NOTES ON THE MULTIPLICITY CONJECTURE

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Abstract. New cases of the multiplicity conjecture are considered.

Introduction

Throughout this paper, we fix a field K and let R be a homogeneous K-algebra. In other words, R is a finitely generated K-algebra, generated over K by elements of degree 1, and hence is isomorphic to S/I where $S = K[x_1, \dots, x_n]$ is a polynomial ring and I a graded ideal contained in $(x_1, \dots, x_n)$. Consider a graded minimal free S-resolution of R:

$$0 \longrightarrow \bigoplus_{j=1}^{b_p} S(-d_{pj}) \longrightarrow \cdots \longrightarrow \bigoplus_{j=1}^{b_1} S(-d_{1j}) \longrightarrow S \longrightarrow 0.$$

The ring R is said to have a *pure resolution* if for all i, the shifts d_{ij} do not depend on j (but only on i). Hence if the resolution is pure, it has the following shape:

$$0 \longrightarrow S^{b_p}(-d_p) \longrightarrow \cdots \longrightarrow S^{b_1}(-d_1) \longrightarrow S \longrightarrow 0.$$

When R is Cohen-Macaulay and has a pure resolution, Huneke and Miller's formular [15] says that the multiplicity of R is given by

$$e(R) = (\prod_{i=1}^{p} d_i)/p!.$$

In general we define $M_i(I) = \max\{d_{ij} : j = 1, \dots, b_i\}$ and $m_i(I) = \min\{d_{ij} : j = 1, \dots, b_i\}$ for $i = 1, \dots p$. When there is no danger of ambiguity, we write M_i and m_i for short.

Huneke and Srinivasan had the following

Conjecture 1. *For each homogeneous Cohen-Macaulay K-algebra R*

$$(\prod_{i=1}^{p} m_i)/p! \leq e(R) \leq (\prod_{i=1}^{p} M_i)/p!.$$

Conjecture 1 has been widely studied and partial results have been obtained. In [13], the first author and Srinivasan showed that this conjecture is true in the following cases: R has a quasi-pure resolution (i.e., $m_i(I) \geq M_{i-1}(I)$); I is a perfect ideal of codimension 2; I is a codimension 3 Gorenstein ideal generated by 5 elements; I is a stable ideal; I is a squarefree strongly stable ideal. Furthermore, Guardo and

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Van Tuly [9] proved that the conjecture holds for powers of complete intersections, Srinivasan [20] proved a stronger bounds for Gorenstein algebras with quasi-pure resolutions. And recently, in [17], Migliore, Nagel and Römer proved a stronger version of Conjecture 1 when R is a codimension 2 or Gorenstein codimension 3 algebra (with no limitations on the number of generators). As a corollary, they showed that in these two cases, the multiplicity $e(R)$ reaches the upper or lower bound if and only if R has a pure resolution.

It was observed in [13], that the lower bound in Conjecture 1 fails in general if R is not Cohen-Macaulay, even one replaces Conjecture 1 the projective dimension by the codimension. In the same paper, the authors had the following stronger conjecture for the upper bound of the multiplicity of R.

Conjecture 2. *Let R be a homogeneous K-algebra of codimension s. Then*

$$e(R) \leq (\prod_{i=1}^{s} M_i)/s!.$$

If the defining ideal of R is stable, or squarefree strongly stable, or if R has a linear resolution, Conjecture 2 is shown to be true in [13]. In addition, Gold [10] proved it for codimension 2 lattice ideals. This was generalized by Römer [19] for all codimension two ideals. In the same paper Römer proved Conjecture 2 for componentwise linear ideals in characteristic 0.

In Section 1 we show that if $I \subset S$ is an ideal of codimension s, not necessarily perfect, for which one has

$$(\prod_{i=1}^{s} m_i)/s! \leq e(R) \leq (\prod_{i=1}^{s} M_i)/s!,$$

and if $f_1, \ldots, f_m$ is a regular sequence modulo I, then the corresponding inequalities are again valid for $(I, f_1, \ldots, f_m)$. One might expect that the proof of this statement is rather simple. But again careful estimates are required to establish the result.

In Section 2 we show that Conjecture 2 is valid in the limit with respect to taking powers of ideal, that is, we show that

$$\lim_{k\to\infty} \frac{e(S/I^k)}{\frac{1}{s!}\prod_{i=1}^{s} M_i(I^k)} \leq 1.$$

Unfortunately, this does not imply that Conjecture 2 holds for all sufficiently high powers of I.

In view of the results by Migliore, Nagel and Römer [17] one is lead to ask the following question: suppose that for a ring R the lower bound in Conjecture 1, or the upper bound in Conjecture 2 is reached. Does this imply that R is Cohen-Macaulay and has a pure resolution?

In Section 3 we show that not only in the cases described by Migliore, Nagel and Römer this improved multiplicity conjecture holds, but also for rings with almost pure resolutions, as well as for rings defined by componentwise linear ideals.

We also recall a recent result of Miró-Roig [18] who showed that graded ideals of maximal minors of maximal grade satisfy Conjecture 1, and we show that again in this case the upper or lower bound is reached if and only if the resolution is pure. This case includes rings whose defining ideal is a power of a graded regular sequence.

1. The multiplicity conjecture and regular sequences

Let $I \subset S$ be a graded ideal of codimension s, $R = S/I$ and $f_1, \dots f_m$ a homogeneous regular sequence of R. Suppose that

$$(\prod_{i=1}^{s} m_i(I))/s! \le e(R) \le (\prod_{i=1}^{s} M_i(I))/s!.$$

Are the corresponding inequalities again valid for for $R/(f_1, \dots, f_m)$? In fact, without assuming that S/I is Cohen-Macaulay, we have the following

Theorem 1.1. *Let $I \subset S$ be a graded ideal of codimension s, $R = S/I$ and $\mathbf{f} = f_1, \dots f_m$ a homogeneous regular sequence of R. Suppose that*

$$(\prod_{i=1}^{s} m_i(I))/s! \le e(R) \le (\prod_{i=1}^{s} M_i(I))/s!.$$

Then

$$(\prod_{i=1}^{s+m} m_i(I, \mathbf{f}))/(s+m)! \le e(R/(\mathbf{f})) \le (\prod_{i=1}^{s+m} M_i(I, \mathbf{f}))/(s+m)!.$$

Proof. By using induction on m, one needs only to show the case $m = 1$. For simplicity, we denote f_1 by f, $M_i(I)$ and $m_i(I)$ by M_i and m_i, respectively. Let $d = \deg f$.

We first show that $e(R/(f)) \le (\prod_{i=1}^{s+1} M_i(I, f))/(s+1)!$. We have $\operatorname{codim}(I, f) = s + 1$ and $e(R/(f)) = e(S/(I, f)) = e(R) \cdot d$.

Let $G_{\bullet}$ be the minimal graded free resolution of R, and $H_{\bullet}$ the minimal graded free resolution of $S/(f)$. Then $F_{\bullet} = G_{\bullet} \otimes H_{\bullet}$ is the minimal graded free resolution of $R/(f) = S/(I, f)$. Hence $F_i = G_i \otimes S \oplus G_{i-1} \otimes R(-d)$, $i = 1, \dots, p+1$, where $p = \operatorname{proj\,dim} S/I$. Therefore $M_i((I, f)) = \max\{M_i, M_{i-1} + d\}$ for $i = 1, \dots, p+1$.

Thus we need to show that

$$(1) \qquad (s+1)d \prod_{i=1}^{s} M_i \le \prod_{i=1}^{s+1} \max\{M_i, M_{i-1} + d\}, \text{ where } M_0 = 0.$$

Set $M_i = id + y_i$ for $1 \le 1 \le s+1$. Then $\max\{M_i, M_{i-1} + d\} = \max\{id + y_i, id + y_{i-1}\} = id + \max\{y_{i-1}, y_i\}$. Let $N = \{i : y_i \ge 0, \quad 1 \le i \le s+1\}$, and let $j = \max\{i : i \in N\}$. In case $N = \emptyset$, we set $j = 0$. Then $y_i < 0$ for all i with $j < i \le s+1$.

We will distinguish two cases:

Case 1. $j = s+1$. We have $\max\{y_s, y_{s+1}\} \geq y_{s+1} \geq 0$. Inequality (1) is equivalent to the inequality

$$(s+1)d \prod_{i=1}^{s} M_i \leq \prod_{i=1}^{s} \max\{M_i, M_{i-1}+d\} \cdot ((s+1)d + \max\{y_s, y_{s+1}\}),$$

which is satisfied since $M_i \leq \max\{M_i, M_{i-1}+d\}$ for $i = 1, \ldots, s$, and because $\max\{y_s, y_{s+1}\} \geq 0$.

Case 2. $j < s+1$. It is suffices to show that

$$(2) \quad \prod_{i=j+1}^{s} (id + y_i) \cdot (s+1)d \leq \prod_{i=j+1}^{s+1} (id + y_{i-1}) = ((j+1)d + y_j) \prod_{i=j+2}^{s+1} (id + y_{i-1}).$$

Set $z_i = y_i/d$. Then inequality (2) is equivalent to the inequality

$$\prod_{i=j+1}^{s} (i + z_i) \cdot (s+1) \leq ((j+1) + z_j) \prod_{i=j+1}^{s} ((i+1) + z_i).$$

Since $id + y_i = M_i > 0$, it follows $0 < i + z_i$ for $i = 1, \ldots, s+1$. Hence we need to show that

$$s+1 \leq ((j+1) + z_j) \cdot \prod_{i=j+1}^{s} (1 + 1/(i + z_i)).$$

Since $z_j = y_j/d \geq 0$ and $z_i < 0$ for all $i = j+1, \ldots, s$, and since $i + z_i = i + y_i/d = (id + y_i)/d = m_i/d > 0$, it follows that $1/(i + z_i) > 1/i$. Therefore

$$s+1 = (j+1) \prod_{i=j+1}^{s} (1 + 1/i) \leq ((j+1) + z_j) \prod_{i=j+1}^{s} (1 + 1/(i + z_i)).$$

Similarly, by taking $j = \max\{i : y_i \leq 0, \quad 0 \leq i \leq s+1\}$, where $y_0 = 0$, and distinguishing the cases $j = s+1$ and $j < s+1$, one sees that $e(R/(f)) \geq (\prod_{i=1}^{s+1} m_i(I, f))/(s+1)!$. □

2. Powers of an ideal

As the main result of this section we want to prove that Conjecture 2 is true in the limit with respect to taking powers of an ideal. To be more precise, we show

Theorem 2.1. *For any graded ideal $I \subset S$ of codimension s we have*

$$\lim_{k \to \infty} \frac{e(S/I^k)}{\frac{1}{s!} \prod_{i=1}^{s} M_i(I^k)} \leq 1.$$

Proof. For the proof of this result we will proceed in several steps.

(i) Let M be a graded S-module of projective dimension p. We set

$$\operatorname{reg}_i(M) = \max\{j : \beta_{ii+j}(M) \neq 0\} \quad \text{for} \quad i = 0, \ldots, p.$$

Then $M_i(I) = \operatorname{reg}_i(S/I) + i$ for $i = 1, \ldots, p$, and

$$\operatorname{reg}(I) = \max\{\operatorname{reg}_i(I) : i = 0, \ldots, p\}$$

is the regularity of I.

Let $L \subset S$ be a graded ideal of codimension s. We claim that $\operatorname{reg}_i(S/L) \geq \operatorname{reg}_{i-1}(S/L)$ for $i = 0, \ldots, s$. In fact, let

$$0 \longrightarrow F_p \xrightarrow{\varphi_p} F_{p-1} \xrightarrow{\varphi_{p-1}} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow S/L \longrightarrow 0$$

be a graded minimal free resolution of S/L. Suppose that $\operatorname{reg}_i(S/L) < \operatorname{reg}_{i-1}(S/L)$ for some $i \leq s$. Then $M_i(L) \leq M_{i-1}(L)$. Let $e \in F_{i-1}$ be a homogeneous basis element of degree $M_{i-1}(L)$, and let f be any homogeneous basis element of F_i. Then $\deg \varphi_i(f) = \deg f \leq M_i(L) \leq M_{i-1}(L)$. Thus if we write $\varphi_i(f)$ as a linear combination of the basis elements of F_{i-1}, the coefficient a of e will be of degree $\deg a \leq M_i(L) - M_{i-1}(L) \leq 0$. This is only possible if $a = 0$, since $\varphi_i(f) \in \mathfrak{m}F_{i-1}$, where $\mathfrak{m}$ is the graded maximal ideal of S.

Now we consider the S-dual of the resolution F. Let e^* be the dual basis element of e, and $\varphi_i^* \colon F_{i-1}^* \to F_i^*$ the map dual to φ_i. Then it follows that $\varphi_i^*(e^*) = 0$. Thus e^* is a cycle of the dual complex. On the other hand, e^* cannot be a boundary, since e^* is a basis element of F_{i-1}^* and since the image of $\varphi_{i-1}^* \colon F_{i-2}^* \to F_{i-1}^*$ is contained in $\mathfrak{m}F_{i-1}^*$. Hence we see that $\operatorname{Ext}_S^{i-1}(S/L, S) = H^{i-1}(F^*) \neq 0$. This contradicts the fact that $\operatorname{Ext}_S^j(S/L, S) = 0$ for $j < s$, since $\operatorname{grade} L = \operatorname{codim} L = s$.

(ii) Cutkosky, Herzog and Trung [6] as well as Kodiyalam [16] showed that

$$\operatorname{reg}_i(I^k) = q_i k + c_i$$

is a linear function for $k \gg 0$. In particular, $\operatorname{reg}(I^k) = qk + c$ for $k \gg 0$. It is also shown that $q = q_0$, see [6, Corollary 3.2].

By (i) we have

$$\operatorname{reg}_0(I^k) \leq \operatorname{reg}_i(I^k) \leq \operatorname{reg}(I^k)$$

for $i = 0, \ldots, s-1$ where $s = \operatorname{codim} I$ ($= \operatorname{codim} I^k$ for all k). Thus (ii) implies

$$qk + c_0 \leq q_i k + c_i \leq qk + c$$

for $i = 0, \ldots, s-1$ and all $k \gg 0$. This implies that $q_i = q$ for $i = 0, \ldots, s-1$.

(iii) From (ii) it follows that

$$\frac{1}{s!} \prod_{i=1}^{s} M_i(I^k) = \frac{q^s}{s!} k^s + \cdots \quad \text{for} \quad k \gg 0.$$

is a polynomial function of degree s whose leading coefficient is $q^s/s!$.

(iv) For $k \gg 0$, the function $s!e(S/I^k)$ is a polynomial function whose leading term is an integer which we denote by $e(I, S)$. In other words,

$$e(S/I^k) = \frac{e(I,S)}{s!} k^s + \cdots \quad \text{for} \quad k \gg 0.$$

For all $k \gg 0$, let $\{P_1, \ldots, P_r\}$ be the (stable) set of minimal prime ideals of S/I^k of height s. The associativity formula of multiplicity ([4, Corollary 4.7.8]) then shows that

$$e(S/I^k) = \sum_{i=1}^{r} \ell(S_{P_i}/I_{P_i}^k) e(S/P_i).$$

Here $\ell(M)$ denotes the length of a module M.

Each S_{P_i} is a regular local ring of dimension s, and I_{P_i} is $P_iS_{P_i}$ primary. Therefore

$$\ell(S_{P_i}/I_{P_i}^k) = \frac{e(I_{P_i}, S_{P_i})}{s!}k^s + \cdots$$

is a polynomial function of degree s for $k \gg 0$, see [4, Proposition 4.6.2], where the numerator $e(I_{P_i}, S_{P_i})$ of the leading coefficient of this polynomial is the multiplicity of S_{P_i} with respect to I_{P_i}. This proves our assertion and also shows that

$$e(I, S) = \sum_P e(I_P, S_P)e(S/P), \qquad (3)$$

where the sum is taken over all asymptotic minimal prime ideals of I.

(v) The theorem will follow once we have shown $e(I, S) \leq q^s$. We first notice that $e(I, S) \leq e(J, S)$ for any ideal $J \subset I$ with $\operatorname{codim} J = \operatorname{codim} I = s$, and that $e(I, S) = e(J, S)$ if J is a reduction ideal of I, that is, if $JI^k = I^{k+1}$ for some k. Indeed, this follows from formula (3) and the fact that the corresponding statements are true for ideals in a local ring which are primary to the maximal ideal, see [4, Lemma 4.6.5].

Now we use the fact, shown by Kodiyalam [16, Theorem 5], that I admits a reduction ideal J with $\operatorname{reg}_0(J) = q$. Hence replacing I by J we may assume that I is generated in degree $\leq q$.

After a base field extension we may assume that K is infinite. Then, since $\operatorname{codim} I = s$, generically chosen q-forms $f_1, \ldots, f_s \in I_q$ will form a regular sequence. Let L be the ideal generated by these forms. Then $e(I, S) \leq e(L, S) = q^s$, as desired. It just remains to establish the last equation. This can be seen as follows: Since the L is generated by a regular sequence each of the factor modules L^{k-1}/L^k is a free S/L-module, whose rank is $\binom{s+k-1}{s-1}$. Hence, since $e(S/L) = q^s$, we see that $e(L^{k-1}/L^k) = q^s\binom{s+k-1}{s-1}$. It follows that

$$e(S/L^k) = q^s \sum_{j=1}^k \binom{s+k-1}{s-1} = \frac{q^s}{s!}k^s + \cdots$$

This implies that $e(L, S) = q^s$. □

Unfortunately Theorem 2.1 does not imply that Conjecture 2 is true for all high enough powers of an ideal, as it is easy to find ideals for which

$$\lim_{k\to\infty} \frac{e(S/I^k)}{\frac{1}{s!}\prod_{i=1}^s M_i(I^k)} = 1.$$

3. The improved multiplicity conjecture

Motivated by the results of Migliore, Nagel and Römer [17], we say that the improved multiplicity conjecture holds, if all standard graded K-algebras R satisfy the multiplicity conjectures, and whenever the bounds are reached, then the defining ideal has a pure resolution and R is Cohen-Macaulay.

In this section we show that for some interesting classes of examples the improved multiplicity conjecture holds.

Generalizing the result [17, Corollary 1.3] we first show

Theorem 3.1. *Let $I \subset S$ be a graded ideal of codimension 2. Then S/I satisfies the improved multiplicity conjecture.*

Proof. Römer proved in [19, Theorem 2.4] that $R = S/I$ satisfies Conjecture 2. Thus it remains to be shown that if $e(R) = (1/2)M_1M_2$, then R is Cohen-Macaulay and has a pure resolution. Once it is shown that R is Cohen-Macaulay, then by [17, Theorem 1.3] we also have that R has a pure resolution. (This last fact also follows from Theorem 3.5 below.)

Let $S = K[x_1, \dots, x_n]$. One may assume that $|K| = \infty$, and that (after a generic change of coordinates) $x_1, \dots, x_n$ is an almost regular sequence on R, i.e, multiplication with x_i on $R_{i-1} = R/(x_1, \dots, x_{i-1})R$ has a finite length kernel for all i. In his proof, Römer showed that

$$e(R) \leq e(R_{n-2}) \leq (1/2)M_1M_2, .$$

He also showed in [19, Lemma 2.3] that $e(R_{n-2}) = e(R) + \text{length}(0 :_{R_{n-3}} x_{n-2})$. Thus if we assume that the upper bound is reached, then x_{n-2} is regular on R_{n-3}. By [12, Proposition 3] this implies that $x_1, \dots, x_{n-2}$ is a regular sequence, and hence R is Cohen-Macaulay. □

We say that I is *componentwise linear* if all ideals spanned by the graded components of I have a linear resolution. Our next class of rings satisfying the improved multiplicity conjecture is the following

Theorem 3.2. *Let $I \subset S$ be a componentwise linear ideal of codimension s. Then for S/I the improved multiplicity conjecture holds.*

Before we prove this theorem we first note

Lemma 3.3. *Let $I \subset S$ be a componentwise linear ideal with a pure resolution. Then I has a linear resolution.*

Proof. We may assume that the base field is infinite. Let $\text{Gin}(I)$ denote the generic initial ideal of I with respect to the reverse lexicographical order. In [1] it is shown that I is componentwise linear if and only if I and $\text{Gin}(I)$ have the same graded Betti-numbers, provided $\operatorname{char} K = 0$, and that $\text{Gin}(I)$ is a strongly stable ideal. Here we only need that I and $\text{Gin}(I)$ have the same graded Betti-numbers and that $\text{Gin}(I)$ is stable. The proof given in [1] shows that this is the case in all characteristics. Indeed, let $I_{\langle j \rangle}$ be the ideal generated by all elements of degree j in I. By our assumption on I, the ideal $I_{\langle j \rangle}$ has a linear resolution. Applying the Bayer-Stillman theorem [3] it follows that $\text{Gin}(I_{\langle j \rangle})$ has a linear resolution, so that $\text{Gin}(I_{\langle j \rangle}) = \text{Gin}(I)_{\langle j \rangle}$. This proves that $\text{Gin}(I)$ is again componentwise linear. Since $\text{Gin}(I_{\langle j \rangle}) = \text{Gin}(I)_{\langle j \rangle}$ is p-Borel and has a linear resolution, Proposition 10 of Eisenbud, Reeves and Totaro [7] implies that $\text{Gin}(I)_{\langle j \rangle}$ is a stable monomial ideal for all j, and hence $\text{Gin}(I)$ is a stable monomial ideal. With the same arguments as in the proof of [11, Theorem 1.1] it then follows that I and $\mathrm{Gin}(I)$ have the same graded Betti-numbers.

Replacing I by $\mathrm{Gin}(I)$ we may as well assume that I is a stable ideal. Since I has a pure resolution, it is in particular generated in one degree and hence has a linear resolution, as it is a stable ideal, see [8]. $\square$

Proof Theorem 3.2. As shown in the preceding lemma, we may assume that I is a stable monomial ideal. Its Betti-numbers do not depend on the characteristic of the base field. Thus we may assume that the base field has characteristic 0. Since I is componentwise linear it follows from [11, Theorem 1.1.] (see also the proof of Lemma 3.3) that $\mathrm{Gin}(I)$ is again componentwise linear and that I and $\mathrm{Gin}(I)$ have the same graded Betti-numbers. Replacing I by $\mathrm{Gin}(I)$ and observing that $\mathrm{Gin}(I)$ is strongly stable since the characteristic of the base field is 0, we may now assume that I is strongly stable.

The proof of the multiplicity conjecture for stable ideals given in [13, Theorem 3.2] is in fact only valid for strongly stable ideals, as it is used there that if $I \subset K[x_1, \ldots, x_n]$ is stable, then $I : x_n$ is stable as well. But this is only true for strongly stable ideals. However, as seen above, the stable ideal may be replaced by a strongly stable ideal. Thus for the rest of our proof we may follows the arguments given in the proof of [13, Theorem 3.2].

We first treat the case that S/I is Cohen-Macaulay. In that case we may assume that $x_n^a \in I$ where $n = \dim_K S_1$, so that in particular S/I is Artinian. In the proof of [13, Theorem 3.2] it is shown that $e(S/I) \leq e(S/(x_1, \ldots, x_n)^a) = (1/n!) \prod_{i=1}^n M_i$. Thus if the upper bound is reached, then $I = (x_1, \ldots, x_n)^a$, and so S/I has a pure resolution.

Now suppose S/I reaches the lower bound. We prove the assertion by induction on the length of S/I. The case $\mathrm{length}(S/I) = 1$ is trivial. So now we assume that $\mathrm{length}(S/I) > 1$. Let $J \subset \bar{S} = K[x_1, \ldots, x_{n-1}]$ be the unique monomial ideal such that $(J, x_n) = (I, x_n)$. The ideals J and $(I : x_n)$ are again strongly stable ideals, and since the multiplicity conjecture holds for strongly stable ideals we have

$$\begin{aligned} e(S/I) &= e(S/(I, x_n)) + e(S/(I : x_n)) \\ &\geq (1/(n-1)!) \prod_{i=1}^{n-1} m_i(J) + (1/n!) \prod_{i=1}^{n} m_i(I : x_n). \end{aligned}$$

It is shown in the proof of [13, Theorem 3.2] that the right hand side of this inequality is greater that or equal to $(1/n!) \prod_{i=1}^n m_i$.

Our assumption implies that $e(S/(I, x_n)) = e(\bar{S}/J) = (1/(n-1)!) \prod_{i=1}^{n-1} m_i(J)$ and $e(S/(I : x_n)) = (1/n!) \prod_{i=1}^n m_i(I : x_n)$. Hence the induction hypothesis yields that both $\bar{S}/J$ and $S/(I : x_n)$ have a pure resolution. Lemma 3.3 then implies that both $\bar{S}/J$ and $S/(I : x_n)$ have a linear resolution. Since S/I is Artinian, it follows that $\bar{S}/J$ and $S/(I : x_n)$ are Artinian, and so there exist numbers a and b such that $J = (x_1, \ldots, x_{n-1})^a$, and $(I : x_n) = (x_1, \ldots, x_n)^b$. Therefore,

$$I = (x_1, \ldots, x_{n-1})^a + (x_1, \ldots, x_n)^b x_n.$$

If $n = 1$, then I has a linear resolution. Thus we now may assume that $n > 1$.

Since $J \subset I \subset (I : x_n)$ it follows that $a \geq b$, and since I is strongly stable it follows that $a \leq b+1$. Suppose that $a = b$. Then

$$\begin{aligned} e(S/I) &= e(\bar{S}/J) + e(S/(I : x_n) \\ &= \binom{n+a-2}{n-1} + \binom{n+a-1}{n}, \end{aligned}$$

so that

$$n!e(S/I) = (2n+a-1)\prod_{i=0}^{n-2}(a+i).$$

On the other hand, $m_i = a+i-1$ for $i = 1, \ldots, n-1$ and $m_n = a+n$. Therefore, $n!e(S/I) \neq \prod_{i=1}^{n} m_i$ for $n > 1$, a contradiction. Hence we conclude that $a = b+1$, and so $I = (x_1, \ldots, x_n)^{b+1}$. In particular, I has a linear resolution.

Finally, let I be an arbitrary strongly stable ideal such that the multiplicity of S/I reaches the upper bound. We want to show that S/I is Cohen-Macaulay.

We may assume that $(I : x_n) \neq I$, because otherwise $I \subset \bar{S} = K[x_1, \ldots, x_{n-1}]$, and we are done by induction on n. Assume S/I is not Cohen-Macaulay. In this case it shown in the proof [13, Theorem 3.7] that

$$e(S/I) = e(S/(I : x_n)) \leq 1/t!\prod_{i=1}^{t} M_i(I : x_n) \leq 1/s!\prod_{i=1}^{s} M_i,$$

where $t = \operatorname{codim} S/(I : x_n) \leq \operatorname{codim} S/I = s$. It is also shown in [13, Lemma 3.6] that $M_i(I : x_n) \leq M_i$ for all i. Since on the other hand, $M_i(I)/i \geq 1$ for all i, our assumption implies that

(i) $t = s$,
(ii) $M_i(I : x_n) = M_i(I)$ for all $i = 1, \ldots, s$, and
(iii) $e(S/(I : x_n)) = 1/s!\prod_{i=1}^{s} M_i(I : x_n)$.

Using Noetherian induction and Lemma 3.3, condition (i) and (iii) yield that $S/(I : x_n)$ is Cohen-Macaulay of codimension s with linear resolution. Since $(I : x_n)$ is strongly stable we conclude therefore that $(I : x_n) = (x_1, \ldots, x_s)^a$ for some integer a.

Let $G(I) = \{u_1, \ldots, u_r, v_1x_n, \ldots, v_tx_n\}$, where the monomials u_i are not divisible by x_n. Then $(I : x_n) = (u_1, \ldots, u_r, v_1, \ldots, v_t)$. Since $M_1(I : x_n) = M_1(I) = a$, it follows that $I = (u_1, \ldots, u_r)$. This contradicts our assumption that $(I : x_n) \neq I$. $\square$

In order to prove the improved multiplicity conjecture in the quasi-pure case we must assume that the considered algebras are Cohen-Macaulay. It would be nice this hypothesis could be dropped from the assumptions.

Theorem 3.4. *Suppose that $I \subset S$ is a Cohen-Macaulay ideal of codimension s with a quasi-pure resolution. Then S/I satisfies the improved multiplicity conjecture.*

Proof. The proof given in [13] yields the assertion. We sketch the arguments. Let

$$0 \longrightarrow \bigoplus_{j=1}^{b_s} S(-d_{sj}) \longrightarrow \cdots \longrightarrow \bigoplus_{j=1}^{b_1} S(-d_{1j}) \longrightarrow S \longrightarrow 0.$$

be the minimal graded free resolution of S/I. There are square matrices A and B derived from the shifts d_{ij} with

$$\det A = \sum_{\substack{1\le j_i\le b_i\\ 1\le i\le s}} \prod_{i=1}^{s} d_{ij_i} V(d_{1j_1},\dots,d_{sj_s}), \tag{4}$$

where $V(d_{1j_1},\dots,d_{sj_s})$ is the Vandermonde with entries given by $d_{1j_1},\dots,d_{sj_s}$, and such that

$$\det A = s!e(S/I)\det B, \tag{5}$$

It is also shown that

$$\det B = \sum_{\substack{1\le j_i\le b_i\\ 1\le i\le s}} V(d_{1j_1},\dots,d_{sj_s}). \tag{6}$$

Using the fact that the resolution is quasi-pure, it follows from (6) that $\det B > 0$. Taking minimum and maximum of the d_{ij_i} in the products in (4) and using (5), one obtains the inequalities

$$(\prod_{i=1}^{s} m_i)\det B \le s!e(S/I)\det B \le (\prod_{i=1}^{s} M_i)\det B.$$

Here the lower, resp. the upper inequality becomes an equality if and only if $m_i = d_{ij}$, resp. $M_i = d_{ij}$ for all i and j. Thus the assertion follows. $\square$

As a last example we consider ideals of maximal minors. For these ideals Miró-Roig has proved Conjecture 1. Inspecting the inequalities in her proof one can also see that improved multiplicity conjecture holds. For the convenience of the reader we give a complete proof of the theorem, similar to that of Miró-Roig, in order to see explicitly that the bounds are reached only when the resolution of the ideal is pure. Independently, also Migliore, Nagel and Römer found a proof of this theorem.

Let $H = (h_{ij})$ be an $m\times n$-matrix with $m \le n$, whose entries are polynomials. We say that H is a *homogeneous matrix* if all minors of H are homogenous polynomials. In particular, the entries of H itself must be homogenous. For each i and j let $d_{ij} = \deg h_{ij}$. Then, since the 2-minors are homogeneous, we get $d_{1j}+d_{i1} = d_{11}+d_{ij}$ for all i and j. Thus if we set $b_i = d_{i1}$ and $a_j = d_{11} - d_{1j}$, then

$$d_{ij} = b_i - a_j \quad \text{for all} \quad i = 1,\dots,m \quad \text{and} \quad j = 1,\dots,n.$$

Conversely, given any sequences of integers $b_1,\dots,b_m$ and $a_1,\dots,a_n$ we obtain the degree matrix of a homogeneous matrix by setting $d_{ij} = b_i - a_j$. After a suitable permutation of the rows and columns, we may assume that $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \cdots \le b_m$. Then this implies that $d_{i+1,j} \ge d_{ij} \ge d_{i,j+1}$ for all i and j. For the rest of this section we will remain with this assumption on the degrees of the entries.

Set $r = n - m$, and let $I_m(H)$ be the ideal of maximal minors of M. Then height $I_m(H) \leq r+1$, and if equality holds then $I_m(H)$ is perfect, see [5, Theorem 2.1 and Theorem 2.7].

We want to prove the following

Theorem 3.5. *Suppose that* height $I_m(H) = r+1$. *Then the improved multiplicity conjecture holds for $S/I_m(H)$.*

Proof. For later calculations it is useful to set $u_{ij} = d_{j,i+j-1}$ for all $i = 1, \dots r+1$ and $j = 1, \dots, m$. Using this notation, we have

$$b_j - a_{i+j-1} = u_{ij}$$

for all i and j in the above range, and since we assume that $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_m$ we have

$$u_{1j} \geq u_{2j} \geq \cdots \geq u_{r+1,j} \quad \text{for} \quad j = 1, \dots, r+1, \tag{7}$$

and

$$u_{ij} \geq u_{i+1,j-1} \quad \text{for all } i,\ j \text{ with } i+j \leq n+m+1,\ i \leq r+1 \text{ and } 1 < j. \tag{8}$$

According to [14, Corollary 6.5] the multiplicity of $R = S/I_m(H)$ is then given by

$$e(R) = \sum_{1 \leq j_1 \leq j_2 \leq \cdots \leq j_{r+1} \leq m} \prod_{i=1}^{r+1} u_{i,j_i}. \tag{9}$$

Since by assumption $I_m(H)$ is perfect, the Eagon-Northcott complex provides a minimal graded free S-resolution of $I_m(H)$. This allows us to compute the numbers $M_i(I_m(H))$. Following [5], the Eagon-Northcott complex resolving $I_m(H)$ can be described as follows: let F an G be finitely generated free S-modules with basis $f_1, \dots, f_m$ and $g_1, \dots, g_n$, resp., and let $\varphi\colon G \to F$ be the linear map with

$$\varphi(g_j) = \sum_{i=1}^{m} h_{ij} f_i, \quad j = 1, \dots, n.$$

denote by $\bigwedge^j G$ the jth exterior power of G, and by $S_i(F)$ the ith symmetric power of F. For $j = 1, \dots, n$ we may then view $\varphi(g_j)$ as an element of the symmetric algebra $S(F) = \bigoplus_i S_i(F)$, and the Koszul complex of $\varphi(g_1), \dots, \varphi(g_n)$ is then given by

$$0 \longrightarrow \bigwedge^{n} G \otimes S(F) \longrightarrow \cdots \longrightarrow \bigwedge^{1} G \otimes S(F) \longrightarrow \bigwedge^{0} G \otimes S(F) \longrightarrow 0.$$

The symmmetric algebra $S(F)$ is graded, and the elements $\varphi(g_j)$ are homogeneous of degree 1 (the coefficients h_{ij} are here considered to be of degree 0). The rth graded component of the Koszul complex

$$0 \longrightarrow \bigwedge^{r} G \otimes S_0(F) \longrightarrow \cdots \longrightarrow \bigwedge^{1} G \otimes S_{r-1}(F) \longrightarrow \bigwedge^{0} G \otimes S_r(F) \longrightarrow 0 \tag{10}$$

is a complex of free S-modules, whose S-dual

$$0 \longrightarrow (\bigwedge^{0} G \otimes S_r(F))^* \longrightarrow \cdots \longrightarrow (\bigwedge^{r-1} G \otimes S_1(F))^* \longrightarrow (\bigwedge^{r} G \otimes S_0(F))^* \longrightarrow 0$$

is the Eagon-Northcott complex resolving $I_m(H)$.

Set $\deg f_i = a - b_i$ for $i = 1, \ldots, m$ and $\deg g_j = a - a_j$ for $j = 1, \ldots, n$. Then the Koszul complex (10) is a graded complex. In order to make the augmentation map $(\bigwedge^r G \otimes S_0(F))^* \to I_m(H)$ homogeneous of degree 0. We let the Eagon-Northcott complex be the dual of complex (10) with respect to $S(-(r-1)a - b)$, where $a = \sum_{i=1}^n a_i$ and $b = \sum_{i=1}^m b_i$.

Now we can compute the M_i for the ideal $I_m(H)$. For a basis element $e \in \bigwedge^{r-k} G \otimes S_k(F)$ we denote by e^* the dual basis element in $(\bigwedge^{r-k} G \otimes S_k(F))^*$. Then the elements

$$(g_{i_1} \wedge g_{i_2} \wedge \cdots \wedge g_{i_{r-k}} \otimes f_{j_1} f_{j_2} \cdots f_{j_k})^*$$

establish a basis of $(\bigwedge^{r-k} G \otimes S_k(F))^*$, and we have

$$\begin{aligned} \deg(g_{i_1} \wedge g_{i_2} \wedge \cdots \wedge g_{i_{r-k}} &\otimes f_{j_1} f_{j_2} \cdots f_{j_k})^* \\ &= (r-1)a + b - \sum_{s=1}^{r-k}(a - a_{i_s}) - \sum_{t=1}^{k}(a - b_{j_t}) \\ &= -a + b + \sum_{s=1}^{r-k} a_{i_s} + \sum_{t=1}^{k} b_{j_t}. \end{aligned}$$

It follows that

$$\begin{aligned} M_{k+1} &= -a + b + a_{m+k+1} + \cdots + a_n + kb_m \\ &= \sum_{i=1}^{m-1}(b_i - a_i) + \sum_{i=0}^{k}(b_m - a_{m+i}) \\ &= \sum_{j=1}^{m-1} u_{1j} + \sum_{i=1}^{k+1} u_{im}. \end{aligned}$$

Thus we need to prove the following inequality

$$(11) \qquad (r+1)! \sum_{1 \le j_1 \le j_2 \le \cdots \le j_{r+1} \le m} \prod_{i=1}^{r+1} u_{i,j_i} \le \prod_{k=0}^{r} (\sum_{j=1}^{m-1} u_{1j} + \sum_{i=1}^{k+1} u_{im}).$$

We use induction on $\min\{r+1, m\}$ to prove this inequality. In case $r = 0$, we have $n = m$, and on both sides of the inequality we have the same expression, namely $\sum_{j=1}^n u_{1j}$. In case $m = 1$, the ideal $I_m(H)$ is generated by the regular sequence $h_{11}, \ldots, h_{1n}$. In this case the inequality is also known to be true, see [13]. (It also follows from the result in the next section).

We now assume that $\min\{r+1,m\}>1$, and decompose the expression for the multiplicity as follows

$$\begin{aligned}(r+1)!\sum_{1\le j_1\le j_2\le\cdots\le j_{r+1}\le m}\prod_{i=1}^{r+1}u_{i,j_i} &= (r+1)!\sum_{1\le j_1\le j_2\le\cdots\le j_{r+1}<m}\prod_{i=1}^{r+1}u_{i,j_i}\\ &+ (r+1)(r!\sum_{1\le j_1\le\cdots\le j_r\le m}\prod_{i=1}^{r}u_{i,j_i})u_{r+1,m}.\end{aligned}$$

Using induction we can replace the second summand by the larger term

$$(r+1)\prod_{k=0}^{r-1}(\sum_{j=1}^{m-1}u_{1j}+\sum_{i=1}^{k+1}u_{im})u_{r+1,m},$$

and obtain the inequality

$$\begin{aligned}(12)\qquad (r+1)!e(R) &\le (r+1)!\sum_{1\le j_1\le j_2\le\cdots\le j_{r+1}<m}\prod_{i=1}^{r+1}u_{i,j_i}\\ &+ (r+1)\prod_{k=0}^{r-1}(\sum_{j=1}^{m-1}u_{1j}+\sum_{i=1}^{k+1}u_{im})u_{r+1,m},\end{aligned}$$

On the other hand, by (7) we get

$$\begin{aligned}(13)\quad \prod_{k=0}^{r}(\sum_{j=1}^{m-1}u_{1j}+\sum_{i=1}^{k+1}u_{im}) &\ge \prod_{k=0}^{r-1}(\sum_{j=1}^{m-1}u_{1j}+\sum_{i=1}^{k+1}u_{im})(\sum_{j=1}^{m-1}u_{1j}+(r+1)u_{r+1,m})\\ &= \prod_{k=0}^{r-1}(\sum_{j=1}^{m-1}u_{1j}+\sum_{i=1}^{k+1}u_{im})(\sum_{j=1}^{m-1}u_{1j})\\ &+ (r+1)\prod_{k=0}^{r-1}(\sum_{j=1}^{m-1}u_{1j}+\sum_{i=1}^{k+1}u_{im})u_{r+1,m}.\end{aligned}$$

Thus comparing (12) and (13) it remains to be shown that

$$(r+1)!\sum_{1\le j_1\le j_2\le\cdots\le j_{r+1}\le m-1}\prod_{i=1}^{r+1}u_{i,j_i}\le\prod_{k=0}^{r-1}(\sum_{j=1}^{m-1}u_{1j}+\sum_{i=1}^{k+1}u_{im})(\sum_{j=1}^{m-1}u_{1j}).$$

By induction hypothesis

$$(r+1)!\sum_{1\le j_1\le j_2\le\cdots\le j_{r+1}\le m-1}\prod_{i=1}^{r+1}u_{i,j_i}\le\prod_{k=0}^{r}(\sum_{j=1}^{m-2}u_{1j}+\sum_{i=1}^{k+1}u_{i,m-1}).$$

Thus the desired inequality follows once we can show that

$$(14)\qquad \prod_{k=0}^{r}(\sum_{j=1}^{m-2}u_{1j}+\sum_{i=1}^{k+1}u_{i,m-1})\le\prod_{k=0}^{r-1}(\sum_{j=1}^{m-1}u_{1j}+\sum_{i=1}^{k+1}u_{im})(\sum_{j=1}^{m-1}u_{1j}).$$

This however is obvious, since the kth factor on left hand side for $k = 0$ is equal to the last factor on the right hand side, and since, due to (8), for $k = 1, \dots, r$, the kth factor on the left hand side $\sum_{j=1}^{m-2} u_{1j} + \sum_{i=1}^{k+1} u_{i,m-1}$ is less than or equal to the $(k-1)$th factor $\sum_{j=1}^{m-1} u_{1j} + \sum_{i=1}^{k} u_{im}$ on the left hand side.

In order to prove the lower inequality, note that

$$\begin{aligned} m_{k+1} &= -a + b + a_1 + \cdots + a_{r-k} + kb_1 \\ &= \sum_{i=2}^{m}(b_i - a_{r+i}) + \sum_{i=0}^{k}(b_1 - a_{r-i+1}) \\ &= \sum_{i=r+1-k}^{r} u_{i1} + \sum_{j=1}^{m} u_{r+1,j}. \end{aligned}$$

Thus we need to prove the following inequality

$$(r+1)! \sum_{1 \le j_1 \le j_2 \le \cdots \le j_{r+1} \le m} \prod_{i=1}^{r+1} u_{i,j_i} \ge \prod_{k=0}^{r}\Big(\sum_{i=r+1-k}^{r} u_{i1} + \sum_{j=1}^{m} u_{r+1,j}\Big). \tag{15}$$

The proof of this inequality is completely analogue to that of inequality (11), since the situation somehow dual to previous case. Indeed, the substitution

$$u_{ij} \mapsto u_{r+2-i,m+1-j} \quad \text{for} \quad i = 1, \dots, r \quad \text{and} \quad j = 1, \dots, m$$

transfers (11) to (15) and *reverses* the inequalities (7) and (8). Thus the lower bound follows from the upper bound.

Now suppose the multiplicity of $S/I_m(H)$ reaches the upper bound. This is only possible if we have equality in (12), (13) and (14).

It follows from the formula for the shifts in the resolution of the Eagon-Northcott complex, that the resolution of $S/I_m(H)$ is pure if and only if $a_1 = a_2 = \cdots = a_n$ and $b_1 = b_2 = \cdots = b_m$, which is equivalent to say that all u_{ij} are equal.

Hence by induction we have equality in (12) if and only if $u_{ij} = u$ for some u and all $i = 1, \dots, r+1$ and $j = 1, \dots, m-1$. On the other hand, we get equality in (13) if and only if $u_{im} = u_{r+1,m}$ for $i = 1, \dots, r$, while equality holds in (14) if and only if $u_{im} = u_{i,m-1} = u$ for $i = 1, \dots, r$. Thus all u_{ij} must be equal to u.

Using the reflection principle of above it also follows that the lower bound for the multiplicity is reached only when the resolution of $S/I_m(H)$ is pure. $\square$

Remarks 3.6. (a) Theorem 3.5 includes the case studied by Guardo and Van Tuly [9], namely that rings whose defining ideal is generated by powers of a homogeneous regular sequence satisfy the improved multiplicity conjecture. In fact, if $f_1, \dots, f_{r+1}$ is a homogeneous regular sequence and $I = (f_1, \dots, f_{r+1})$, then I^m is the ideal of maximal minors of the $m \times m + r$-matrix whose ith main diagonal has all entries f_i for $i = 1, \dots, r+1$, while all other entries of the matrix are 0.

(b) It can be easily seen from the proof of Theorem 1.1 that $e(R/(\mathbf{f}))$ reaches the upper bound if and only if there exists an integer d such that $M_i = id$ and $\deg f_i = d$ for all i.

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