Proposition 4.10) is the only horizontal component of $D(y_1) \cap D(y_2)$. In all other cases there does not exist any horizontal component of $D(y_1) \cap D(y_2)$.*

The proof will be given in the next section.

5 Calculation of intersection multiplicities

In view of Proposition 4.4 (and its proof) it is enough to calculate (if at least one of j_1, j_2, j_3 is odd) $(D(j_i), D(j_l), D(j_k))$ or (if at least one of j_i, j_k is odd) $(D(j_i), D(j_k), Z(j_l))$, where $\{i, k, l\} = \{1, 2, 3\}$. Using Lemma 4.1 we see (as before) that

$$(D(j_i), D(j_l), D(j_k)) = \chi(\mathcal{O}_{D(j_i) \cap D(j_l)} \otimes_{\mathcal{O}_{D(j_i)}}^{\mathbb{L}} \mathcal{O}_{D(j_i) \cap D(j_k)}),$$

which we write as $((D(j_i) \cap D(j_l)), (D(j_i) \cap D(j_k)))$ (intersection multiplicity in $D(j_i)$). The corresponding analogous formula holds if one replaces $D(j_l)$ by $Z(j_l)$. We may and will assume that j_i is odd (see below). For any odd $j \in V'$ with $\nu_p(j^2) \geq 1$ by Corollary 3.11 the special fibre of the regular formal scheme $D(j)$ is a scheme and its underlying reduced subscheme is a union of copies of $\mathbb{P}_{\mathbb{F}}^1$. The same reasoning as in [D] now shows that the intersection number of two divisors $E = E_1 + E_2$ and F in $D(j)$ (defined as usual as the Euler-Poincaré characteristic of their structure sheaves) is bilinear i.e. satisfies $(E, F) = (E_1, F) + (E_2, F)$ provided that the support of $E \cap F$ is contained in $D(j)_p$ and is proper over \mathbb{F} .

Next we compute the intersection multiplicity of two of the projective lines in $D(j)_p$.

Lemma 5.1 *Let $j \in V'$ be odd with $\nu_p(j^2) \geq 1$ and let $\mathbb{P}_{[\Lambda]}, \mathbb{P}_{[\Lambda']} \subset D(j)_p$. Then their intersection multiplicity in $D(j)$ has the value:*

$$(\mathbb{P}_{[\Lambda]}, \mathbb{P}_{[\Lambda']}) = \begin{cases} 0, & \text{if } \mathbb{P}_{[\Lambda]} \text{ and } \mathbb{P}_{[\Lambda']} \text{ do not intersect} \\ 1, & \text{if } \mathbb{P}_{[\Lambda]} \text{ and } \mathbb{P}_{[\Lambda']} \text{ intersect in precisely one point} \\ -(p+1), & \text{if } [\Lambda] = [\Lambda'] \text{ and } \nu_p(j^2) = 1 \\ -2p, & \text{if } [\Lambda] = [\Lambda'] \text{ and } \nu_p(j^2) > 1 \text{ and } \mathbb{P}_{[\Lambda]} \subset Z(j/p) \\ -p, & \text{if } [\Lambda] = [\Lambda'] \text{ and } \nu_p(j^2) > 1 \text{ and } \mathbb{P}_{[\Lambda]} \not\subset Z(j/p). \end{cases}$$

Proof. The first case is obvious. In the second case we have to compute the length of the local ring of the intersection point of $\mathbb{P}_{[\Lambda]}$ and $\mathbb{P}_{[\Lambda']}$. We find a special endomorphism $y \in V'_p$ with $\nu_p(y^2) = 1$ and such that $Z(y)$ contains $\mathbb{P}_{[\Lambda]}$ and $\mathbb{P}_{[\Lambda']}$. But then the claim follows from [KR1], Lemma 4.7. The same lemma also shows the third case. To compute the intersection multiplicity in the fourth and in the fifth case, we use the fact that $(\mathbb{P}_{[\Lambda]}, D(j)_p) = 0$. This follows as in the proof of Proposition 4.4 from the exact sequence

$$0 \longrightarrow \mathcal{O}_{D(j)} \xrightarrow{p} \mathcal{O}_{D(j)} \longrightarrow \mathcal{O}_{D(j)}/(p) \longrightarrow 0.$$

Let p^a be the multiplicity of $\mathbb{P}_{[\Lambda]}$ in $D(j)_p$ (see Corollary 3.11). We write $j^2 = p^b j_0$, where $\nu_p(j_0^2) = 1$. Suppose first that $\mathbb{P}_{[\Lambda]}$ does not belong to $Z(j_0)$. Then using the second case and Corollary 3.11 we therefore get in the fourth case

$$0 = (\mathbb{P}_{[\Lambda]}, D(j)_p) = p^a \cdot (\mathbb{P}_{[\Lambda]}, \mathbb{P}_{[\Lambda]}) + p^2 \cdot p^{a-1} + p^{a+1}.$$

Thus $(\mathbb{P}_{[\Lambda]}, \mathbb{P}_{[\Lambda]}) = -2p$.

If $\mathbb{P}_{[\Lambda]} \subset Z(j_0)$, then we calculate (still in the fourth case)

$$0 = p^a \cdot (\mathbb{P}_{[\Lambda]}, \mathbb{P}_{[\Lambda]}) + (p^2 - p) \cdot p^{a-1} + (p+1) \cdot p^a.$$

(Note that there are $p+1$ projective lines intersecting $\mathbb{P}_{[\Lambda]}$ which also belong to $Z(j_0)$ and there are $p^2 - p$ projective lines intersecting $\mathbb{P}_{[\Lambda]}$ which do not belong to $Z(j_0)$.) Thus again $(\mathbb{P}_{[\Lambda]}, \mathbb{P}_{[\Lambda]}) = -2p$.

In the fifth case we calculate (using $a = 0$):

$$0 = (\mathbb{P}_{[\Lambda]}, \mathbb{P}_{[\Lambda]}) + p.$$

Thus $(\mathbb{P}_{[\Lambda]}, \mathbb{P}_{[\Lambda]}) = -p$. □

In the sequel we will use the following terminology.

If j is an arbitrary special endomorphism such that $j^2 \neq 0$, then *the boundary of $Z(j)$* is the set of all $\mathbb{P}_{[\Lambda]}$ which belong to $Z(j)$ but not to $Z(j/p)$ and which do not contain any points of $Core(j)$ (see section 3). It will be denoted by $B(j)$.

Also recall from section 3 that if j is an odd special endomorphism and $j = p^b j_0$, where $\nu_p(j_0^2) = 1$, then $Cent(j)$ is the set of all $\mathbb{P}_{[\Lambda]}$ which are contained in $Z(j_0)$.

If E is a divisor in $D(j)$, then we denote by E_v the vertical part of E , i.e. the part having support in the special fibre.

For any $\mathbb{P}_{[\Lambda]}$ and any $r \in \{1, 2, 3\}$ we denote by $\text{mult}_r(\mathbb{P}_{[\Lambda]})$ the multiplicity of $\mathbb{P}_{[\Lambda]}$ in $D(j_r)_p$ given by Corollary 3.11.

Further we denote again by χ the quadratic residue character of \mathbb{Z}_p^\times resp. $\mathbb{Z}_{(p)}^\times$.

Lemma 5.2 *Suppose that $y_1, y_2 \in V'$ such that $y_1 \perp y_2$ and that $a_1 := \nu_p(y_1^2) \geq 1$ is odd and that $a_2 := \nu_p(y_2^2) \geq 2$. Write $y_i^2 = \varepsilon_i p^{a_i}$. Suppose that in case that $a_1 = a_2$ we have $\chi(-\varepsilon_1 \varepsilon_2) = -1$. Let $\mathbb{P}_{[\Lambda]} \subset D(y_1) \cap D(y_2)$ and suppose that $\mathbb{P}_{[\Lambda]} \notin B(y_2)$. In case that a_2 is even and that $a_2 < a_1$ or $a_1 = 1$ suppose further that $\mathbb{P}_{[\Lambda]}$ does not contain the point in $Core(y_2)$. Then $(\mathbb{P}_{[\Lambda]}, (D(y_2) \cap D(y_1))_v) = 0$ (intersection multiplicity in $D(y_1)$).*

Proof. Suppose first that $a_1 \geq 3$. We distinguish (a priori) 4 times 3 cases. The four cases are given by the distinguishing whether $\mathbb{P}_{[\Lambda]} \in Cent(y_1)$ or not and whether $\mathbb{P}_{[\Lambda]} \in Cent(y_2)$ or not. Let p^{m_i} be the multiplicity of $\mathbb{P}_{[\Lambda]}$ in $D(y_i)_p$ given by Corollary 3.11. For each of the four cases one distinguishes whether $m_1 < m_2$ or $m_1 = m_2$ or $m_1 > m_2$.

For example, if $\mathbb{P}_{[\Lambda]} \in Cent(y_1)$ and $m_1 < m_2$, then (using Propositions 4.6, 4.7 and Lemma 5.1)

$$(\mathbb{P}_{[\Lambda]}, (D(y_2) \cap D(y_1))_v) = (-2p)p^{m_1} + (p+1)p^{m_1} + (p^2 - p)p^{m_1-1} = 0.$$

Let us consider the case $\mathbb{P}_{[\Lambda]} \notin Cent(y_1) \cup Cent(y_2)$ and $m_1 = m_2$. Then there are p^2 projective lines intersecting $\mathbb{P}_{[\Lambda]}$ such that each is contained in $D(y_1)_p$ and $D(y_2)_p$ with multiplicity p^{m_1-1} . There is one projective line intersecting $\mathbb{P}_{[\Lambda]}$ which is contained in $D(y_1)_p$ and $D(y_2)_p$ with multiplicity p^{m_1+1} . Let $a = \min\{a_1, a_2\}$. Now using Lemma 5.1 and Proposition 4.7 we compute

$$\begin{aligned} & (\mathbb{P}_{[\Lambda]}, (D(y_2) \cap D(y_1))_v) = \\ & \left(\frac{a+1}{2} - m_1\right)(-2p)p^{m_1} + \left(\frac{a+1}{2} - m_1 - 1\right)p^{m_1+1} + p^2 \left(\frac{a+1}{2} - m_1 + 1\right)p^{m_1-1} = 0. \end{aligned}$$

The other cases are proved analogously, as well as the case $a_1 = 1$ which is even simpler (use lemma 3.5). □

In the sequel we will say that two projective lines have distance d , if the corresponding vertices in the building have distance d .

Proof of Proposition 4.11. Denote by $(D(y_1) \cap D(y_2))_{vh}$ the divisor in $D(y_1)$ which is given by the part of $(D(y_1) \cap D(y_2))$ having support in the special fibre and, in case that $a_1 > a_2$ and that a_2 is even, includes additionally the horizontal part of $D(y_1) \cap D(y_2)$ passing through the point in $Core(y_2)$ (see Proposition 4.10).

Suppose that $x \in (D(y_1) \cap D(y_2))(\mathbb{F})$ such that in case that $a_1 > a_2$ and that a_2 is even $\{x\}$ is not equal to $Core(y_2)$. Then we must show that there is no horizontal component of $D(y_1) \cap D(y_2)$ passing through x . We choose $y \in V'_p$ such that $\nu_p(y^2) = 1$ and $y \perp y_1, y_2$ and $x \in Z(y)(\mathbb{F})$. By the results of [T] we know the value of $(D(y_1), D(y_2), Z(y))$. We also know by loc. cit. the structure of $D(y_1) \cap Z(y)$ (Lemma 3.5). Using this we calculate (below) $((D(y_1) \cap D(y_2))_{vh}, (D(y_1) \cap Z(y)))$ (intersection multiplicity in $D(y_1)$) and observe that it equals $(D(y_1), D(y_2), Z(y))$. If there was a horizontal component of $D(y_1) \cap D(y_2)$ passing through x , then it would cause a strictly positive contribution to the intersection multiplicity which could not be compensated by any other contributions of horizontal components. Hence $(D(y_1) \cap D(y_2))_{vh} = (D(y_1) \cap D(y_2))$.

Now we come to the calculation of $((D(y_1) \cap D(y_2))_{vh}, (D(y_1) \cap Z(y)))$.

We distinguish cases according as a_2 is even or odd and if $a_1 < a_2$ or $a_1 = a_2$ or $a_1 > a_2$. First we handle the case that a_2 even and $a_2 < a_1$.

For each $\mathbb{P}_{[\Lambda]} \subset D(y_1) \cap Z(y)$ we compute the contribution to $((D(y_1) \cap D(y_2))_v, (D(y_1) \cap Z(y))_v)$ coming from the part of $(D(y_1) \cap Z(y))_v$ which has support in $\mathbb{P}_{[\Lambda]}$. Then we add all these contributions. (More precisely, this means the following. If $\mathbb{P}_{[\Lambda]}$ is contained in the divisor $D(y_1) \cap Z(y)$ in $D(y_1)$ (then automatically with multiplicity $m = 1$, see Lemma 3.5), then we compute $((D(y_1) \cap D(y_2))_v, \mathbb{P}_{[\Lambda]})$ as intersection multiplicity in $D(y_1)$.) By Lemma 5.2 the contribution of $\mathbb{P}_{[\Lambda]}$ is 0 if $\mathbb{P}_{[\Lambda]} \subset D(y_2)$ and $\mathbb{P}_{[\Lambda]} \notin B(y_2)$ and $Core(y_2) \notin \mathbb{P}_{[\Lambda]}$. If $\mathbb{P}_{[\Lambda]} \subset D(y_1) \cap Z(y)$ and $\mathbb{P}_{[\Lambda]} \in B(y_2)$, then the contribution to $((D(y_1) \cap D(y_2))_v, (D(y_1) \cap Z(y)))$ coming from the part of $(D(y_1) \cap D(y_2))_v$ which has support in $\mathbb{P}_{[\Lambda]}$ is $-2p + p = -p$. (where $-2p$ comes from the self-intersection and p comes from the intersection with the unique projective line $\mathbb{P}_{[\Lambda']}$ in $D(y_1) \cap D(y_2)$ which intersects $\mathbb{P}_{[\Lambda]}$. By Proposition 4.6 it has multiplicity p in $D(y_2) \cap D(y_1)$.) Now any such $\mathbb{P}_{[\Lambda]}$ is intersected in $D(y_1)$ by p additional projective lines ($\neq \mathbb{P}_{[\Lambda']}$) which belong to $Z(y)$ but not to $D(y_2)$. Each of these p projective lines contributes with the value 1 to the intersection multiplicity. Therefore together with $\mathbb{P}_{[\Lambda]}$ they contribute with the value 0 to the intersection multiplicity. Next we come to the contribution of the two projective lines passing through the point in $Core(y_2)$. If $a_2 \geq 4$ each contributes with the value

$$(-2p) \cdot p^{a_2/2-1} + p^2 \cdot p^{a_2/2-2} + p^{a_2/2-1} = -p^{a_2/2} + p^{a_2/2-1}.$$

In case $a_2 = 2$ the contribution is $-2p + 1$. In this case we have to add the contributions of the p projective lines which do not belong to $D(y_2)$ but to $Z(y)$ and intersect $\mathbb{P}_{[\Lambda]}$. Thus we have in both cases

$$((D(y_1) \cap D(y_2))_v, (D(y_1) \cap Z(y))_v) = -2p^{a_2/2} + 2p^{a_2/2-1}.$$

By Lemma 3.5 we have $(D(y_1) \cap Z(y))_v = (D(y_1) \cap Z(y))$. Using Proposition 4.10 we finally get

$$((D(y_1) \cap D(y_2))_{vh}, (D(y_1) \cap Z(y))) = 0.$$

By [T], Theorem 5.1 (and its proof) we see that this is already the value of $((D(y_1) \cap D(y_2)), (D(y_1) \cap Z(y)))$. Thus $((D(y_1) \cap D(y_2))_{vh} = ((D(y_1) \cap D(y_2)))$.

Next we treat the case that a_2 is odd. Then by Lemma 4.8 and our assumption we only need to treat the case $a_2 > a_1$ and $\chi(-\varepsilon_1 \varepsilon_2) = 1$. Then again $(D(y_1) \cap Z(y))_v = (D(y_1) \cap Z(y))$

(Lemma 3.5) and we only have to calculate $((D(y_1) \cap D(y_2))_v, (D(y_1) \cap Z(y))_v)$. We write $y^2 = \varepsilon p$.

Claim $((D(y_1) \cap D(y_2))_v, (D(y_1) \cap Z(y))_v)$ equals 0 if $\chi(-\varepsilon\varepsilon_1) = 1$, and it equals $-2(p-1)p^{\frac{a_1-1}{2}}$ if $\chi(-\varepsilon\varepsilon_1) = -1$.

The first claim follows from Lemma 5.2.

In the second case we compute for each projective line in $(D(y_1) \cap Z(y))_v$ the contribution to the intersection multiplicity coming from the part of $(D(y_1) \cap Z(y))_v$ having support in this projective line. By Lemma 5.2 we only have to consider such projective lines which are in $B(y_2)$ or which intersect a projective line in $B(y_2)$ and are not in $D(y_2)$. If such a projective line lies in $B(y_1) \cap B(y_2)$ (and lies in $Z(y)$), then its contribution is $(-p)(\frac{a_1+1}{2}) + p(\frac{a_1+1}{2} - 1) = -p$. There are precisely $2(p-1)p^{\frac{a_1-1}{2}-1}$ such projective lines. (There are precisely two projective lines in the apartment of projective lines belonging to $Cent(y_1) \cap Cent(y_2)$ for which the distance to $B(y_1)$ (defined as the corresponding distance in the building) is $\frac{a_1-1}{2}$. For each of these two, there are $(p-1)p^{\frac{a_1-1}{2}-1}$ projective lines with the described properties which have distance $\frac{a_1-1}{2}$ to the projective line in the apartment.)

For all other projective lines in $B(y_2)$, their respective contributions add to zero with the contributions of the p projective lines in $Z(y)$ which intersect it but are not contained in $D(y_2)$ (as explained in the case that a_2 is even and $a_2 < a_1$). This yields the claim.

Again, in both cases of the claim, this is also equal to the intersection multiplicity $((D(y_1) \cap D(y_2))_v, (D(y_1) \cap Z(y))_v)$ given by [T], Theorem 5.1.

Finally, the case that $a_1 < a_2$ and a_2 is even is treated in the same way as the second case of the above claim (the intersection multiplicity is also again $-2(p-1)p^{\frac{a_1-1}{2}}$). □

Now we are ready to start the intersection calculus. We fix the notation $a_r = \nu_p(j_r^2)$ and write $j_r^2 = \varepsilon_r p^{a_r}$. By Proposition 4.2 we may assume that j_1, j_2, j_3 are pairwise perpendicular to each other. From now on we make this assumption (unless otherwise mentioned). Further we assume that $2 \leq a_1 \leq a_2 \leq a_3$. To calculate $(D(j_i), D(j_l), D(j_k)) = ((D(j_l) \cap D(j_i)), (D(j_k) \cap D(j_i)))$ (intersection multiplicity in $D(j_i)$) we want to use Propositions 4.6 - 4.11 and Lemmas 3.2, 5.1 and 5.2. Note that we always find $i \in \{1, 2, 3\}$ such that j_i is odd. This follows from [Ku1], 1.16, see also (5.1) below.

Proposition 5.3 *We have the following identities of intersection multiplicities.*

- (i) *If a_1 is even and a_2 is odd, then $(D(j_1), D(j_2), D(j_3)) = 2p^{(a_1+a_2-3)/2}(p-1)$.*
- (ii) *If a_1 and a_3 are odd and a_2 is even, then $(D(j_1), D(j_2), D(j_3)) = 0$.*
- (iii) *If a_1 and a_2 are odd and a_3 is even, then*

$$(D(j_1), D(j_2), D(j_3)) = -2p^{(a_1+a_2-4)/2}(\frac{a_1+1}{2}p - \frac{a_1-1}{2})(p-1).$$
- (iv) *If a_1 is odd and a_2 and a_3 are even and $a_2 < a_3$, then $(D(j_1), D(j_2), D(j_3)) = 0$.*
- (v) *If a_1 is odd and a_2 and a_3 are even and $a_2 = a_3$, then*

$$(D(j_1), D(j_2), D(j_3)) = -2p^{(a_1+a_2-3)/2}(\frac{a_1+1}{2}p - \frac{a_1-1}{2}).$$

The case that a_1 and a_2 are even and the case that a_1 and a_2 and a_3 are odd will be treated later.

Proof. Let us consider the first case. In view of Propositions 4.10 and 4.11 we have to show that $(D(j_1) \cap D(j_2))_v, D(j_3) \cap D(j_2))_v = 0$ (intersection multiplicity in $D(j_2)$). But this follows from Lemma 5.2. The same reasoning shows the second and the fourth case.

Let us consider the third case. For each $\mathbb{P}_{[\Lambda]}$ belonging to $D(j_3)$ and to $D(j_1)$ we compute the contribution to the intersection multiplicity which comes from the part of $D(j_3) \cap D(j_1)$ which has support in $\mathbb{P}_{[\Lambda]}$ and we add the contributions for the several projective lines. (More precisely, if $D(j_3) \cap D(j_1)$ contains $\mathbb{P}_{[\Lambda]}$ with multiplicity m (Propositions 4.6, 4.7), we compute the intersection $(m \cdot \mathbb{P}_{[\Lambda]}, D(j_2) \cap D(j_1))$ in $D(j_1)$ and add the contributions of the several $\mathbb{P}_{[\Lambda]}$.) By Lemma 5.2 we only need to consider such $\mathbb{P}_{[\Lambda]}$ which belong to $B(j_2)$ or which do not belong to $D(j_2)$. First we assume that $a_1 < a_2$.

First we consider the contributions of such $\mathbb{P}_{[\Lambda]}$, which belong to $B(j_2) \cap B(j_3)$. This contribution is $-2p + p = -p$ if $\mathbb{P}_{[\Lambda]}$ does not belong to $B(j_1)$ and it is $(-p) \cdot (\frac{a_1+1}{2})^2 + (\frac{a_1+1}{2} - 1) \cdot \frac{a_1+1}{2} \cdot p$ if it belongs to $B(j_1)$. Next it is easy to see that there are precisely $2(p-1)p^{\frac{a_2-a_1}{2}-1} \cdot p \cdot (p^2)^{\frac{a_1-1}{2}-1}$ projective lines belonging to $B(j_2) \cap B(j_3)$ but not to $B(j_1)$: We start with one of the two projective lines in the apartment of projective lines belonging to $Cent(j_1)$ and to $Cent(j_2)$ such that $D(j_2)_p$ and $D(j_3)_p$ contain this projective line with the same multiplicity (given by Corollary 3.11). There are $p-1$ neighboring projective lines belonging to $Cent(j_1)$ but not to $Cent(j_2)$. For each such choice there are p new neighboring projective lines belonging to $Cent(j_1)$. We continue walking in $Cent(j_1)$ and enlarging our distance to the mentioned apartment. After $\frac{a_2-a_1}{2} - 1$ such steps, the distances to $B(j_1)$ and to $B(j_2)$ are equal. Now there are $p \cdot (p^2)^{\frac{a_1-1}{2}-1}$ possible ways to go on until $B(j_2)$ but not to $B(j_1)$ (in each step enlarging the distance to the apartment by one). Similarly there are precisely $2(p-1)p^{\frac{a_2-a_1}{2}-1} \cdot (p^2 - p) \cdot (p^2)^{\frac{a_1-1}{2}-1}$ projective lines belonging to $B(j_1) \cap B(j_2) \cap B(j_3)$.

Next it is easy to see that the contribution of all $\mathbb{P}_{[\Lambda]}$ which do not belong to $B(j_3)$ sum up to zero. (More precisely, one checks the following fact for any projective line $\mathbb{P}_{[\Lambda']}$ which lies in the apartment of projective lines belonging to $Cent(j_1) \cap Cent(j_2)$ and for which the multiplicities of $\mathbb{P}_{[\Lambda']}$ in $D(j_2)_p$ and in $D(j_3)_p$ are different: The contributions coming from all projective lines $\mathbb{P}_{[\Lambda]}$ in $B(j_2)$ (or intersecting it) with the property that $\mathbb{P}_{[\Lambda']}$ is the projective line in the apartment such that its distance to $\mathbb{P}_{[\Lambda]}$ is minimal, sum up to 0.)

Thus

$$\begin{aligned} (D(j_1), D(j_2), D(j_3)) &= 2(p-1)p^{\frac{a_2-a_1}{2}-1} \cdot p \cdot (p^2)^{\frac{a_1-1}{2}-1} \cdot (-p) \\ &\quad + 2(p-1)p^{\frac{a_2-a_1}{2}-1} \cdot (p^2 - p) \cdot (p^2)^{\frac{a_1-1}{2}-1} \\ &\quad \times ((-p) \cdot (\frac{a_1+1}{2})^2 + (\frac{a_1+1}{2} - 1) \cdot \frac{a_1+1}{2} \cdot p) \\ &= -2p^{(a_1+a_2-4)/2} (\frac{a_1+1}{2}p - \frac{a_1-1}{2})(p-1). \end{aligned}$$

In case $a_1 = a_2$ the calculation is almost the same, we have only to replace the above formulas for the number of projective lines which belong to $B(j_2) \cap B(j_3)$ but not to $B(j_1)$ resp. which belong to $B(j_1) \cap B(j_2) \cap B(j_3)$ by the expressions $2(p-1) \cdot (p^2)^{\frac{a_1-1}{2}-1}$ resp. $2(p-1)^2 \cdot (p^2)^{\frac{a_1-1}{2}-1}$. The resulting expression for $(D(j_1), D(j_2), D(j_3))$ does not change.

The fifth case is done analogously. \square

Next we come to the case that a_1 and a_2 are even and (hence) a_3 is odd. We intersect in $D(j_3)$. Now there is a horizontal component in $D(j_1) \cap D(j_3)$ passing through the unique point in $Core(j_1) = Core(j_2)$ and also one of $D(j_2) \cap D(j_3)$ passing through the point in $Core(j_1) = Core(j_2)$. If we proceed as before, then we are faced with the problem that we have to compute the intersection multiplicity of these two horizontal components in $D(j_3)$. To avoid this problem, we proceed as follows. First we choose $\gamma \in \mathbb{Z}_{(p)}$ such that $\nu_p(j_2^2 + \gamma^2 j_1^2) = a_2 + 2$ (which is possible, as follows from the fact that $\chi(\varepsilon_1 \varepsilon_2) = \chi(-1)$, which in turn follows

from the form of the matrix S' given in section 3). Now let $j'_2 = j_2 + \gamma j_1$. By Proposition 4.2, we get $(Z(j_1), Z(j_2), Z(j_3)) = (Z(j_1), Z(j'_2), Z(j_3))$. Using also Proposition 4.4 we also see $(D(j_1), Z(j_2), D(j_3)) = (D(j_1), Z(j'_2), D(j_3))$. It follows from the construction of j'_2 that $Core(j_1) \neq Core(j'_2)$ and that $Core(j'_2)$ consists of a supersingular point lying on one of the projective lines passing through the point in $Core(j_1)$.

Proposition 5.4 *Suppose that a_1 and a_2 are even. Then*

$$(D(j_1), Z(j_2), D(j_3)) = p^{\frac{a_1+a_2}{2}} + p^{\frac{a_1+a_2}{2}-1} - 2p^{a_1-1}.$$

Proof. We compute the value of $(D(j_1), Z(j'_2), D(j_3))$. We write $(D(j_1), Z(j'_2), D(j_3)) = ((D(j_1) \cap D(j_3)), (Z(j'_2) \cap D(j_3)))$ as intersection multiplicity in $D(j_3)$. Assume first that $a_2 > 2$ and $a_2 < a_3 - 1$. We write $(D(j_1) \cap D(j_3)) = v_1 + h_1$, where v_1 denotes the vertical part (i.e. the part with support in the special fibre) of $D(j_1) \cap D(j_3)$ as divisor in $D(j_3)$ and h_1 denotes the horizontal part. Similarly we write $Z(j'_2) \cap D(j_3) = v_2 + h_2$. Then $(D(j_1), Z(j'_2), D(j_3)) = (v_1, h_2) + (h_1, v_2) + (v_1, v_2)$. (Note that by the above construction of j'_2 , the horizontal part h_1 and h_2 do not intersect.) Using Propositions 4.10 and 4.6 we compute

$$(v_1, h_2) = (p^{\frac{a_1}{2}-1} + p^{\frac{a_1}{2}-2})(1 + (p-1) + \dots + (p-1)p^{\frac{a_2}{2}}) = (p^{\frac{a_1}{2}-1} + p^{\frac{a_1}{2}-2})p^{\frac{a_2}{2}+1}.$$

(Here the first summand 1 in the second parenthesis claims that the horizontal component of $Z(j'_2/p^{a_2/2+1}) \cap D(j_3)$ intersects the two projective lines which contain $Cent(j'_2)$ each with multiplicity 1. This follows by combining Lemma 3.4 and point 3 of Lemma 3.5.)

Similarly, we compute

$$(h_1, v_2) = (p-1)p^{\frac{a_1}{2}-1}(1 + (1+p) + \dots + (p^{\frac{a_2}{2}-1} + p^{\frac{a_2}{2}})).$$

Next one checks that the contribution to (v_1, v_2) coming from the part of $(Z(j'_2) \cap D(j_3))_v$ which has support in the two projective lines passing through the point of $Core(j_1)$ is exactly $-(h_1, v_2)$.

To calculate the rest of (v_1, v_2) we write $v_2 = (Z(j'_2) \cap D(j_3))_v = \sum_r (D(j'_2/p^r) \cap D(j_3))_v$ and see (using Lemma 5.2) that for each r we only have to consider the contributions coming from the projective lines in $(D(j'_2/p^r) \cap D(j_3))_v$ which are in $B(j_1)$ or which intersect a projective line in $B(j_1)$. Further these contributions add to zero if $r \neq \frac{a_2-a_1}{2}, \frac{a_2-a_1}{2} - 1$. In each of these two cases these contributions add to $(-p)(p^2)^{\frac{a_1}{2}-1}$. Adding everything we get the desired result. The cases that $a_1 = 2$ and/or $a_2 = a_3 - 1$ are computed analogously. \square

Next we treat the case that $a_1 a_2 a_3$ is odd. Here we are faced with the problem that in case $a_i = a_k$ and $\chi(-\varepsilon_i \varepsilon_k) = 1$ we cannot apply Propositions 4.7 and 4.11 to get information about the structure of $D(j_i) \cap D(j_k)$. But if in this case $a_l \neq a_i, a_k$ we can compute $((D(j_i) \cap D(j_l)), (D(j_k) \cap D(j_l)))$ as intersection multiplicity in $D(j_l)$. Thus the only case where a problem arises, is the case that $a_1 = a_2 = a_3 =: a$ and at least two of the three expressions $\chi_{rs} := \chi(-\varepsilon_r \varepsilon_s)$ are equal to 1. Suppose $\chi_{ij} = \chi_{ik} = 1$. But then the matrix $T = \text{diag}(\varepsilon_1 p^a, \varepsilon_2 p^a, \varepsilon_3 p^a)$ is $\text{GL}_3(\mathbb{Z}_{(p)})$ -equivalent to a matrix of the form $\text{diag}(\varepsilon_i p^a, \eta_k \varepsilon_k p^a, \eta_l \varepsilon_l p^a)$, where $\eta_l, \eta_k \in \mathbb{Z}_{(p)}^\times$ such that $\chi(\eta_l) = \chi(\eta_k) = -1$. Then $\chi(-\varepsilon_i \eta_k \varepsilon_k) = \chi(-\varepsilon_i \eta_l \varepsilon_l) = -1$. Since we are finally interested in the value of $(Z(j_1), Z(j_2), Z(j_3))$ and since for this we may replace T by $\text{diag}(\varepsilon_i p^a, \eta_k \varepsilon_k p^a, \eta_l \varepsilon_l p^a)$ (Proposition 4.2), we are reduced to the case that $\chi_{ij} = \chi_{ik} = -1$.

Proposition 5.5 *Suppose that $a_1 a_2 a_3$ is odd. Then we have the following identities of intersection multiplicities.*

(i) *If $a_1 = a_2 = a_3 =: a$ and $\chi_{12} = \chi_{13} = -1$ and $\chi_{23} = 1$, then*

$$(D(j_1), D(j_2), D(j_3)) = -\frac{a+1}{2}p^a + 3\frac{a+1}{2}p^{a-1} - (a-1)p^{a-2}.$$

(ii) *If $a_1 = a_2 = a_3 =: a$ and $\chi_{12} = \chi_{13} = \chi_{23} = -1$, then*

$$(D(j_1), D(j_2), D(j_3)) = -\frac{a+1}{2}p^a + 3\frac{a+1}{2}p^{a-1} - 2(a-1)p^{a-2}.$$

(iii) *If $a_1 \leq a_2 < a_3$ and $\chi_{12} = -1$, then*

$$(D(j_1), D(j_2), D(j_3)) = -2p^{\frac{a_1+a_2-4}{2}}(p-1)\left(\frac{a_1+1}{2}p - \frac{a_1-1}{2}\right).$$

(iv) *If $a_1 = a_2 < a_3$ and $\chi_{12} = 1$, then*

$$(D(j_1), D(j_2), D(j_3)) = 2p^{a_1-1}.$$

(v) *If $a_1 < a_2 = a_3$ and $\chi_{12} = \chi_{13} = 1$, then*

$$(D(j_1), D(j_2), D(j_3)) = -p^{\frac{a_1+a_2-4}{2}}(p+1)\left(\frac{a_1+1}{2}p - \frac{a_1-1}{2}\right).$$

(vi) *If $a_1 < a_2 = a_3$ and $\chi_{12} = -1$ and $\chi_{13} = 1$, then*

$$(D(j_1), D(j_2), D(j_3)) = -p^{\frac{a_1+a_2-4}{2}}(p-1)\left(\frac{a_1+1}{2}p - \frac{a_1-1}{2}\right).$$

(vii) *If $a_1 < a_2 < a_3$ and $\chi_{12} = 1$, then*

$$(D(j_1), D(j_2), D(j_3)) = 0.$$

Note that the calculation of the intersection multiplicity $(Z(j_1), Z(j_2), Z(j_3))$ in case $a_1 < a_2 = a_3$ and $\chi_{13} = -1$ can be reduced to the case case $a_1 < a_2 = a_3$ and $\chi_{13} = 1$ by the same reasoning as the one before the proposition.

Proof. In case (i) we write $(D(j_1), D(j_2), D(j_3)) = ((D(j_1) \cap D(j_2)), (D(j_1) \cap D(j_3)))$, and we compute for each projective line in $D(j_1) \cap D(j_3)$ the contribution to the intersection multiplicity $((D(j_1) \cap D(j_2)), (D(j_1) \cap D(j_3)))$ which comes from the part of $D(j_1) \cap D(j_3)$ which has support in that projective line. By Lemma 5.2 we only have to consider such projective lines, which are in $B(j_2)$, or which are not in $D(j_2)$ but intersect an element in $B(j_2)$.

First we consider projective lines in $B(j_1) \cap B(j_2) \cap B(j_3)$. Then each contributes with the value $-p \cdot \frac{a+1}{2}$ (Proposition 4.7) and there are precisely $(p^2 - 3p + 2)(p^2)^{\frac{a-1}{2}-1}$ such. Next, there are $(p-3)(p^2)^{\frac{a-1}{2}-1}$ projective lines which are in $B(j_2) \cap B(j_3)$ but not in $B(j_1)$. Each contributes the value $-p$. Further, there are $(p-1)(p^2)^{\frac{a-1}{2}-1}$ projective lines which are in $B(j_1) \cap B(j_2)$ and $D(j_3)$ but not in $B(j_3)$. Each contributes with value $-p$.

Next we count the contributions coming from the projective lines in $B(j_2)$ which are not in $B(j_1) \cup B(j_3)$. For each such we also add the contributions coming from the projective lines in $D(j_1) \cap D(j_3)$ which intersect it but are not in $D(j_2)$. Now, if $\mathbb{P}_{[\Lambda]} \in B(j_2)$ but not

in $B(j_1) \cup B(j_3)$ and if $\text{mult}_1(\mathbb{P}_{[\Lambda]}) \neq \text{mult}_3(\mathbb{P}_{[\Lambda]})$, then the contribution coming from $\mathbb{P}_{[\Lambda]}$ (including the contributions coming from the projective lines in $D(j_1) \cap D(j_3)$ which intersect $\mathbb{P}_{[\Lambda]}$ but are not in $D(j_2)$) is easily seen to be 0. Next for each m between 1 and $\frac{a-1}{2}$ we count the contributions coming from such $\mathbb{P}_{[\Lambda]} \in B(j_2)$ but not in $B(j_1) \cup B(j_3)$ for which $\text{mult}_1(\mathbb{P}_{[\Lambda]}) = \text{mult}_3(\mathbb{P}_{[\Lambda]}) = p^m$. In case $m = \frac{a-1}{2}$ there are precisely two such (lying in the apartment of projective lines lying in $\text{Cent}(j_1) \cap \text{Cent}(j_3)$). The contribution for each of the two is then $(-2p)p^{\frac{a-1}{2}} + p \cdot p^{\frac{a-1}{2}} + p^{\frac{a-1}{2}} + (2p-2)p^{\frac{a-1}{2}-1} + 2(p^2-2p+1)p^{\frac{a-1}{2}-1}$. (The first two summands come from the projective line itself, the rest comes from the projective lines in $D(j_1) \cap D(j_3)$ which intersect it but do not belong to $D(j_2)$.) Thus the contribution coming from the case $m = \frac{a-1}{2}$ is $2((-2p)p^{\frac{a-1}{2}} + p \cdot p^{\frac{a-1}{2}} + p^{\frac{a-1}{2}} + (2p-2)p^{\frac{a-1}{2}-1} + 2(p^2-2p+1)p^{\frac{a-1}{2}-1}) = 2p^{\frac{a-1}{2}}(p-1)$. Similarly, for $m < \frac{a-1}{2}$, the contribution becomes $2(p^2-2p+1)(p^2)^{\frac{a-1}{2}-m-1}((-2p)p^m(\frac{a-1}{2}-m) + p \cdot p^m(\frac{a-1}{2}-m) + p^2p^{m-1}(\frac{a-1}{2}-m+1)) = 2(p^2-2p+1)p^{a-m-2}$. Now summing everything up, we get the desired result.

Case (ii) is done analogously.

In case (iv) we write $(D(j_1), D(j_2), D(j_3)) = ((D(j_1) \cap D(j_3)), (D(j_2) \cap D(j_3)))$ (intersection multiplicity in $D(j_3)$). As usual, for each projective line in $(D(j_1) \cap D(j_3))$ we compute the contribution to intersection multiplicity coming from the part of $(D(j_1) \cap D(j_3))$ which has support in that projective line. We only have to consider projective lines in $B(j_2)$ or intersecting a projective line in $B(j_2)$. There are no projective lines in $B(j_2) \cap B(j_3)$ which belong to $D(j_1)$. We count the contribution coming from the projective lines in $B(j_1) \cap B(j_2)$. Since $\chi_{12} = 1$ there are precisely $(p^2-2p-1)(p^2)^{\frac{a_1-1}{2}-1}$ such and each contributes with the value $-p$ to the intersection multiplicity. Similarly one checks that the projective lines in $B(j_2)$ which are not in $B(j_1)$ together with the projective lines which intersect such altogether contribute with the value $(p^2-1)p^{a_1-2}$. Summing everything up gives the claimed result.

In case (v) we write $(D(j_1), D(j_2), D(j_3)) = ((D(j_1) \cap D(j_2)), (D(j_1) \cap D(j_3)))$ (intersection multiplicity in $D(j_1)$). Again we have to compute the contribution coming from the part of $(D(j_1) \cap D(j_3))$ which has support in a projective line in $B(j_2)$ or a projective line intersecting one in $B(j_2)$. Since $\chi_{13} = 1$ it follows that, if $\mathbb{P}_{[\Lambda]} \in B(j_2)$ and $\mathbb{P}_{[\Lambda]} \subset D(j_1) \cap D(j_3)$, then $\mathbb{P}_{[\Lambda]} \in B(j_3)$. Now it is easy to see that if in this situation $\mathbb{P}_{[\Lambda]} \notin B(j_1)$, then the contribution coming from $\mathbb{P}_{[\Lambda]}$ is $-p$ and that there are precisely $(p+1) \cdot p^{\frac{a_2-a_1}{2}-1} \cdot p \cdot (p^2)^{\frac{a_1-1}{2}-1}$ such projective lines. Similarly, if $\mathbb{P}_{[\Lambda]} \in B(j_1)$, then the contribution coming from $\mathbb{P}_{[\Lambda]}$ is $\frac{a_1+1}{2}(-p)$ and there are precisely $(p+1) \cdot p^{\frac{a_2-a_1}{2}-1} \cdot (p^2-p) \cdot (p^2)^{\frac{a_1-1}{2}-1}$ such projective lines. Summing up all contributions, we obtain the claim.

In case (iii) we proceed similarly. Writing $(D(j_1), D(j_2), D(j_3)) = ((D(j_1) \cap D(j_2)), (D(j_1) \cap D(j_3)))$ (intersection multiplicity in $D(j_1)$), we have to compute the contribution coming from the part of $(D(j_1) \cap D(j_3))$ which has support in a projective line in $B(j_2)$ or a projective line intersecting one in $B(j_2)$. Now we consider the apartment of projective lines corresponding to the intersection $\text{Cent}(j_1) \cap \text{Cent}(j_2)$. For any projective line in $B(j_2)$ (or intersecting it) there is a unique projective line in the apartment, such that the distance of the two projective lines (i.e. the distance of the corresponding vertices in the building) is minimal. Now we consider the two projective lines $\mathbb{P}_{[\Lambda_0]}, \mathbb{P}_{[\Lambda_1]}$ in the apartment such that $\text{mult}_3(\mathbb{P}_{[\Lambda_0]}) = \text{mult}_2(\mathbb{P}_{[\Lambda_0]})$ and $\text{mult}_3(\mathbb{P}_{[\Lambda_1]}) = \text{mult}_2(\mathbb{P}_{[\Lambda_1]})$. Then by the same reasoning as in case (v) the contribution coming from all projective lines in $B(j_2)$ whose distance to the apartment is the distance to $\mathbb{P}_{[\Lambda_0]}$ or $\mathbb{P}_{[\Lambda_1]}$ gives the claimed intersection multiplicity. One easily checks that for any other projective line $\mathbb{P}_{[\Lambda]}$ in the apartment the contributions coming from all projective lines $\mathbb{P}_{[\Lambda']}$ in $B(j_2)$ (or intersecting it) with the property that $\mathbb{P}_{[\Lambda]}$ is the projective line in the apartment such that its distance to $\mathbb{P}_{[\Lambda']}$ is minimal, sum up to 0.

Case (vi) is proved analogously to case (v).

Case (vii) follows from Lemma 5.2. \square

We now drop the assumption that T is diagonal. We want to obtain an expression for $(Z(j_1), Z(j_2), Z(j_3))$. Suppose that T is $\mathrm{GL}_3(\mathbb{Z}_{(p)})$ -equivalent to $\mathrm{diag}(\varepsilon_1 p^{a_1}, \varepsilon_2 p^{a_2}, \varepsilon_3 p^{a_3})$, where $\varepsilon_i \in \mathbb{Z}_{(p)}^\times$ for all i and $a_1 \leq a_2 \leq a_3$. (Now the restriction that $2 \leq a_1, a_2, a_3$ is dropped.) To state the result, we introduce the following invariants of T , comp. [W]. Let

$$\tilde{\xi} = \begin{cases} \chi(-\varepsilon_1 \varepsilon_2) & \text{if } a_1 \equiv a_2 \pmod{2}, \\ 0 & \text{if } a_1 \not\equiv a_2 \pmod{2}, \end{cases}$$

and let

$$\sigma = \begin{cases} 2 & \text{if } a_1 \equiv a_2 \pmod{2}, \\ 1 & \text{if } a_1 \not\equiv a_2 \pmod{2}. \end{cases}$$

Further, let

$$\eta = \begin{cases} +1 & \text{if } T \text{ is isotropic over } \mathbb{Q}_p, \\ -1 & \text{if } T \text{ is anisotropic over } \mathbb{Q}_p. \end{cases}$$

To distinguish whether T is isotropic or anisotropic over \mathbb{Q}_p , we recall the following fact (see [W], p. 189). Let $i, j \in \{1, 2, 3\}$ with $i \neq j$ and $a_i \equiv a_j \pmod{2}$, and define $k \in \{1, 2, 3\}$ by $\{i, j, k\} = \{1, 2, 3\}$. Then T is isotropic over \mathbb{Q}_p if and only if $\chi(-\varepsilon_i \varepsilon_j) = 1$ or $a_k \equiv a_j \pmod{2}$. On the other hand, since T is represented by V' we have

$$-1 = (-1)^{a_1+a_2+a_3} \chi(-1)^{a_1+a_2+a_3+a_1a_2+a_2a_3+a_1a_3} \chi(\varepsilon_1)^{a_2+a_3} \chi(\varepsilon_2)^{a_1+a_3} \chi(\varepsilon_3)^{a_1+a_2}, \quad (5.1)$$

see [KR2], section 7 or [Ku1], (1.16). Using this we can decide, whether T is isotropic or anisotropic over \mathbb{Q}_p , and we can also determine $\tilde{\xi}$ from a_1, a_2, a_3 , except in case that $a_1 a_2 a_3$ is odd (then $\tilde{\xi} = -1$ and $\tilde{\xi} = 1$ is possible).

Theorem 5.6 *Suppose that T is $\mathrm{GL}_3(\mathbb{Z}_{(p)})$ -equivalent to $\mathrm{diag}(\varepsilon_1 p^{a_1}, \varepsilon_2 p^{a_2}, \varepsilon_3 p^{a_3})$, where $\varepsilon_i \in \mathbb{Z}_{(p)}^\times$ for all i and $a_1 \leq a_2 \leq a_3$. Then there is the following explicit expression for the intersection multiplicity $(Z(j_1), Z(j_2), Z(j_3))$.*

$$\begin{aligned} (Z(j_1), Z(j_2), Z(j_3)) &= - \sum_{i=0}^{a_1} \sum_{j=0}^{\frac{a_1+a_2-\sigma}{2}-i} p^{i+j} (-1)^i (i+2j) \\ &\quad - \eta \sum_{i=0}^{a_1} \sum_{j=0}^{\frac{a_1+a_2-\sigma}{2}-i} p^{\frac{a_1+a_2-\sigma}{2}-j} (-1)^{a_3+\sigma+i} (a_3 + \sigma + i + 2j) \\ &\quad - \tilde{\xi}^2 p^{\frac{a_1+a_2-\sigma}{2}+1} \sum_{i=0}^{a_1} \sum_{j=0}^{a_3-a_2+2\sigma-4} \tilde{\xi}^j (-1)^{a_2-\sigma+i+j} (a_2 - \sigma + 2 + i + j). \end{aligned}$$

Proof. We already know by Proposition 4.2 that $(Z(j_1), Z(j_2), Z(j_3))$ only depends on the $\mathrm{GL}_3(\mathbb{Z}_{(p)})$ -equivalence class of T . Thus we may assume that $T = \mathrm{diag}(\varepsilon_1 p^{a_1}, \varepsilon_2 p^{a_2}, \varepsilon_3 p^{a_3})$. Next by comparing with [KR2], Proposition 6.2 resp. [T], Theorem 5.1 we get the claim in case $a_1 = 0$ resp. $a_1 = 1$. Now by induction on $a_1 + a_2 + a_3$ we see (using Proposition 4.4) that it is enough to show that the above formula predicts the same values of $(D(j_1), D(j_2), D(j_3))$ resp. $(D(j_1), Z(j_2), D(j_3))$ as the propositions of this section. This is in all cases checked by a straightforward calculation. \square

6 The connection to representation densities and Eisenstein series

In this section we want to express the local intersection multiplicity $(Z(j_1), Z(j_2), Z(j_3))$ in terms of certain representation densities and the global intersection multiplicity $\chi_T(Z_1, Z_2, Z_3)$ in terms of the derivative of the T -th Fourier coefficient a certain Eisenstein series.

First we recall that, for $S \in \text{Sym}_m(\mathbb{Z}_p)$ and $U \in \text{Sym}_n(\mathbb{Z}_p)$ with $\det(S) \neq 0$ and $\det(U) \neq 0$, the representation density is defined as

$$\alpha_p(S, U) = \lim_{t \rightarrow \infty} p^{-tn(2m-n-1)/2} \left| \{x \in M_{m,n}(\mathbb{Z}/p^t\mathbb{Z}); S[x] - U \in p^t \text{Sym}_n(\mathbb{Z}_p)\} \right|.$$

Given S as above, let

$$S_r = \begin{pmatrix} S & & \\ & 1_r & \\ & & -1_r \end{pmatrix}.$$

Then there is a rational function $A_{S,U}(X) \in \mathbb{Q}(X)$ of X such that

$$\alpha_p(S_r, U) = A_{S,U}(p^{-r}).$$

One defines

$$\alpha'_p(S, U) = \frac{\partial}{\partial X}(A_{S,U}(X))|_{X=1}.$$

(Comp. [KR1].) Let $S = \text{diag}(1, -1, 1, -\Delta)$. (Since T is represented by the space $V'(\mathbb{Q}_p)$, it is not represented by $V(\mathbb{Q}_p)$ resp. S , see [Ku1], Proposition 1.3.)

Theorem 6.1 *There is the following relation between intersection multiplicities and representation densities:*

$$(Z(j_1), Z(j_2), Z(j_3)) = -\frac{p^4}{(p^2+1)(p^2-1)} \alpha'_p(S, T).$$

Proof. Since both sides only depend on the $\text{GL}_3(\mathbb{Z}_{(p)})$ -equivalence class of T , we may assume that T is diagonal and equals $\text{diag}(\varepsilon_1 p^{a_1}, \varepsilon_2 p^{a_2}, \varepsilon_3 p^{a_3})$, where $\varepsilon_i \in \mathbb{Z}_{(p)}^\times$ for all i and $a_1 \leq a_2 \leq a_3$. We compute the expression on the right hand side and compare with Theorem 5.6. The right hand side can be expressed explicitly, combining a result of Katsurada and a result of Shimura. This was already done in [T] in case $a_1 = 1$, but the reasoning carries over. More precisely, by the same reasoning as in the proof of [T], Theorem 5.1, the function $A_{S,T}(X)$ is given as follows.

$$A_{S,T}(X) = (1 + p^{-2}X)(1 - p^{-2}X^2)\tilde{F}_p(T; -X),$$

where $\tilde{F}_p(T; X)$ is given by the following expression.

$$\begin{aligned} \tilde{F}_p(T; X) &= \sum_{i=0}^{a_1} \sum_{j=0}^{(a_1+a_2-\sigma)/2-i} p^{i+j} X^{i+2j} \\ &+ \eta \sum_{i=0}^{a_1} \sum_{j=0}^{(a_1+a_2-\sigma)/2-i} p^{(a_1+a_2-\sigma)/2-j} X^{a_3+\sigma+i+2j} \\ &+ \tilde{\xi}^2 p^{(a_1+a_2-\sigma+2)/2} \sum_{i=0}^{a_1} \sum_{j=0}^{a_3-a_2+2\sigma-4} \tilde{\xi}^j X^{a_2-\sigma+2+i+j}. \end{aligned}$$

Here, the invariants $\eta, \sigma, \tilde{\xi}$ are defined as in the last section. This yields an explicit expression for the right hand side of the formula in the statement of the theorem which is straightforward to calculate, and one checks that it is the same expression as the one given in Theorem 5.6. \square

Next we come to the comparison of the global intersection multiplicity $\chi_T(Z_1, Z_2, Z_3)$ and the derivative of the T -th Fourier coefficient of a certain Eisenstein series.

First we shortly recall some notations and facts introduced in [KR2]. See [Ku1] and [KR2], section 7 for details.

Let K'_p be the stabilizer in $G'(\mathbb{Q}_p)$ of a superspecial lattice L in X^{HB} (see section 2). Define $V'(\mathbb{Z}_p) = \text{End}(L, F) \cap V'_p$ and $V(\mathbb{Z}_p) = \Lambda \otimes \mathbb{Z}_p$, where Λ is the fixed self-dual lattice of section 1. Let $K' = K'_p K^p \subset G'(\mathbb{A}_f)$. Let $\varphi_f^p = \text{char}(\omega_1 \times \omega_2 \times \omega_3) \in S(V(\mathbb{A}_f^p)^3)$ and $\varphi_p = \text{char}(V(\mathbb{Z}_p)^3) \in S(V(\mathbb{Q}_p)^3)$ and $\varphi'_p = \text{char}(V'(\mathbb{Z}_p)^3) \in S(V'(\mathbb{Q}_p)^3)$.

Let $\chi = \chi_V$ be the quadratic character of $\mathbb{A}^\times / \mathbb{Q}^\times$ associated to V , i.e., $\chi(x) = (x, \det(V))_{\mathbb{A}}$, where $(\ , \)_{\mathbb{A}}$ denotes the global Hilbert symbol. To φ_p one associates $\Phi_p \in I_3(0, \chi_{V_p})$ (see [Ku1] for an explanation of the induced representation $I_3(0, \chi_{V_p})$ of $\text{Sp}_6(\mathbb{Q}_p)$, and more generally of the induced representation $I_3(s, \chi)$ of $\text{Sp}_6(\mathbb{A})$). Analogously to φ_f^p one associates Φ_f^p and to φ'_p one associates Φ'_p .

Then Φ_p is completed to an incoherent standard section $\Phi(s) = \Phi_\infty^2(s) \cdot \Phi_f^p(s) \cdot \Phi_p(s) \in I_3(s, \chi)$ (see [Ku1]), where $\Phi_\infty^2(s)$ is associated to the Gaussian in $S(V'(\mathbb{R})^3)$. Denote by $pr(K')$ the image of K' under the projection $pr : G'(\mathbb{A}_f) \rightarrow SO(V')(\mathbb{A}_f)$. Analogously one defines $pr(K)$. Let $\underline{x} = (j_1, j_2, j_3) \in V'^3$ be the triple of our fixed three special endomorphisms (section 2) so that the matrix of the quadratic form with respect to \underline{x} is our fixed T . Finally let (having chosen Haar measures on $G'(\mathbb{A}_f)$ and $Z'(\mathbb{A}_f)$)

$$O_T(\varphi'_f) = \int_{Z'(\mathbb{A}_f) \backslash G'(\mathbb{A}_f)} \varphi'_f(g^{-1} \underline{x} g) dg.$$

Analogously one defines $O_T(\varphi_f^p)$ and $O_T(\varphi'_p)$. Let $P \subset \text{Sp}_6$ be the Siegel parabolic subgroup (see introduction). We consider the following Eisenstein series on $\text{Sp}_6(\mathbb{A})$,

$$E(g, s, \Phi) = \sum_{\gamma \in P(\mathbb{Q}) \backslash \text{Sp}_6(\mathbb{Q})} \Phi(\gamma g, s).$$

Suppose that $\omega_1 \times \omega_2 \times \omega_3$ is locally centrally symmetric (i.e. invariant under the action of $\mu_2(\mathbb{A}_f^p)$). Then for $h \in \text{Sp}_6(\mathbb{R})$ the derivative in $s = 0$ of $E_T(h, s, \Phi)$ (see also introduction) can be expressed as

$$\begin{aligned} E'_T(h, 0, \Phi) &= \text{vol}(SO(V')(\mathbb{R})) \cdot pr(K') \cdot W_T^2(h) \\ &\times \frac{W'_{T,p}(e, 0, \Phi_p)}{W_{T,p}(e, 0, \Phi'_p)} \cdot \text{vol}(K')^{-1} \text{vol}(Z(\mathbb{Q}) \backslash Z(\mathbb{A}_f)) \cdot O_T(\varphi'_f). \end{aligned}$$

See [KR2] and [Ku1] for an explanation of the Whittaker functions W_T^2 and $W_{T,p}$. With these preparations we are ready to state the final result.

Theorem 6.2 *Let $h \in \text{Sp}_6(\mathbb{R})$ and suppose that $\omega_1 \times \omega_2 \times \omega_3$ is locally centrally symmetric. Then there is the following relation between Eisenstein series and intersection multiplicities.*

$$E'_T(h, 0, \Phi) = -\frac{1}{2} \log(p) \cdot \kappa \cdot \chi_T(Z_1, Z_2, Z_3) \cdot W_T^2(h),$$

where κ is the volume constant with value $\kappa = \text{vol}(SO(V')(\mathbb{R})) \text{vol}(pr(K))$.

Proof. By the above formula for $E'_T(h, 0, \Phi)$ we can write

$$E'_T(h, 0, \Phi) = c_1 \cdot W_T^2(h) \frac{O_T(\varphi'_p)}{W_{T,p}(e, 0, \Phi'_p)} W'_{T,p}(e, 0, \Phi_p) O_T(\varphi_f^p)$$

for some constant c_1 (meaning that it is independent of T and h and $\omega_1 \times \omega_2 \times \omega_3$). Further we can write $W'_{T,p}(e, 0, \Phi_p) = c_2 \cdot \alpha'_p(S, T)$ for some constant c_2 (see the proof of [KR2], Proposition 7.2). Using Theorem 6.1 we see that $W'_{T,p}(e, 0, \Phi_p) = c_3 \cdot (Z(j_1), Z(j_2), Z(j_3))$ for some constant c_3 . Combining this with Proposition 2.2, we obtain

$$W'_{T,p}(e, 0, \Phi_p) O_T(\varphi_f^p) = c_4 \cdot \chi_T(Z_1, Z_2, Z_3)$$

for some constant c_4 . Next we claim that the quotient $\frac{O_T(\varphi'_p)}{W_{T,p}(e, 0, \Phi'_p)}$ is a constant. The proof for this is the same as the proof of formula (5.3.33) in the proof of Proposition 5.3.3 in [KRY]. Thus we obtain

$$E'_T(h, 0, \Phi) = c \cdot \chi_T(Z_1, Z_2, Z_3) \cdot W_T^2(h)$$

for some constant c . It remains to prove that $c = -\frac{1}{2} \text{vol}(SO(V')(\mathbb{R})) \text{vol}(pr(K)) \log(p)$. But by [KR2], Theorem 7.3, this is true provided that T is not divisible by p . Since this claim is independent of T , it is always true. This ends the proof. \square

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