

ON THE RADICAL OF A MONOMIAL IDEAL

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ABSTRACT. Algebraic and combinatorial properties of a monomial ideal and its radical are compared.

1. INTRODUCTION

There are simple examples of Cohen-Macaulay ideals whose radical is not Cohen-Macaulay. The first such example is probably due to Hartshorne [5], who proved that in positive characteristic the toric ring $K[s^4, s^3t, st^3, t^4]$ is a set theoretic complete intersection. With CoCoA or other computer algebra systems many other examples, also in characteristic zero, can be constructed. The following example due Conca was computed with CoCoA: let $S = K[x_1, x_2, x_3, x_4, x_5]$ and $J = (x_2^2 - x_4x_5, x_1x_3 - x_3x_4, x_3x_4 - x_1x_5) \subset S$. Then S/J is a 2-dimensional Cohen-Macaulay ring, $\sqrt{J} = (x_1x_3 - x_1x_5, x_3x_4 - x_1x_5, x_2^2 - x_4x_5, x_1^2x_2 - x_1x_2x_4, x_2x_3^2 - x_2x_3x_5)$ and S/\sqrt{J} is *not* Cohen-Macaulay. Indeed, the depth of S/\sqrt{J} equals 1. On the other hand it is well-known that the Cohen-Macaulay property of a monomial ideal is inherited by its radical. The reason is that the radical of a monomial ideal is essentially obtained by polarization and localization. This observation, was communicated to the third author by David Eisenbud. Both operations, polarization and localization, preserve the Cohen-Macaulay property. An explicit proof of this fact can be found in [11]. The purpose of this paper is to exploit this idea and to show that many other nice properties are inherited by the radical of a monomial ideal.

2. THE COMPARISON

For the proof of the main result of this paper we need some preparation. We begin with the following extension [10, Theorem 1.1] of Hochster's formula [1, Theorem 5.3.8] describing the local cohomology of a monomial ideal.

Let K be a field, $S = K[x_1, \dots, x_n]$ the polynomial ring and $I \subset S$ a monomial ideal. The unique minimal monomial system of generators of I is denoted by $G(I)$. For $i = 1, \dots, n$ we set

$$t_i = \max\{\nu_i(u) : u \in G(I)\},$$

where for a monomial $u \in S$, $u = x_1^{a_1} \cdots x_n^{a_n}$ we set $\nu_i(u) = a_i$ for $i = 1, \dots, n$.

For $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$, we set

$$G_a = \{i : 1 \leq i \leq n, a_i < 0\},$$

1991 *Mathematics Subject Classification.* 13D02, 13P10, 13D40, 13A02.

and define the simplicial complex $\Delta_a(I)$ whose faces are the sets $L \setminus G_a$ with $G_a \subset L$, and such that L satisfies the following condition: for all $u \in G(I)$ there exists $i \notin L$ such that $\nu_i(u) > a_i \geq 0$.

Notice that the inequality $a_i \geq 0$ in the definition of $\Delta_a(I)$ follows from the condition $i \notin L \supset G_a$. It is included only for the reader's convenience.

With the notation introduced one has

Theorem 2.1 (Takayama [10]). *Let $I \subset S$ be a monomial ideal. Then the Hilbert series of the local cohomology modules of S/I with respect to the \mathbb{Z}^n -grading is given by*

$$\text{Hilb}(H_{\mathfrak{m}}^i(S/I), \mathbf{t}) = \sum_{F \in \Delta} \sum_a \dim_K \tilde{H}_{i-|F|-1}(\Delta_a(I); K) \mathbf{t}^a$$

where Δ is the simplicial complex corresponding to the Stanley-Reisner ideal \sqrt{I} , and the second sum is taken over all $a \in \mathbb{Z}^n$ such that $a_i \leq t_i - 1$ for all i , and $G_a = F$.

As a first application of this theorem we have

Corollary 2.2. *Let $I \subset S$ be a monomial ideal. Then*

$$a(S/I) \leq \sum_{i=1}^n t_i - n,$$

where $a(S/I)$ is the a -invariant of S/I .

Proof. By Theorem 2.1, we know that $H_{\mathfrak{m}}^i(R)_a = 0$ for all i and for all $a \in \mathbb{Z}^n$ such that $a_i > t_i - 1$ for some i . Thus in particular, if $d = \dim R$, then $H_{\mathfrak{m}}^d(R)_j = 0$ for $j > \sum_{i=1}^n t_i - n$. \square

We say that S/I has *maximal a -invariant* if the upper bound in Corollary 2.2 is attained, that is, if $a(S/I) = \sum_{i=1}^n t_i - n$.

For our main theorem the next corollary is important.

Corollary 2.3. *Let $I \subset S$ be a monomial ideal. Then we have the following isomorphisms of K -vector spaces*

$$H_{\mathfrak{m}}^i(S/I)_a \cong H_{\mathfrak{m}}^i(S/\sqrt{I})_a$$

for all $a \in \mathbb{Z}^n$ with $a_i \leq 0$ for $1 \leq i \leq n$.

Proof. Consider the multigraded Hilbert series of $H_{\mathfrak{m}}^i(S/I)$ and $H_{\mathfrak{m}}^i(S/\sqrt{I})$. Let $a \in \mathbb{Z}^n$ be such that $a_i \leq 0$ for all $1 \leq i \leq n$. Then by Theorem 2.1, we have

$$\begin{aligned} \dim_K H_{\mathfrak{m}}^i(S/I)_a &= \dim_K \tilde{H}_{i-|F|-1}(\Delta_a(I); K), \quad \text{and} \\ \dim_K H_{\mathfrak{m}}^i(S/\sqrt{I})_a &= \dim_K \tilde{H}_{i-|F|-1}(\Delta_a(\sqrt{I}); K), \end{aligned}$$

For a monomial u we set $\text{supp}(u) = \{i: x_i \text{ divides } u\}$. Now since for every $u \in G(I)$ there exists $v \in G(\sqrt{I})$ such that $\text{supp}(u) \supset \text{supp}(v)$, and since for every $v \in G(\sqrt{I})$ there exists $u \in G(I)$ such that $\text{supp}(v) = \text{supp}(u)$, it follows that $\Delta_a(I) = \Delta_a(\sqrt{I})$. Thus we have $\dim_K H_{\mathfrak{m}}^i(S/I)_a = \dim_K H_{\mathfrak{m}}^i(S/\sqrt{I})_a$. \square

Let M be a graded S -module. For the convenience of the reader we recall the following two concepts which generalize the Cohen-Macaulay property and non-pure shellability of simplicial complexes.

The following definition is due to Stanley [9, Section II, 3.9]:

Definition 2.4. Let M be a finitely generated graded S -module. The module M is *sequentially Cohen-Macaulay* if there exists a finite filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \dots \subset M_r = M$$

of M by graded submodules of M such that each quotient M_i/M_{i-1} is CM, and $\dim M_1/M_0 < \dim M_2/M_1 < \dots < \dim M_r/M_{r-1}$.

It is known (see for example [6, Corollary 1.7]) that if M is sequentially Cohen-Macaulay, then the filtration given in the definition is uniquely determined. We call it the *attached filtration* of the sequentially Cohen-Macaulay module M .

The uniqueness of the filtration is seen as follows: suppose $\text{depth } M = t$, then M_1 is the image of the natural map $\text{Ext}_S^{n-t}(\text{Ext}_S^{n-t}(M, \omega_S), \omega_S) \rightarrow M$. Here $\omega_S = S(-n)$ is the canonical module of S . Then one notices that M/M_1 is again sequentially Cohen-Macaulay and uses induction on the length of the attached sequence.

In case M is a cyclic module, say, $M = S/I$, with attached filtration $0 = M_0 \subset M_1 \subset \dots$, each of the the modules M_i is an ideal in S/I , and hence is of the form I_i/I for certain (uniquely determined) ideals $I_i \subset S$. Thus S/I is sequentially Cohen-Macaulay, if and only if there exists a chain of graded ideals

$$I = I_0 \subset I_1 \subset I_2 \subset \dots \subset I_r = S$$

such that each factor module I_{i+1}/I_i is Cohen-Macaulay with

$$\dim I_{i+1}/I_i < \dim I_{i+2}/I_{i+1}$$

for $i = 0, \dots, r-2$. Moreover if this property is satisfied, then this chain of ideals is uniquely determined.

In the particular case that I is a monomial ideal, the natural map

$$\text{Ext}_S^{n-t}(\text{Ext}_S^{n-t}(S/I, \omega_S), \omega_S) \rightarrow S/I$$

is a homomorphism of multigraded S -modules. This implies that the attached chain of ideals of the sequentially Cohen-Macaulay module S/I is a chain of monomial ideals.

Now let us briefly describe the other concept which was introduced by Dress [4]:

Definition 2.5. Let M be a finitely generated graded S -module. A filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \dots \subset M_r = M$$

of M by graded submodules of M is called *clean* if for all $i = 1, \dots, r$ there exists a minimal prime ideal P_i of M such that $M_i/M_{i-1} \cong S/P_i$. The module M is called *clean* if it has a clean filtration.

Again, if $M = S/I$ is cyclic, then S/I is clean if there exists a chain of ideals $I = I_0 \subset I_1 \subset I_2 \subset \dots \subset I_{r-1} \subset I_r = S$ such that $I_{i+1}/I_i \cong S/P_i$ with P_i a minimal prime ideal of I . In other words, for all $i = 0, \dots, r-1$ there exists $f_{i+1} \in I_{i+1}$ such

that $I_{i+1} = (I_i, f_{i+1})$ and $P_i = I_i : f_{i+1}$. In case I is a monomial ideal we require that all f_i are monomials.

Dress [4] shows that a Stanley-Reisner ideal I_Δ is clean if and only the simplicial complex Δ is non-pure shellable in the sense of Björner and Wachs [3].

In the proof of our main theorem we use polarization, as indicated in the introduction. Let $I = (u_1, \dots, u_m)$ with $u_i = x_1^{a_{i1}} \cdots x_n^{a_{in}}$. We fix some number i with $1 \leq i \leq n$, introduce a new variable y , and set $v_k = x_1^{a_{k1}} \cdots x_i^{a_{ki}-1} y \cdots x_n^{a_{kn}}$ if $a_{ki} > 1$, and $v_k = u_k$ otherwise. We call $J = (v_1, \dots, v_m)$ the *1-step polarization of I with respect to the variable x_i* . The element $y - x_i$ is regular on $S[y]/J$ and $(S[y]/J)/(y - x_i)(S[y]/J) \cong S/I$, see [1, Lemma 4.2.16].

Let as above $t_i = \max\{\nu_i(u_j) : j = 1, \dots, m\}$, and set $t = \sum_{i=1}^n t_i - n$. Then it is clear that if we apply t suitable 1-step polarizations, we end up with a squarefree monomial ideal I^p , which is called the *complete polarization of I* .

Now we are ready to present the main result of this section.

Theorem 2.6. *Let K be a field, $S = K[x_1, \dots, x_n]$ the polynomial ring over K , and $I \subset S$ a monomial ideal. Suppose that S/I satisfies one of the following properties: S/I is (i) Cohen-Macaulay, (ii) Gorenstein, (iii) sequentially Cohen-Macaulay, (iv) generalized Cohen-Macaulay, (v) Buchsbaum, (vi) clean, or (vii) level and has maximal a -invariant. Then S/\sqrt{I} satisfies the corresponding property.*

Proof. We first use the trick, mentioned in the introduction, to show that the Betti-numbers $\beta_i(I)$ of I do not increase when passing to \sqrt{I} .

We denote by I^p the complete polarization of I . Let T be the polynomial ring in the variables that are needed to polarize I . Then I^p is a squarefree monomial ideal in T with $\beta_i(I^p) = \beta_i(I)$ for all i . It is easy to see that if we localize at the multiplicative set N generated by the *new* variables which are needed to polarize I , one obtains $I^p T_N = (\sqrt{I}) T_N$. Since localization is an exact functor, the localized free resolution will be a possibly non-minimal free resolution of $(\sqrt{I}) T_N$. Since the extension $S \rightarrow T_N$ is flat, the desired inequality follows.

Proof of (i) and (ii): The inequality $\beta_i(\sqrt{I}) \leq \beta_i(I)$ implies that $\text{depth } S/\sqrt{I} \geq \text{depth } S/I$. On the other hand, $\dim S/I = \dim S/\sqrt{I}$. This implies that S/\sqrt{I} is Cohen-Macaulay, if S/I is so.

Suppose now that S/I is Gorenstein. Then $\beta_q(S/I) = 1$ where q is the codimension of I , see [1, Theorem 3.3.7 and Corollary 3.3.9]. Therefore, $\beta_q(S/\sqrt{I}) \leq 1$. Since I and \sqrt{I} have the same codimension, we see that $\beta_q(S/\sqrt{I}) > 0$, and hence $\beta_q(S/\sqrt{I}) = 1$. Again using [1, Theorem 3.3.7 and Corollary 3.3.9] we conclude that S/\sqrt{I} is Gorenstein. This fact follows also from [2, Corollary 3.4].

Proof of (iii): Since S/I is sequentially Cohen-Macaulay there exists a chain of monomial ideals

$$I = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_k = S$$

such that I_{j+1}/I_j is Cohen-Macaulay for all $j = 0, \dots, k-1$ and such that $\dim I_1/I_0 < \dim I_2/I_1 < \cdots < \dim I_k/I_{k-1}$.

Suppose x_1^a with $a > 1$ divides a generator of I . Then we apply a 1-step polarization for x_1 to all the ideals I_i , and obtain a chain of ideals $J = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_k = \tilde{S}$ where $\tilde{S} = S[y]$. It follows that $y - x_1$ is \tilde{S}/J_i -regular and $(\tilde{S}/J_i)/(y - x_1)(\tilde{S}/J_i) \cong S/I_i$ for all i . Therefore $y - x_1$ is J_{i+1}/J_i -regular, and $(J_{i+1}/J_i)/(y - x_1)(J_{i+1}/J_i) \cong I_{i+1}/I_i$. Thus J is sequentially Cohen-Macaulay.

Since the complete polarization I_i^p of the ideals I_i for $i = 1, \dots, k$, is obtained by a sequence of 1-step polarizations, it follows that I^p is sequentially Cohen-Macaulay. As I_i^p/I_{i+1}^p is Cohen-Macaulay, we conclude as in the proof of (i) that $\sqrt{I_{i+1}^p}/\sqrt{I_i^p}$ is Cohen-Macaulay of the same dimension as I_{i+1}/I_i . This shows that \sqrt{I} is sequentially Cohen-Macaulay.

Proof of (iv) and (v): Assuming that S/I is generalized Cohen-Macaulay or Buchsbaum, one has that S/I is equidimensional and that $H_m^i(S/I)_j = 0$ for all $i < \dim S/I$, and all but finitely many j . Since I and \sqrt{I} have the same minimal prime ideals, it follows that \sqrt{I} is again equidimensional.

Let \mathbb{Z}_-^n be the set of all $a \in \mathbb{Z}^n$ such that $a_i \leq 0$ for $i = 1, \dots, n$. By Corollary 2.3, $H_m^i(S/I)_a = H_m^i(S/\sqrt{I})_a$ for all $a \in \mathbb{Z}_-^n$. Moreover, by Hochster's formula, $H_m^i(S/\sqrt{I})_a = 0$ for all $a \notin \mathbb{Z}_-^n$. Therefore, $\dim_K H_m^i(S/\sqrt{I})_j \leq \dim_K H_m^i(S/I)_j$ for all $j \leq 0$ and $H_m^i(S/J)_j = 0$ for $j > 0$. It is known [8] that a squarefree monomial ideal is Buchsbaum if and only if it is generalized Cohen-Macaulay. Thus (iv) and (v) follow.

Proof of (vi): Assuming that S/I is clean, there exists a chain of monomial ideals $I = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_{r-1} \subset I_r = S$ such that $I_{i+1}/I_i \cong S/P_i$ with P_i a minimal prime ideal of I . We claim that $\sqrt{I_{i+1}}/\sqrt{I_i} = S/P_i$, if $\sqrt{I_{i+1}} \neq \sqrt{I_i}$. This then implies that S/\sqrt{I} is clean, since the prime ideals P_i are also minimal prime ideals of \sqrt{I} .

In order to prove this claim we introduce some notation: let $u = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and $v = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ be two monomials. Then we set

$$u : v = \prod_{i=1}^n x_i^{\max\{a_i - b_i, 0\}}, \quad \text{and} \quad u_{red} = \prod_{\substack{i \\ a_i > 0}} x_i.$$

We then have

$$(1) \quad (u : v)_{red} = (u_{red} : v_{red}) \prod_{\substack{i \\ a_i > b_i > 0}} x_i.$$

Note that if I is a monomial ideal with monomial generators u_1, \dots, u_m , then

$$\sqrt{I} = ((u_1)_{red}, \dots, (u_m)_{red}) \quad \text{and} \quad I : v = (u_1 : v, \dots, u_m : v).$$

Back to the proof of our claim, our assumption implies that for all $i = 0, \dots, r - 1$ there exists a monomial $v_{i+1} \in I_{i+1}$ such that $I_{i+1} = (I_i, v_{i+1})$ and $P_i = I_i : v_{i+1}$. Suppose $P_i = (x_{i_1}, \dots, x_{i_s})$. Then $P_i = I_i : v_{i+1}$ if and only if

- (a) for all $j = 1, \dots, s$ there exists $u \in I_i$ such that $u : v_{i+1} = x_{i_j}$, and

(b) for all monomial generators $w \in I_i$ there exists an integer j with $1 \leq j \leq s$ such that $x_{i_j} | (w : v_{i+1})$.

We need to show that $P_i = \sqrt{I_i} : (v_{i+1})_{red}$, if $(v_{i+1})_{red} \notin \sqrt{I_i}$, and prove this by checking (a) and (b) for the pair $\sqrt{I_i}$ and $(v_{i+1})_{red}$.

Let j be an integer with $1 \leq j \leq s$. Then there exists $u \in I_i$ such that $u : v_{i+1} = x_{i_j}$. Suppose $u = \prod_{k=1}^n x_k^{a_k}$ and $v_{i+1} = \prod_{k=1}^n x_k^{b_k}$, then (1) implies that $x_{i_j} = (u : v_{i+1})_{red} = (u_{red} : (v_{i+1})_{red})w$ where $w = \prod_{k, a_k > b_k > 0} x_k$. Suppose x_{i_j} divides w , then $u_{red} : (v_{i+1})_{red} = 1$. This implies that $(v_{i+1})_{red} \in \sqrt{I_i}$, a contradiction. Therefore $u_{red} : (v_{i+1})_{red} = x_{i_j}$, and this proves (a). The argument also shows that $b_{i_j} = 0$ for $j = 1, \dots, s$.

For the proof of (b), let $w \in I_i$ be a monomial generator. Then there exists an integer j with $1 \leq j \leq s$ such that $x_{i_j} | (w : v_{i+1})$. It follows that x_{i_j} divides $(w : v_{i+1})_{red}$. Let $w = \prod_{k=1}^n x_k^{c_k}$. Then (1) implies that x_{i_j} divides $(w_{red} : (v_{i+1})_{red}) \prod_{k, c_k > b_k > 0} x_k$. However, $b_{i_j} = 0$, as we have seen in the proof of (a). Therefore, x_{i_j} divides $(w_{red} : (v_{i+1})_{red})$. Since $\sqrt{I_i}$ is generated by the monomials w_{red} where the monomials w are the generators of I_i , condition (b) follows.

Proof of (vii): By assumption S/I is level. This means that S/I is Cohen-Macaulay and that all generators of the canonical module $\omega_{S/I}$ of S/I have the same degree, say g . In this situation the a -invariant $a(S/I)$ of S/I is just $-g$, see [1, Section 3.6]. Suppose $d = \dim S/I$; then I has a graded minimal free resolution \mathbb{F} of length $q = n - d - 1$ with $F_q = S^b(-c)$. Since $\omega_{S/I}$ may be represented as the cokernel of $F_{q-1}^* \rightarrow F_q^*$, which is dual of the map $F_q \rightarrow F_{q-1}$ with respect to $S(-n)$, it follows that $a(S/I) = c - n$.

For $i = 1, \dots, n$ we set again

$$t_i = \max\{\nu_i(u) : u \in G(I)\}.$$

By Corollary 2.2, one has the upper bound $a(S/I) \leq \sum_{i=1}^n t_i - n$. Since we assume that S/I has maximal a -invariant, the upper bound is reached. Let $I^p \subset T$ the complete polarization of I . This polarization requires precisely $t = \sum_{i=1}^n t_i - n$ 1-step polarizations. It follows that S/I is obtained from T/I^p as a residue class ring modulo a regular sequence of linear forms of length t . From the above description of the a -invariant we now conclude that $a(T/I^p) = a(S/I) - t = 0$. Let \mathbb{G} be the multigraded minimal free resolution of the squarefree monomial ideal I^p . Since $\text{proj dim } I^p = \text{proj dim } I = q$, and since $a(T/I^p) = 0$, we see that $G_q = T(-m)^b$, where $m = n + t = \dim T$. This implies that G_q as a multigraded module is isomorphic to $T(-e)^b$ where $e = (1, 1, \dots, 1)$.

For $i = 1, \dots, m$ let e_i be the i th canonical basis vector of \mathbb{Z}^m . Then $e = \sum_{i=1}^m e_i$, and we may assume that $\deg x_i = e_i$ for $i = 1, \dots, n$, while the new variables have the multidegrees e_i with $i = n + 1, \dots, m$. We define a new multigrading on T and T/I^p : for an element f of multidegree a we set $\deg' f = \pi(a)$, where $\pi : \mathbb{Z}^m \rightarrow \mathbb{Z}^n$ is the projection onto the first n components of \mathbb{Z}^m .

As above, let N be the multiplicative set generated by the t new variables which are needed to polarize I . Then $I^p T_N = \sqrt{I} T_N$, and localization with respect to N preserves the new multigrading since $\deg' f = 0$ for all $f \in N$. Therefore \mathbb{G}_N is,

with respect to the new grading, a multigraded free T_N -resolution of $\sqrt{I}T_N$ with $(G_q)_N = T_N(-1, \dots, -1)^b$ and $(-1, \dots, -1) \in \mathbb{Z}^n$.

Let \mathbb{H} be the multigraded minimal free S -resolution of \sqrt{I} . Then $\mathbb{H}T_N$ is the minimal multigraded free T_N -resolution of $\sqrt{I}T_N$. A comparison with the (possibly non-minimal) graded free T_N -resolution \mathbb{G}_N shows that H_q is a direct summand of copies of $S(-1, \dots, -1)$. Since S/I and S/\sqrt{I} are Cohen-Macaulay of the same dimension, we see that $q = \text{proj dim } I = \text{proj dim } \sqrt{I}$. Therefore all summands in the last step of the resolution \mathbb{H} of S/\sqrt{I} have the same shift. This shows that S/\sqrt{I} is level. \square

Remark 2.7. In Theorem 2.6(i) (or (iv)), it suffices to require that I is an arbitrary homogeneous (generalized) Cohen-Macaulay ideal whose radical \sqrt{I} is a monomial ideal, i.e. we do not need to require that I itself is a monomial ideal.

Indeed it is enough to prove that there is a surjective homomorphism $H_{\mathfrak{m}}^i(S/I) \rightarrow H_{\mathfrak{m}}^i(S/\sqrt{I})$ for all i . The natural surjective map $S/I \rightarrow S/\sqrt{I}$ induce for all i commutative diagrams

$$\begin{array}{ccc} \text{Ext}^i(S/\sqrt{I}, S) & \longrightarrow & \text{Ext}^i(S/I, S) \\ \downarrow & & \downarrow \\ H_{\sqrt{I}}^i(S) & \longrightarrow & H_I^i(S). \end{array}$$

Since $H_{\sqrt{I}}^i(S) \cong H_I^i(S)$ and since $\text{Ext}^i(S/\sqrt{I}, S) \rightarrow H_{\sqrt{I}}^i(S)$ is an essential extension (see [12]), it follows that $\text{Ext}_S^i(S/\sqrt{I}, S) \rightarrow \text{Ext}_S^i(S/I, S)$ is injective for all i . Hence the desired conclusion follows by local duality.

On the other hand, as for the Gorenstein property, we must assume that I is a monomial ideal. For example, $I = (xy + yz, xz)$ is a complete intersection, hence, a Gorenstein ideal, while $\sqrt{I} = (xy, yz, xz)$ is not Gorenstein.

3. THE INVERSE PROBLEM

The results of the previous section indicate the following question: for a subset $F \subset [n]$, let P_F be the prime ideal generated by the x_i with $i \in F$. The minimal prime ideals of a squarefree I are all of this form, and since I is a radical ideal it is the intersection of its minimal prime ideals, say, $I = \bigcap_{i=1}^r P_{F_i}$ with $F_i \subset [n]$.

Suppose I is Cohen-Macaulay. For which exponents a_{ij} is the ideal

$$J = \bigcap_{i=1}^r (x_j^{a_{ij}} : j \in F_i)$$

again Cohen-Macaulay?

Of course if we raise the x_i uniformly to some power, say x_i is replaced by $x_i^{a_i}$ everywhere in the intersection, then the resulting ideal J is the image of the flat map $S \rightarrow S$ with $x_i \mapsto x_i^{a_i}$ for all i . Thus in this case J will be Cohen-Macaulay, if I is so. On the other hand, if we allow arbitrary exponents, the question seems to be quite delicate, and we do not know a general answer. However, if we require that for *all* choices of exponents the resulting ideal is again Cohen-Macaulay, a complete answer is possible.

We need a definition to state the next result. Let L be a monomial ideal. Lyubeznik [7] defines the *size* of L as follows: let $L = \bigcap_{j=1}^r Q_j$ be an irredundant primary decomposition of L , where the Q_i are monomial ideals. Let h be the height of $\sum_{j=1}^r Q_j$, and denote by v the minimum number t such that there exist j_1, \dots, j_t with $\sqrt{\sum_{i=1}^t Q_{j_i}} = \sqrt{\sum_{j=1}^r Q_j}$. Then $\text{size } L = v + (n - h) - 1$.

Since for monomial ideals the operations of forming sums and taking radicals can be exchanged, the numbers v and h , and hence the size of L depends only on the associated prime ideals of L .

We shall need the following result of Lyubeznik [7, Proposition 2]:

Lemma 3.1. *Let L be a monomial ideal in S . Then $\text{depth } S/L \geq \text{size } L$.*

Now we can state the main result of this section.

Theorem 3.2. *Let $I \subset S = K[x_1, \dots, x_n]$ be a Cohen-Macaulay squarefree monomial ideal, and write*

$$I = \bigcap_{i=1}^r P_{F_i},$$

where the sets $F_i \subset [n]$ are pairwise distinct, and all have the same cardinality c .

For $i = 1, \dots, r$ and $j = 1, \dots, c$ we choose integers $a_{ij} \geq 1$, and set

$$Q_{F_i} = (x_j^{a_{ij}} : j \in F_i) \quad \text{for } i = 1, \dots, r.$$

Then the following conditions are equivalent:

(a) for all choices of the integers a_{ij} the ideal

$$J = \bigcap_{i=1}^r Q_{F_i}$$

is Cohen-Macaulay;

(b) for each subset $A \subset [r]$, the ideal $I_A = \bigcap_{i \in A} P_{F_i}$ is Cohen-Macaulay;

(c) $\text{height } P_{F_i} + P_{F_j} = c + 1$ for all $i \neq j$;

(d) for $r \geq 2$ either $|\bigcup_{i=1}^r F_i| = c + 1$, or $|\bigcap_{i=1}^r F_i| = c - 1$;

(e) after a suitable permutation of the elements of $[n]$ we either have

$$F_i = \{1, \dots, i - 1, i + 1, \dots, c, c + 1\} \quad \text{for } i = 1, \dots, r,$$

or

$$F_i = \{1, \dots, c - 1, c - 1 + i\} \quad \text{for } i = 1, \dots, r;$$

(f) $\text{size } I = \dim S/I$;

(g) S/L is Cohen-Macaulay for any monomial ideal L such that $\text{Ass } L = \text{Ass } I$.

Proof. (a) \Rightarrow (b): Let $Q_{F_i} = (x_j^2 : j \in F_i)$ if $i \in A$, and $Q_{F_i} = P_{F_i}$ if $i \notin A$. By assumption, $J = \bigcap_{i=1}^r Q_{F_i}$ is Cohen-Macaulay. Hence the complete polarization J^p of J is again Cohen-Macaulay. We have $J^p = \bigcap_{i=1}^r Q_{F_i}^p$ with $Q_{F_i}^p = (x_j y_j : j \in F_i)$ if

$i \in A$, and $Q_{F_i}^p = P_{F_i}$ if $i \notin A$. Let N be the multiplicative set generated by all the variables x_i . Then J_N^p is Cohen-Macaulay, and hence

$$J_N^p = \bigcap_{i \in A} (y_j : j \in F_i).$$

This shows that $I_A = \bigcap_{i \in A} P_{F_i}$ is Cohen-Macaulay.

(b) \Rightarrow (c): Consider the exact sequence

$$0 \longrightarrow S/(P_{F_i} \cap P_{F_j}) \longrightarrow S/P_{F_i} \oplus S/P_{F_j} \longrightarrow S/(P_{F_i} + P_{F_j}) \longrightarrow 0.$$

The rings S/P_i and S/P_j are Cohen-Macaulay of dimension $n - c$, while $S/(P_{F_i} + P_{F_j})$ is Cohen-Macaulay of dimension $n - d$ where d is the height of $P_{F_i} + P_{F_j}$. The exact sequence yields that $S/(P_{F_i} \cap P_{F_j})$ is Cohen-Macaulay if and only if $d = c + 1$.

Since by assumption $S/P_{F_i} \cap P_{F_j}$ is Cohen-Macaulay for all $i \neq j$, the assertion follows.

(c) \Rightarrow (d): We must show: given a collection of subsets $F_1, \dots, F_r \subset [n]$ with

- (i) $|F_i| = c$ for all i ;
- (ii) $|F_i \cup F_j| = c + 1$ for all $i \neq j$.

Then either $|\bigcup_{i=1}^r F_i| = c + 1$, or $|\bigcap_{i=1}^r F_i| = c - 1$.

Suppose this is not the case. Then, since $|F_1 \cap F_2| = c - 1$ and $|F_1 \cup F_2| = c + 1$, there exist integers i and j such that $F_1 \cap F_2 \not\subset F_i$, and $F_j \not\subset F_1 \cup F_2$. The conditions (i) and (ii) then imply that there exists an element $x \in F_1 \cap F_2$ such that $F_1 \cup F_2 \setminus \{x\} = F_i$, and an element $y \in F_j \setminus (F_1 \cup F_2)$ such that $F_j = \{y\} \cup (F_1 \cap F_2)$. It follows that $F_i \cup F_j = (F_1 \cup F_2) \cup \{y\}$. This contradicts (ii).

(d) \Rightarrow (e): Assume that $|\bigcup_{i=1}^r F_i| = c + 1$. After a suitable permutation of the elements of $[n]$ we may assume that $\bigcup_{i=1}^r F_i = \{1, \dots, c + 1\}$. Since $|F_i| = c$, there exists $j_i \in \{1, \dots, c + 1\}$ such that $F_i = \{1, \dots, c + 1\} \setminus \{j_i\}$. Since the sets F_i are pairwise distinct it follows that $j_i \neq j_k$ for $i \neq k$. Thus after applying again suitable permutation we may assume that $j_i = i$ for $i = 1, \dots, r$.

The second statement follows similarly.

(e) \Rightarrow (f): In the first case, $v = 2$ and $h = (c + 1)$, while in the second case, $v = r$ and $h = c - 1 + r$. Thus in both cases $\text{size } I = n - c = \dim S/I$.

(f) \Rightarrow (g): By Lemma 3.1 and the remark preceding the lemma, we have

$$\text{depth } S/L \geq \text{size } L = \text{size } I = \dim S/I = \dim S/L.$$

Hence S/L is Cohen-Macaulay.

Finally the implication (g) \Rightarrow (a) is trivial. \square

Corollary 3.3. *With notation as above, the following conditions are equivalent:*

- (a) J is a Gorenstein ideal for all choices of the integers a_{ij} ;
- (b) $r = 1$ or $c = 1$.

Proof. If $r = 1$ or $c = 1$, then J is complete intersection for all choices of the integers a_{ij} . Thus (b) implies (a).

Conversely suppose condition (b) is not satisfied. We assume that $c > 1$, and have to show that $r = 1$. By Theorem 3.2 we have $|\bigcap_{i=1}^r F_i| = c - 1$ or $|\bigcup_{i=1}^r F_i| = c + 1$.

In the first case we may assume that $F_i = \{1, \dots, c-1, i+c-1\}$ for $i = 1, \dots, r$. Assume $r > 1$, and let $Q_{F_1} = (x_1^2, x_2, \dots, x_c)$ and $Q_{F_i} = P_{F_i}$ for $i \geq 2$. Then $J = \bigcap_{i=1}^r Q_{F_i} = (x_1^2, x_1x_2, \prod_{i=0}^{r-1} x_{c+i})$ is not Gorenstein, a contradiction.

In the second case suppose that $r \geq 3$. With the same argument as in the proof of Theorem 3.2 it follows that $I_A = \bigcap_{i \in A} P_{F_i}$ is a Gorenstein ideal for all subsets $A \subset [r]$. Therefore $P_{F_1} \cap P_{F_2} \cap P_{F_3}$ is Gorenstein. We may assume that $F_1 = \{1, 2, \dots, c\}$, $F_2 = \{2, 3, \dots, c+1\}$ and $F_3 = \{1, 3, 4, \dots, c+1\}$. Then $P_{F_1} \cap P_{F_2} \cap P_{F_3} = (x_1x_2, x_1x_{c+1}, x_2x_{c+1}, x_3, \dots, x_c)$ is not Gorenstein, a contradiction.

On the other hand, if $r = 2$, then $|\bigcap_{i=1}^r F_i| = c-1$, and we are again in the first case. Thus we must have that $r = 1$. \square

Remark 3.4. From a view point of Stanley-Reisner rings, the ideal I in the first case of condition (e) in Theorem 3.2 corresponds to an iterated cone of a 0-dimensional simplicial complex. In this case it is known that S/I itself is Gorenstein if the corresponding 0-dimensional simplicial complex consists of at most 2 points, see [9, Theorem 5.1(e)]. The corollary also follows from this fact.

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