

ON THE REGULARITY OF LOCAL COHOMOLOGY OF BIGRADED ALGEBRAS

AHAD RAHIMI

ABSTRACT. The Hilbert functions and the regularity of the graded components of local cohomology of a bigraded algebra are considered. Explicit bounds for these invariants are obtained for bigraded hypersurface rings.

INTRODUCTION

In this paper we study algebraic properties of the graded components of local cohomology of a bigraded K -algebra. Let P_0 be a Noetherian ring, $P = P_0[y_1, \dots, y_n]$ be the polynomial ring over P_0 with the standard grading and $P_+ = (y_1, \dots, y_n)$ the irrelevant graded ideal of P . Then for any finitely generated graded P -module M , the local cohomology modules $H_{P_+}^i(M)$ are naturally graded P -module and each graded component $H_{P_+}^i(M)_j$ is a finitely generated P_0 -module. In case $P_0 = K[x_1, \dots, x_m]$ is a polynomial ring, the K -algebra P is naturally bigraded with $\deg x_i = (1, 0)$ and $\deg y_i = (0, 1)$. In this situation, if M is a finitely generated bigraded P -module, then each of the modules $H_{P_+}^i(M)_j$ is a finitely generated graded P_0 -module.

We are interested in the Hilbert functions and the Castelnuovo-Mumford regularity of these modules.

In Section 1 we introduce the basic facts concerning graded and bigraded local cohomology and give a description of the local cohomology of a graded (bigraded) P -module from its graded (bigraded) P -resolution.

In Section 2 we use a result of Gruson, Lazarsfeld and Peskine on the regularity of reduced curves, in order to show that the regularity of $H_{P_+}^i(M)_j$ as a function in j is bounded provided that $\dim_{P_0} M/P_+M \leq 1$.

The rest of the paper is devoted to study of the local cohomology of a hypersurface ring $R = P/fP$ where $f \in P$ is a bihomogeneous polynomial.

In Section 3 we prove that the Hilbert function of the top local cohomology $H_{P_+}^n(R)_j$ is a nonincreasing function in j . If moreover, the ideal $I(f)$ generated by all coefficients of f is \mathfrak{m} -primary where \mathfrak{m} is the graded maximal ideal of P_0 , then by a result of Katzman and Sharp the P_0 -module $H_{P_+}^i(R)_j$ is of finite length. In particular, in this case the regularity of $H_{P_+}^i(R)_j$ is also a nonincreasing function in j .

In the following section we compute the regularity of $H_{P_+}^i(R)_j$ for a special class of hypersurfaces. For the computation we use in an essential way a result of Stanley

and J. Watanabe. They showed that a monomial complete intersection has the strong Lefschetz property. Stanley used the hard Lefschetz theorem, while Watanabe representation theory of Lie algebras to prove this result. Using these facts the regularity and the Hilbert function of $H_{P_+}^i(P/f_\lambda^r P)_j$ can be computed explicitly. Here $r \in \mathbb{N}$ and $f_\lambda = \sum_{i=1}^n \lambda_i x_i y_i$ with $\lambda_i \in K$. As a consequence we are able to show that $H_{P_+}^{n-1}(P/f^r P)_j$ has a linear resolution and its Betti numbers can be computed. We use these results in the last section to show that for any bigraded hypersurface ring $R = P/fP$ for which $I(f)$ is \mathfrak{m} -primary, the regularity of $H_{P_+}^i(R)_j$ is linearly bounded in j .

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1. BASIC FACTS ABOUT GRADED AND BIGRADED LOCAL COHOMOLOGY

Let P_0 be a Noetherian ring, and let $P = P_0[y_1, \dots, y_n]$ be the polynomial ring over P_0 in the variables y_1, \dots, y_n . We let $P_j = \bigoplus_{|b|=j} P_0 y^b$ where $y^b = y_1^{b_1} \dots y_n^{b_n}$ for $b = (b_1, \dots, b_n)$, and where $|b| = \sum_i b_i$. Then P be a standard graded P_0 -algebra and P_j is a free P_0 -module of rank $\binom{n+j-1}{n-1}$.

In most cases we assume that P_0 is either a local ring with residue class field K , or $P_0 = K[x_1, \dots, x_m]$ is the polynomial ring over the field K in the variables x_1, \dots, x_m .

We always assume that all P -modules considered here are finitely generated and graded. In case that P_0 is a polynomial ring, then P itself is bigraded, if we assign to each x_i the bidegree $(1, 0)$ and to each y_j the bidegree $(0, 1)$. In this case we assume that all P -modules are even bigraded. Observe that if M is bigraded, and if we set

$$M_j = \bigoplus_i M_{(i,j)}$$

Then $M = \bigoplus_j M_j$ is a graded P -module and each graded component M_j is a finitely generated graded P_0 -module, with grading $(M_j)_i = M_{(i,j)}$ for all i and j .

Now let $S = K[y_1, \dots, y_n]$. Then $P = P_0 \otimes_K K[y_1, \dots, y_n] = P_0 \otimes_K S$. Let $P_+ := \bigoplus_{j>0} P_j$ be the irrelevant graded ideal of the P_0 -algebra P .

Next we want to compute the graded P -modules $H_{P_+}^i(P)$. Observe that there are isomorphisms of graded R -modules

$$\begin{aligned} H_{P_+}^i(P) &\cong \varinjlim_{k \geq 0} \text{Ext}_P^i(P/(P_+)^k, P) \\ &\cong \varinjlim_{k \geq 0} \text{Ext}_{P_0 \otimes_K S}^i(P_0 \otimes_K S/(y)^k, P_0 \otimes_K S) \\ &\cong P_0 \otimes_K \varinjlim_{k \geq 0} \text{Ext}_P^i(S/(y)^k, S) \\ &\cong P_0 \otimes_K H_{(y)}^i(S). \end{aligned}$$

Since $H_{S_+}^i(S) = 0$ for $i \neq n$, we get

$$H_{P_+}^i(P) = \begin{cases} P_0 \otimes_k H_{(y)}^n(S) & \text{for } i = n, \\ 0 & \text{for } i \neq n. \end{cases}$$

Let M be a graded S -module. We write $M^\vee = \text{Hom}_K(M, K)$ and consider M^\vee a graded S -module as follows: for $\varphi \in M^\vee$ and $f \in S$ we let $f\varphi$ be the element in M^\vee with

$$f\varphi(m) = \varphi(fm) \quad \text{for all } m \in M,$$

and define the grading by setting $(M^\vee)_j := \text{Hom}_K(M_{-j}, K)$ for all $j \in \mathbb{Z}$.

Let ω_S be the canonical module of S . Note that $\omega_S = S(-n)$, since S is a polynomial ring in n indeterminates. By the graded version of the local duality theorem, see [1, Example 13.4.6] we have $H_{S_+}^n(S)^\vee = S(-n)$ and $H_{S_+}^i(S) = 0$ for $i \neq n$. Applying again the functor $(-)^\vee$ we obtain

$$H_{S_+}^n(S) = \text{Hom}_K(S(-n), K) = \text{Hom}_K(S, K)(n).$$

We can thus conclude that

$$H_{S_+}^n(S)_j = \text{Hom}_k(S, K)_{n+j} = \text{Hom}_K(S_{-n-j}, K) \quad \text{for all } j \in \mathbb{Z}.$$

Let $S_l = \bigoplus_{|a|=l} Ky^a$. Then

$$\text{Hom}_K(S_{-n-j}, K) = \bigoplus_{|a|=-n-j} Kz^a,$$

where $z \in \text{Hom}_K(S_{-n-j}, K)$ is the K -linear map with

$$z^a(y^b) = \begin{cases} z^{a-b}, & \text{if } b \leq a, \\ 0, & \text{if } b \not\leq a. \end{cases}$$

Here we write $b \leq a$ if $b_i \leq a_i$ for $i = 1, \dots, n$. Therefore $H_{S_+}^n(S)_j = \bigoplus_{|a|=-n-j} Kz^a$, and this implies that

$$(1) \quad H_{P_+}^n(P)_j = P_0 \otimes_K H_{(y)}^n(S)_j = \bigoplus_{|a|=-n-j} P_0 z^a.$$

Hence we see that $H_{P_+}^n(P)_j$ is free P_0 -module of rank $\binom{-j-1}{n-1}$. Moreover, if P_0 is graded

$$H_{P_+}^n(P)_{(i,j)} = \bigoplus_{|a|=-n-j} (P_0)_i z^a = \bigoplus_{\substack{|a|=i \\ |b|=-n-j}} Kx^a z^a.$$

The next theorem describes how the local cohomology of a graded P -module can be computed from its graded free P -resolution

Theorem 1.1. *Let M be a finitely generated graded P -module. Let \mathbb{F} be a graded free P -resolution of M . Then we have graded isomorphisms*

$$H_{P_+}^{n-i}(M) \cong H_i(H_{P_+}^n(\mathbb{F})).$$

Proof. Let

$$\mathbb{F} : \cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow 0.$$

Applying the functor $H_{P_+}^n$ to \mathbb{F} , we obtain the complex

$$H_{P_+}^n(\mathbb{F}) : \cdots \rightarrow H_{P_+}^n(F_2) \rightarrow H_{P_+}^n(F_1) \rightarrow H_{P_+}^n(F_0) \rightarrow 0.$$

We see that

$$H_{P_+}^n(M) = \text{Coker}(H_{P_+}^n(F_1) \rightarrow H_{P_+}^n(F_0)) = H_0(H_{P_+}^n(\mathbb{F})),$$

since $H_{P_+}^i(N) = 0$ for each $i > n$ and all finitely generated P -modules N .

We define the functors:

$$\mathcal{F}(M) := H_{P_+}^n(M) \quad \text{and} \quad \mathcal{F}_i(M) := H_{P_+}^{n-i}(M).$$

The functors \mathcal{F}_i are additive, covariant and strongly connected, i.e. for each short exact sequence $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ one has the long exact sequence

$$0 \cdots \rightarrow \mathcal{F}_i(U) \rightarrow \mathcal{F}_i(V) \rightarrow \mathcal{F}_i(W) \rightarrow \mathcal{F}_{i-1}(U) \rightarrow \cdots \rightarrow \mathcal{F}_0(V) \rightarrow \mathcal{F}_0(W) \rightarrow 0.$$

Moreover, $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{F}_i(F) = H_{P_+}^{n-i}(F) = 0$ for all $i > 0$ and all free P -modules F . Therefore, the theorem follows from the dual version of [1, Theorem 1.3.5]. \square

Note that if M is a finitely generated bigraded P -module. Then $H_{P_+}^n(M)$ with natural grading is also a finitely generated bigraded P -module, and hence in Theorem 1.1 we have bigraded isomorphisms

$$H_{P_+}^{n-i}(M) \cong H_i(H_{P_+}^n(\mathbb{F})).$$

2. REGULARITY OF THE GRADED COMPONENTS OF LOCAL COHOMOLOGY FOR MODULES OF SMALL DIMENSION

Let $P_0 = K[x_1, \dots, x_m]$, and M be a finitely generated graded P_0 -module. By Hilbert's syzygy theorem, M has a graded free resolution over P_0 of the form

$$0 \rightarrow F_k \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0,$$

where $F_i = \bigoplus_{j=1}^{t_i} P_0(-a_{ij})$ for some integers a_{ij} . Then the Castelnuovo-Mumford regularity $\text{reg}(M)$ of M is the nonnegative integer

$$\text{reg } M \leq \max_{i,j} \{a_{ij} - i\}$$

with equality holding if the resolution is minimal. If M is an Artinian graded P_0 -module, then

$$\text{reg}(M) = \max\{j : M_j \neq 0\}.$$

We also use the following characterization of regularity

$$\text{reg}(M) = \min\{\mu : M_{\geq \mu} \text{ has a linear resolution}\}.$$

Let M be a finitely generated bigraded P -module, thus $H_{P_+}^i(M)_j$ is a finitely generated graded P_0 -module. Let $f_{i,M}$ be the numerical function given by

$$f_{i,M}(j) = \text{reg } H_{P_+}^i(M)_j$$

for all j . In this section we show that $f_{i,M}$ is bounded provided that M/P_+M has Krull dimension ≤ 1 . There are some explicit examples which show that the condition $\dim_{P_0} M/P_+M \leq 1$ is indispensable. We postpone the example to Section 4. First one has the following

Lemma 2.1. *Let M be a finitely generated graded P -module. Then $\dim_{P_0} M_i \leq \dim_{P_0} M/P_+M$ for all i .*

Proof. Let $r = \min\{j : M_j \neq 0\}$. We prove the lemma by induction on $i \geq r$. Let $i = r$. Note that

$$M/P_+M = M_r \oplus M_{r+1}/P_1M_r \oplus \cdots .$$

It follows that M_r is a direct summand of the P_0 -module M/P_+M , so that $\dim_{P_0} M_r \leq \dim_{P_0} M/P_+M$. We now assume that $i > r$ and $\dim_{P_0} M_j \leq \dim_{P_0} M/P_+M$, for $j = r, \dots, i-1$. We will show that $\dim_{P_0} M_i \leq \dim_{P_0} M/P_+M$. We consider the exact sequence of P_0 -modules

$$0 \rightarrow P_1M_{i-1} + \cdots + P_{i-r}M_r \rightarrow M_i \xrightarrow{\varphi} (M/P_+M)_i \rightarrow 0.$$

By the induction hypothesis, one easily deduces that

$$\dim_{P_0} \sum_{j=1}^{i-r} P_jM_{i-j} \leq \dim_{P_0} M/P_+M,$$

and since $(M/P_+M)_i$ is a direct summand of M/P_+M it also has dimension $\leq \dim_{P_0} M/P_+M$. Therefore, by the above exact sequence, $\dim M_i \leq \dim_{P_0} M/P_+M$, too. \square

The following lemma is needed for the proof next proposition.

Lemma 2.2. *Let M be a finitely generated graded P -module. Then there exists an integer i_0 such that*

$$\text{Ann}_{P_0} M_i = \text{Ann}_{P_0} M_{i+1} \quad \text{for all } i \geq i_0$$

Proof. Since $P_1M_i \subseteq M_{i+1}$ for all i and M is a finitely generated P -module, there exists an integer t such that $P_1M_i = M_{i+1}$ for all $i \geq t$. This implies that $\text{Ann}_{P_0} M_t \subseteq \text{Ann}_{P_0} M_{t+1} \subseteq \dots$. Since P_0 is noetherian, there exists an integer k such that $\text{Ann}_{P_0} M_{t+k} = \text{Ann}_{P_0} M_i$ for all $i \geq t+k = i_0$. \square

Proposition 2.3. *Let M be a finitely generated graded P -module. Then*

$$\dim_{P_0} H_{P_+}^i(M)_j \leq \dim_{P_0} M_j \quad \text{for all } i \text{ and } j \gg 0.$$

Proof. Let $P_+ = (y_1, \dots, y_n)$. Then by [1, Theorem 5.1.19] we have

$$H_{P_+}^i(M) \cong H^i(C(M)\bullet) \quad \text{for all } i \geq 0$$

where $C(M)\bullet$ denote the (extended) Čech complex of M with respect to y_1, \dots, y_n defined as follows:

$$C(M)\bullet : 0 \rightarrow C(M)^0 \rightarrow C(M)^1 \rightarrow \cdots \rightarrow C(M)^n \rightarrow 0$$

with

$$C(M)^t = \bigoplus_{1 \leq i_1 < \dots < i_t \leq n} M_{y_{i_1} \dots y_{i_t}},$$

and where the differentiation $C(M)^t \longrightarrow C(M)^{t+1}$ is given on the component

$$M_{y_{i_1} \dots y_{i_t}} \longrightarrow M_{y_{j_1} \dots y_{j_{t+1}}}$$

to be the homomorphism

$$(-1)^{s-1} \text{nat} : M_{y_{i_1} \dots y_{i_t}} \longrightarrow (M_{y_{i_1} \dots y_{i_t}})_{y_{j_s}},$$

if $\{i_1, \dots, i_t\} = \{j_1, \dots, \hat{j}_s, \dots, j_{t+1}\}$ and 0 otherwise. We set $\mathcal{I} = \{i_1, \dots, i_t\}$ and $y_{\mathcal{I}} = y_{i_1} \dots y_{i_t}$. For $m/y_{\mathcal{I}}^k \in M_{y_{\mathcal{I}}}$, m homogeneous, we set $\deg m/y_{\mathcal{I}}^k = \deg m - \deg y_{\mathcal{I}}^k$. Then we can define a grading on $M_{y_{\mathcal{I}}}$ by setting

$$(M_{y_{\mathcal{I}}})_j = \{m/y_{\mathcal{I}}^k \in M_{y_{\mathcal{I}}} : \deg m/y_{\mathcal{I}}^k = j\} \quad \text{for all } j.$$

In view of Lemma 2.2 there exists an ideal $I \subseteq P_0$ and an integer j_0 such that $\text{Ann}_{P_0} M_j = I$ for all $j \geq j_0$. We now claim that $I \subseteq \text{Ann}_{P_0} (M_{y_{\mathcal{I}}})_j$ for all $j \geq j_0$. Let $a \in I$ and $m/y_{\mathcal{I}}^k \in (M_{y_{\mathcal{I}}})_j$ for some integer k . We may choose an integer l such that

$$\deg m + \deg y_{\mathcal{I}}^l = \deg my_{\mathcal{I}}^l = t \geq j_0.$$

Thus $am/y_{\mathcal{I}}^k = amy_{\mathcal{I}}^l/y_{\mathcal{I}}^{k+l} = 0$, because $my_{\mathcal{I}}^l \in M_t$. Thus we have

$$\dim_{P_0} (M_{y_{\mathcal{I}}})_j = \dim_{P_0} P_0 / \text{Ann}(M_{y_{\mathcal{I}}})_j \leq \dim_{P_0} P_0 / I = \dim_{P_0} M_j.$$

Since $H_{P_+}^i(M)_j$ is a subquotient of the j -th graded component of $C(M)^i$, the desired result follows. \square

Now we can state the main result of this section as follows

Theorem 2.4. *Let M be a finitely generated bigraded P -module such that*

$$\dim_{P_0} M/P_+M \leq 1.$$

Then for all i the functions $f_{i,M}(j) = \text{reg } H_{P_+}^i(M)_j$ are bounded.

In a first step we prove the following

Proposition 2.5. *Let M be a finitely generated bigraded P -module with*

$$\dim_{P_0} M/P_+M \leq 1.$$

Then the function $f_{n,M}(j) = \text{reg } H_{P_+}^n(M)_j$ is bounded above.

Proof. By the bigraded version of Hilbert's syzygy theorem, M has a bigraded free resolution of the form

$$\mathbb{F} : 0 \rightarrow F_k \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

where $F_i = \bigoplus_{k=1}^{t_i} P(-a_{ik}, -b_{ik})$. Applying the functor $H_{P_+}^n(-)_j$ to this resolution yields a graded complex of free P_0 -modules

$$H_{P_+}^n(\mathbb{F})_j : 0 \rightarrow H_{P_+}^n(F_k)_j \rightarrow \dots \rightarrow H_{P_+}^n(F_1)_j \rightarrow H_{P_+}^n(F_0)_j \rightarrow H_{P_+}^n(M)_j \rightarrow 0.$$

Theorem 1.1, together with Proposition 2.3, Lemma 2.1 and our assumption imply that for $j \gg 0$ we have

$$\dim_{P_0} H_i(H_{P_+}^n(\mathbb{F})_j) = \dim_{P_0} H_{P_+}^{n-i}(M)_j \leq \dim_{P_0} M/P_+M \leq 1 \leq i \quad \text{for all } i \geq 1.$$

Moreover we know that

$$H_{P_+}^n(M) = H_0(H_{P_+}^n(\mathbb{F})).$$

Then by a theorem of Lazardsfeld [6, Lemma 1.6], see also [4, Theorem 12.1], one has

$$\text{reg } H_{P_+}^n(M)_j = \text{reg } H_0(H_{P_+}^n(\mathbb{F}))_j \leq \max\{b_i(H_{P_+}^n(\mathbb{F})_j) - i \quad \text{for all } i \geq 0\}$$

where $b_i(H_{P_+}^n(\mathbb{F})_j)$ is the maximal degree of the generators of $H_{P_+}^n(F_i)_j$. Note that

$$H_{P_+}^n(F_i)_j = \bigoplus_{k=1}^{t_i} \bigoplus_{|a|=-n-j+b_{ik}} P_0(-a_{ik})z^a.$$

Thus we conclude that

$$\text{reg } H_{P_+}^n(M)_j \leq \max_{i,k} \{a_{ik} - i\} = c \quad \text{for } j \gg 0,$$

as desired. \square

Next we want to give a lower bound for the functions $f_{i,M}$. We first prove

Proposition 2.6. *Let*

$$\mathbb{G} : 0 \rightarrow G_p \xrightarrow{d_p} G_{p-1} \rightarrow \cdots \rightarrow G_1 \xrightarrow{d_1} G_0 \rightarrow 0,$$

be a complex of free P_0 -modules, where $G_i = \bigoplus_j P_0(-a_{ij})$ for all $i \geq 0$. Let $m_i = \min_j \{a_{ij}\}$. Then

$$\text{reg } H_i(\mathbb{G}) \geq m_i.$$

Proof. Since $H_i(\mathbb{G}) = \text{Ker } d_i / \text{Im } d_{i+1}$ and $\text{Ker } d_i \subseteq G_i$ for all $i \geq 0$, it follows that

$$\begin{aligned} \text{reg } H_i(\mathbb{G}) &\geq \text{largest degree of generators of } H_i(\mathbb{G}) \\ &\geq \text{lowest degree of generators of } H_i(\mathbb{G}) \\ &\geq \text{lowest degree of generators of } \text{Ker } d_i \\ &\geq \text{lowest degree of generators of } G_i \\ &= m_i, \end{aligned}$$

as desired. \square

Corollary 2.7. *Let M be a finitely generated bigraded P -module. Then for each i , the function $f_{i,M}$ is bounded below.*

Proof. Let \mathbb{G} be the complex $H_{P_+}^n(\mathbb{F})_j$ in the proof of Proposition 2.5, then the assertion follows from Proposition 2.6. \square

Proof of Theorem 2.4. Because of Corollary 2.7 it suffices to show that for each i , $f_{i,M}$ is bounded above.

There exists an exact sequence $0 \rightarrow U \rightarrow F \xrightarrow{\varphi} M \rightarrow 0$ of finitely generated bigraded P -modules where F is free. This exact sequence yields the exact sequence of P_0 -modules

$$0 \rightarrow H_{P_+}^{n-1}(M)_j \rightarrow H_{P_+}^n(U)_j \rightarrow H_{P_+}^n(F)_j \xrightarrow{\varphi} H_{P_+}^n(M)_j \rightarrow 0.$$

Let $K_j := \text{Ker } \varphi$. We consider the exact sequences

$$0 \rightarrow K_j \rightarrow H_{P_+}^n(F)_j \rightarrow H_{P_+}^n(M)_j \rightarrow 0$$

$$0 \rightarrow H_{P_+}^{n-1}(M)_j \rightarrow H_{P_+}^n(U)_j \rightarrow K_j \rightarrow 0.$$

Thus we have

$$(2) \quad \text{reg } K_j \leq \max\{\text{reg } H_{P_+}^n(F)_j, \text{reg } H_{P_+}^n(M)_j + 1\}$$

$$(3) \quad \text{reg } H_{P_+}^{n-1}(M)_j \leq \max\{\text{reg } H_{P_+}^n(U)_j, \text{reg } K_j + 1\}.$$

Let $F = \bigoplus_{i=1}^k P(-a_i, -b_i)$, then

$$H_{P_+}^n(F)_j = \bigoplus_{i=1}^k \bigoplus_{|a|=-n-j+b_i} P_0(-a_i)z^a.$$

Therefore, $\text{reg } H_{P_+}^n(F)_j = \max_i \{a_i\}$. By Proposition 2.7, the functions $f_{n,M}$ and $f_{n,U}$ are bounded above, so that, by the inequalities (2) and (3), $f_{n-1,M}$ is bounded above. To complete our proof, for $i > 1$ we see that $H_{P_+}^{n-i}(M)_j \cong H_{P_+}^{n-i+1}(U)_j$. Thus $f_{n-i,M} = f_{n-i+1,U}$ for $i > 1$. By induction on $i > 1$ all $f_{i,M}$ are bounded above, as required.

3. THE HILBERT FUNCTION OF THE COMPONENTS OF THE TOP LOCAL COHOMOLOGY OF A HYPERSURFACE RING

Let R be a hypersurface ring. In this section we want to show that the Hilbert function P_0 -module $H_{P_+}^n(R)_j$ is a nonincreasing function in j . Let $f \in P$ be a bihomogeneous of degree (a, b) . Write $f = \sum_{\substack{|\alpha|=a \\ |\beta|=b}} c_{\alpha\beta} x^\alpha y^\beta$ where $c_{\alpha\beta} \in K$. We may also write $f = \sum_{|\beta|=b} f_\beta y^\beta$ where $f_\beta \in P_0$ with $\deg f_\beta = a$. The monomials y^β which $|\beta| = b$ are ordered lexicographically induced by $y_1 > y_2 > \cdots > y_n$. We consider the hypersurface ring $R = P/fP$. From the exact sequence

$$0 \rightarrow P(-a, -b) \xrightarrow{f} P \rightarrow P/fP \rightarrow 0,$$

we get an exact sequence of P_0 -modules

$$\bigoplus_{|c|=-n-j+b} P_0(-a)z^c \xrightarrow{f} \bigoplus_{|c|=-n-j} P_0z^c \rightarrow H_{P_+}^n(R)_j \rightarrow 0.$$

We also order the bases elements z^c lexicographically induced by $z_1 > z_2 > \cdots > z_n$. Applying f to the bases elements we obtain $fz^c = \sum_{|\beta|=b} f_\beta z^{\beta-c}$, where $z^{\beta-c} = 0$ if $\beta \not\leq c$ componentwise. With respect to these bases the map of free P_0 -modules

is given by a $\binom{-j-1}{n-1} \binom{-j+b-1}{n-1}$ matrix which we denote by U_j . This matrix also describes the image of this map as submodule of the free module F_j where $F_j = \bigoplus_{|c|=-n-j} P_0 z^c$, so that $H_{P_+}^n(R)_j$ is just $\text{Coker } f = F_j/U_j$. Note that $H_{P_+}^n(R)_j = 0$ for all $j > -n$.

Let B_d denote the set of all monomials of degree d in the indeterminates z_1, \dots, z_n . Let $h = \sum_{v \in B_{-n-j}} h_v v \in U_j$ where $h_v \in P_0$ for all v . Then $h_u u$ is called the *initial term* of h if $h_u \neq 0$ and $h_v = 0$ for all $v > u$, and we set $\text{in}(h) = h_u u$. The polynomial $h_u \in P_0$ is called the *initial coefficient* and the monomial u is called the *initial monomial* of h .

Now for a monomial $u \in B_{-n-j}$ we denote $U_{j,u}$ the set of elements in U_j whose initial monomial is u , and we denote by $I_{j,u}$ the ideal generated by the initial coefficients of the elements in $U_{j,u}$.

Note that

$$U_j \setminus \{0\} = \bigcup_{u \in B_{-n-j}} U_{j,u}.$$

We fix the lexicographical order introduced above, and let $\text{in}(U_j)$ be the submodule generated by $\{\text{in}(h) : h \in U_j\}$. Then

$$(4) \quad \text{in}(U_j) = \bigoplus_{u \in B_{-n-j}} I_{j,u} u.$$

Proposition 3.1. *With the above notation we have*

$$I_{j,u} = I_{j-1, z_1 u} \quad \text{for all } j \leq -n \quad \text{and } u \in B_{-n-j}.$$

Proof. Let $h_0 \in I_{j,u}$. Then there exists $h \in U_j$ such that $h = h_0 u + \text{lower terms}$. We set $k = -n - j + b$, for short. Since h is in the image of f , we may also write $h = \sum_{|c|=k} f_c f z^c$ where $f_c \in P_0$ and $f z^c = \sum_{\beta \leq c} f_\beta z^{c-\beta}$. We define $g = \sum_{|c|=k} f_c f z^{c+e_1}$ where $f z^{c+e_1} = \sum_{\beta \leq c+e_1} f_\beta z^{c+e_1-\beta}$ and $e_1 = (1, 0, \dots, 0)$. We see that $g \in U_{j-1}$. We may write

$$g = \sum_{|c|=k} f_c \sum_{\beta \leq c} f_\beta z^{c+e_1-\beta} + \sum_{|c|=k} f_c \sum_{\substack{\beta \leq c \\ \beta \leq c+e_1}} f_\beta z^{c+e_1-\beta}.$$

Thus we conclude that $g = z_1 h + h_1$ where

$$h_1 = \sum_{|c|=k} f_c \sum_{\substack{\beta \leq c \\ \beta \leq c+e_1}} f_\beta z^{c+e_1-\beta}.$$

We now claim that h_1 does not contain z_1 as a factor. For each $\alpha \in \mathbb{N}^n$ we denote by $\alpha(i)$ the i -th component of α . Assume that $(c + e_1 - \beta)(1) > 0$ for some β appearing in the sum of h_1 . Then $c(1) \geq \beta(1)$. Moreover, if $i > 1$, then $(c + e_1 - \beta)(i) \geq 0$ implies that $c(i) \geq \beta(i)$. Hence $c(i) \geq \beta(i)$ for all i , a contradiction. It follows that $\text{in}(g) = \text{in}(h) z_1$. Therefore $h_u \in I_{j-1, z_1 u}$.

Conversely, suppose $h_0 \in I_{j-1, z_1 u}$. Then there exists $g \in U_{j-1}$ such that $g = h_0 z_1 u + \text{lower terms}$. We may write $g = \sum_{|c|=k} f'_c f z^{c+e_1}$ where $f'_c \in P_0$ and $f z^{c+e_1} =$

$\sum_{\beta \leq c+e_1} f_\beta z^{c+e_1-\beta}$. Thus

$$g = \sum_{|c|=k} f'_c \sum_{\beta \leq c} f_\beta z^{c+e_1-\beta} + \sum_{|c|=k} f'_c \sum_{\substack{\beta \leq c \\ \beta \leq c+e_1}} f_\beta z^{c+e_1-\beta}.$$

As above we see that $g = z_1 f' +$ lower terms, where $f' = \sum_{|c|=k} f'_c f z^c$. We see that $f' \in U_j$, and that $\text{in}(f')z_1 = \text{in}(g) = h_0 z_1 u$. Therefore, $\text{in}(f') = h_0 u$, and hence $h_0 \in I_{j,u}$. \square

Let M and N be graded P_0 -modules. We denote by $\text{Hilb}(M) = \sum_{i \in \mathbb{Z}} \dim_K M_i t^i$ the Hilbert-series of M . We write $\text{Hilb}(M) \leq \text{Hilb}(N)$ when $\dim_K M_i \leq \dim_K N_i$ for all i .

Let F be a free P_0 -module with basis $\beta = \{u_1, \dots, u_r\}$. Let U be a graded submodule of F . For $f \in U$, we write $f = \sum_{i=1}^r f_i u_i$ where $f_i \in P_0$. We set $\text{in}(f) = f_j u_j$ where $f_j \neq 0$ and $f_i = 0$ for all $i < j$. We also set $\text{in}(U)$ be the submodule of F generated by all $\text{in}(f)$ such that $f \in U$. Let I be a homogeneous ideal of P_0 . We say that set of homogeneous elements of P_0 forms a K -basis for P_0/I if its image forms a K -basis for P_0/I . Now we can state the following version of Macaulay's theorem, whose proof we include for the convenience of the reader.

Lemma 3.2. *With notation as above we have*

$$\text{Hilb}(F/U) = \text{Hilb}(F/\text{in}(U)).$$

Proof. As in (4) we have $\text{in}(U) = \bigoplus_{i=1}^r I_{u_i} u_i$ where I_{u_i} is the ideal generated by all $f_i \in P_0$ such that there exists $f \in F$ with $\text{in}(f) = f_i u_i$. Thus we have $F/\text{in}(U) = \bigoplus_{i=1}^r P_0/I_{u_i}$. For each j let β_j be a set of homogeneous elements $h_{ij} \in P_0$ which forms a K -basis of P_0/I_{u_j} . Then $\beta = \{\beta_1 u_1, \dots, \beta_r u_r\}$ is a homogeneous K -basis of $F/\text{in}(U)$. To complete our proof we will show that β is also a K -basis of F/U . We first show that the elements of β in F/U are linearly independent. Suppose that in F/U , we have $\sum_{i,j} a_{ij} h_{ij} u_j = 0$ with $a_{ij} \in K$. Thus $\sum_{j=1}^r (\sum_i a_{ij} h_{ij}) u_j \in U$. We set $h_j = \sum_i a_{ij} h_{ij}$, so that $h_1 u_1 + \dots + h_r u_r \in U$. If all $h_j = 0$, then $a_{ij} = 0$ for all i and j , as required. Assume that $h_j \neq 0$ for some j , and let k be the smallest integer such that $h_k \neq 0$. It follows that $h_k u_k + h_{k+1} u_{k+1} + \dots \in U$, so that $h_k \in I_k$, and hence $\sum_i a_{ik} h_{ik} = 0$ modulo I_k . Since h_{ik} are part of a K -basis of P_0/I_k , it follows that $a_{ik} = 0$ for all i , and hence $h_k = 0$, a contradiction.

Now we want to show that each element in F/U can be written as a K -linear combination of elements of β . Let $f + U \in F/U$ where $f \in F$. Thus there exists $f_i \in P_0$ such that $f = \sum_{i=1}^r f_i u_i$. Since $f_1 + I_{u_1} \in P_0/I_{u_1}$, there exists $\lambda_{i1} \in K$ such that $f_1 + I_{u_1} = \sum_i \lambda_{i1} (h_{i1} + I_{u_1})$, so that $f_1 = \sum_i \lambda_{i1} h_{i1} + h_{u_1}$ for some $h_{u_1} \in I_{u_1}$. Hence

$$f = \sum_i \lambda_{i1} h_{i1} u_1 + h_{u_1} u_1 + \sum_{i=2}^r f_i u_i.$$

We set

$$f' = f - \sum_i \lambda_{i1} h_{i1} u_1 = h_{u_1} u_1 + \sum_{i=2}^r f_i u_i.$$

Since $h_{u_1} \in I_{u_1}$, there exist $g_2, \dots, g_r \in P_0$ such that $h_{u_1}u_1 + \sum_{i=2}^r g_i u_i \in U$. Therefore, $h_{u_1}u_1 = -\sum_{i=2}^r g_i u_i$ modulo U . Hence it follows that

$$f' = -\sum_{i=2}^r g_i u_i + \sum_{i=2}^r f_i u_i = \sum_{i=2}^r f'_i u_i \quad \text{modulo } U.$$

Here $f'_i = -g_i + f_i$ for $i = 2, \dots, r$. By induction on the number of summands, we may assume that $\sum_{i=2}^r f'_i u_i$ is a linear combination of elements of β modulo U . Since f differs from f' only by a linear combination of elements of β , the assertion follows. \square

Now we are able to prove that the Hilbert-series of the P_0 -module $H_{P_+}^n(R)_j$ is a nonincreasing function in j .

Theorem 3.3. *Let $R = P/fP$ be a hypersurface ring. Then*

$$\text{Hilb}(H_{P_+}^n(R)_{j-1}) \geq \text{Hilb}(H_{P_+}^n(R)_j) \quad \text{for all } j \leq -n.$$

Proof. Let $F_j = \bigoplus_{u \in B_{-n-j}} P_0 u$ where $u = z_1^{a_1} \dots z_n^{a_n}$ with $\sum_{i=1}^n a_i = -n - j$. In view of (4) we have $F_j/\text{in}(U_j) = \bigoplus_{u \in B_{-n-j}} P_0/I_{j,u}$. By Lemma 3.2 we know that F_j/U_j and $F_j/\text{in}(U_j)$ have the same Hilbert function. Thus Proposition 3.1 implies that for all $j \leq -n$ we have

$$\begin{aligned} \text{Hilb}(H_{P_+}^n(R)_j) &= \text{Hilb}(F_j/U_j) = \sum_i \dim_K \left(\bigoplus_{u \in B_{-n-j}} P_0/I_{j,u} \right) i^i \\ &= \sum_i \sum_{u \in B_{-n-j}} \dim_K(P_0/I_{j,u}) i^i \\ &= \sum_i \sum_{u \in B_{-n-j}} \dim_K(P_0/I_{j-1, z_1 u}) i^i \\ &= \sum_i \sum_{\substack{v \in B_{-n-j+1} \\ a_1 > 0}} \dim_K(P_0/I_{j-1, v}) i^i \\ &\leq \sum_i \sum_{v \in B_{-n-j+1}} \dim_K(P_0/I_{j-1, v}) i^i \\ &= \sum_i \dim_K \left(\bigoplus_{v \in B_{-n-j+1}} P_0/I_{j-1, v} \right) i^i = \text{Hilb}(H_{P_+}^n(R)_{j-1}), \end{aligned}$$

as desired. \square

Corollary 3.4. *Let R be the hypersurface ring P/fP such that the P_0 -module $H_{P_+}^n(R)_j$ has finite length for all j . Then*

$$\text{reg } H_{P_+}^n(R)_{j-1} \geq \text{reg } H_{P_+}^n(R)_j \quad \text{for all } j \leq -n.$$

Proof. The assertion follows from the fact that

$$\text{reg } H_{P_+}^n(R)_j = \text{deg Hilb}(H_{P_+}^n(R)_j).$$

\square

Now one could ask when P_0 -module $H_{P_+}^n(R)_j$ is of finite length. To answer this question we need some preparation. Let A be a Noetherian ring and M be a finitely generated A -module with presentation

$$A^m \xrightarrow{\varphi} A^n \rightarrow M \rightarrow 0.$$

Let U be the corresponding matrix of the map φ and $I_{n-i}(U)$ for $i = 0, \dots, n-1$ be the ideal generated by the $(n-i)$ -minors of matrix U . Then $\text{Fitt}_i(M) := I_{n-i}(U)$ is called the i -th Fitting ideal of M . We use the convention that $\text{Fitt}_i(M) = 0$ if $n-i > \min\{n, m\}$, and $\text{Fitt}_i(M) = A$ if $i \geq n$. In particular, we obtain $\text{Fitt}_r(M) = 0$ if $r < 0$, $\text{Fitt}_0(M)$ is generated by the n -minors of U , and $\text{Fitt}_{n-1}(M)$ is generated by all entries of U . Note that $\text{Fitt}_i(M)$ is an invariant on M , i.e. independent of the presentation. By [5, Proposition 20.7] we have $\text{Fitt}_0(M) \subseteq \text{Ann } M$ and if M can be generated by r element, then $(\text{Ann } M)^r \subseteq \text{Fitt}_0(M)$. Thus we can conclude that $\sqrt{\text{Fitt}_0(M)} = \sqrt{\text{Ann } M}$. Therefore

$$(5) \quad \dim M = \dim A / \text{Ann } M = \dim A / I_n(U).$$

Now we can state the following

Proposition 3.5. *Let R be the hypersurface ring P/fP , and $I(f)$ the ideal of generated by all the coefficients of f . Then $\dim_{P_0} H_{P_+}^n(R)_j \leq \dim P_0/I(f)$. In particular, if $I(f)$ is \mathfrak{m} -primary where $\mathfrak{m} = (x_1, \dots, x_n)$. Then P_0 -module $H_{P_+}^n(R)_j$ is of finite length for $j \leq -n$.*

Proof. As we have already seen, $H_{P_+}^n(R)_j$ has P_0 -presentation

$$P_0^{n_1}(-a) \xrightarrow{\varphi} P_0^{n_0} \rightarrow H_{P_+}^n(R)_j \rightarrow 0,$$

where $n_0 = \binom{-j-1}{n-1}$ and $n_1 = \binom{-j+b-1}{n-1}$. In view of (5) we have $\dim_{P_0} H_{P_+}^n(R)_j = \dim P_0/I_{n_0}(U_j)$ where U_j is the corresponding matrix of the map φ . By [8, Lemma 1.4] we have $\sqrt{I(f)} \subseteq \sqrt{I_{n_0}(U_j)}$. It follows that $\dim_{P_0} H_{P_+}^n(R)_j \leq \dim P_0/I(f)$. Since $I(f)$ is \mathfrak{m} -primary it follows that $\dim P_0/I(f) = 0$. Therefore $\dim_{P_0} H_{P_+}^n(R)_j = 0$, and hence $H_{P_+}^n(R)_j$ has finite length, as required. \square

4. THE REGULARITY OF THE GRADED COMPONENTS OF LOCAL COHOMOLOGY FOR A SPECIAL CLASS OF HYPERSURFACES

Let $A = \bigoplus_{i=0}^n A_i$ be a standard graded Artinian K -algebra, where K is a field of characteristic 0. We say that A has the weak Lefschetz property if there is a linear form l of degree 1 such that the multiplication map $A_i \xrightarrow{l} A_{i+1}$ has maximal rank for all i . This means the corresponding matrix has maximal rank, i.e., l is either injective or surjective. Such an element l is called a weak Lefschetz element on A . We also say that A has the strong Lefschetz property if there is a linear form l of degree 1 such that the multiplication map $A_i \xrightarrow{l^k} A_{i+k}$ has maximal rank for all i and k . Such an element l is called a strong Lefschetz element on A . Note that the set of all weak Lefschetz elements on A is a Zariski-open subset of the affine space A_1 , and the same holds for the set of all strong Lefschetz elements on A . For an algebra A as above, we say that A has the strong Stanley property (SSP) if there

exists $l \in A_1$ such that $l^{n-2i} : A_i \rightarrow A_{n-i}$ is bijective for $i = 0, 1, \dots, [n/2]$. Note that the Hilbert function of standard graded K -algebra satisfying the weak Lefschetz property is unimodal. Stanley [9] and Watanabe [10] proved the following result: Let a_1, \dots, a_n be the integers such that $a_i \geq 1$ and assume as always in this section that $\text{char } K = 0$. Then $A = K[x_1, \dots, x_n]/(x_1^{a_1}, \dots, x_n^{a_n})$ has the strong Lefschetz property.

Theorem 4.1. *Let $r \in \mathbb{N}$ and $f_\lambda = \sum_{i=1}^n \lambda_i x_i y_i$ with $\lambda_i \in K$ and $n \geq 2$, and assume that $\text{char } K = 0$. Then there exists a Zariski open subset $V \subset K^n$ such that for all $\lambda = (\lambda_1, \dots, \lambda_n) \in V$ one has*

$$\text{reg } H_{P_+}^n(P/f_\lambda^r P)_j = -n - j + r - 1.$$

Proof. We first prove the theorem in the case that $f = f_{(1, \dots, 1)} = \sum_{i=1}^n x_i y_i$, and set $R = P/f^r P$. From the exact sequence $0 \rightarrow P(-r, -r) \xrightarrow{f^r} P \rightarrow R \rightarrow 0$, we get an exact sequence of P_0 -modules,

$$(6) \quad \bigoplus_{|b|=-n-j+r} P_0(-r)z^b \xrightarrow{f^r} \bigoplus_{|b|=-n-j} P_0 z^b \rightarrow H_{P_+}^n(R)_j \rightarrow 0.$$

Note that $H_{P_+}^n(R)_j$ is generated by elements of degree 0 and the ideal generated by the coefficients of f is \mathfrak{m} -primary. By Proposition 3.5, we need only to show that

$$(a) [H_{P_+}^n(R)_j]_{-n-j+r-1} \neq 0, \quad \text{and (b) } [H_{P_+}^n(R)_j]_{-n-j+r} = 0.$$

Let $k = -n - j$ for short. For the proof of (a), we take the $(k+r-1)$ -th component of the exact sequence (6), and obtain the exact sequence of K -vector spaces

$$\bigoplus_{\substack{|a|=k-1 \\ |b|=k+r}} Kx^a z^b \xrightarrow{f^r} \bigoplus_{\substack{|a|=k+r-1 \\ |b|=k}} Kx^a z^b \rightarrow [H_{P_+}^n(R)_j]_{k+r-1} \rightarrow 0.$$

We set $V_{\alpha, \beta} := \bigoplus_{\substack{|a|=\alpha \\ |b|=\beta}} Kx^a z^b$. Hence one has $\dim_K V_{k-1, k+r} = \binom{n+k-2}{k-1} \binom{n+k+r-1}{k+r}$ which is less than $\dim_K V_{k+r-1, k} = \binom{n+k+r-2}{k+r-1} \binom{n+k-1}{k}$ for $n \geq 2$. Thus f^r is not surjective, so (a) follows. For the proof of (b), we take the $(k+r)$ -th component of the exact sequence (6), and obtain the exact sequence of K -vector spaces

$$\bigoplus_{\substack{|a|=k \\ |b|=k+r}} Kx^a z^b \xrightarrow{f^r} \bigoplus_{\substack{|a|=k+r \\ |b|=k}} Kx^a z^b \rightarrow [H_{P_+}^n(R)_j]_{k+r} \rightarrow 0.$$

Note that $\dim_K V_{k, k+r} = \dim_K V_{k+r, k}$. We will show that f^r is an isomorphism, then we are done. We fix $c \in \mathbb{N}_0^n$ such that $c = (c_1, \dots, c_n)$ where $c_i \geq 0$. We set

$$V_{\alpha, \beta}^c := \bigoplus_{\substack{|a|=\alpha \\ |b|=\beta \\ a+b=c}} Kx^a z^b \quad \text{and} \quad A_i^c := \bigoplus_{\substack{|a|=i \\ a \leq c}} Kx^a.$$

We define $\varphi : V_{k, k+r}^c \rightarrow A_k^c$ by setting $\varphi(x^a z^b) = x^a$. Note that φ is an isomorphism of K -vector spaces. Let $A^c = \bigoplus_{i=0}^{|c|} A_i^c$. We can define an algebra structure on A^c .

For $x^s, x^t \in A^c$ we define

$$x^s x^t = \begin{cases} x^{s+t} & \text{if } s+t \leq c, \\ 0 & \text{if } s+t \not\leq c. \end{cases}$$

A K -basis of A^c is given by all monomials x^a with $a \leq c$. It follows that

$$A^c = K[x_1, \dots, x_n]/(x_1^{c_1+1}, \dots, x_n^{c_n+1}).$$

Now we see that the map

$$V_{k,k+r} = \bigoplus_{|c|=2k+r} V_{k,k+r}^c \xrightarrow{f^r} \bigoplus_{|c|=2k+r} V_{k+r,k}^c = V_{k+r,k}$$

is an isomorphism if and only if restriction map $f' := f^r|_{V_{k,k+r}^c} : V_{k,k+r}^c \longrightarrow V_{k+r,k}^c$ is an isomorphism for all c with $|c| = 2k+r$.

For each such c we have a commutative diagram

$$\begin{array}{ccc} V_{k,k+r}^c & \xrightarrow{f'} & V_{k+r,k}^c \\ \downarrow & & \downarrow \\ A_k^c & \xrightarrow{l^r} & A_{k+r}^c, \end{array}$$

with $l = x_1 + x_2 + \dots + x_n \in A_1^c$ and where $A_k^c \xrightarrow{l^r} A_{k+r}^c$ is multiplication by l^r in the K -algebra A^c . Since the socle degree of A^c equals $s = 2k+r$, we have $k+r = s-k$. Therefore the multiplication map $l^r : A_k \rightarrow A_{s-k}$ with $r = s-2k$ is an isomorphism by the strong Stanley property of the algebra A^c , see [10, Corollary 3.5]

Now if we replace f by f_λ , then the corresponding linear form in the above commutative diagram is the form $l_\lambda = \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n$. It is known that the property of l_λ to be a weak Lefschetz element is an open condition, that is, there exists a Zariski open set $V \subset K^n$ such that l_λ is a weak Lefschetz element. This open set is not empty since $\lambda = (1, \dots, 1) \in V$. Since any weak Lefschetz element satisfies (SSP), we can replace in the above proof f by f_λ for each $\lambda \in V$, and obtain the same conclusion. \square

Remark 4.2. It is now the time that to show Theorem 2.4 may fail without the assumption that $\dim_{P_0} M/P_+M \leq 1$. In case of Theorem 4.1 we have $M = R = P/f_\lambda^r P$, and so $M/P_+M = P_0$. Therefore in that case $\dim_{P_0} M/P_+M = \dim_{P_0} P_0 = n \geq 2$, and in fact $f_{n,R}$ is not bounded.

Now in the Theorem 4.1, we want to compute the Hilbert function of the P_0 -module $H_{P_+}^n(R)_j$.

Corollary 4.3. *With the assumption of Theorem 4.1, we have*

$$\dim_K(H_{P_+}^n(R)_j)_i = \begin{cases} \binom{n+i-1}{i} \binom{-j-1}{-n-j}, & \text{if } i \leq r, \\ \binom{n+i-1}{i} \binom{-j-1}{-n-j} - \binom{n+i-r-1}{i-r} \binom{-j+r-1}{-n-j+r}, & \text{if } r \leq i \leq -n-j+r-1. \end{cases}$$

Proof. We set $-n - j = k$, for short. We take i -th component of exact sequence (6), and obtain the exact sequence of K -vector space

$$\bigoplus_{\substack{|a|=i-r \\ |b|=k+r}} Kx^a z^b \xrightarrow{f^r} \bigoplus_{\substack{|a|=i \\ |b|=k}} Kx^a z^b \rightarrow [H_{P_+}^n(R)_j]_i \rightarrow 0.$$

If $i \leq r$, from the above exact sequence we see that

$$\dim_K(H_{P_+}^n(R)_j)_i = \dim_K V_{i,k} = \binom{n+i-1}{i} \binom{-j-1}{-n-j}.$$

Now let $r \leq i \leq -n - j + r - 1$. First one has $\dim_K V_{i-r,k+r} < \dim_K V_{i,k}$. We claim that f^r is injective, then we are done. We see that the map

$$V_{i-r,k+r} = \bigoplus_{|c|=i+k} V_{i-r,k+r}^c \xrightarrow{f^r} \bigoplus_{|c|=i+k} V_{i,k}^c = V_{i,k}$$

where $f^r(V_{i-r,k+r}^c) \subset V_{i,k}^c$ is injective if and only if restriction map $f' := f^r|_{V_{i-r,k+r}^c} : V_{i-r,k+r}^c \rightarrow V_{i,k}^c$ is injective for all c with $|c| = i + k$.

For each such c we have a commutative diagram

$$\begin{array}{ccc} V_{i-r,k+r}^c & \xrightarrow{f'} & V_{i,k}^c \\ \downarrow & & \downarrow \\ A_{i-r}^c & \xrightarrow{l^r} & A_i^c \end{array}$$

with $l = x_1 + x_2 + \dots + x_n \in A_1^c$. Since $i < -n - j + r$, then $i < |c| - (i - r)$ and by the weak Lefschetz property the algebra A^c is unimodal. Therefore $\dim_K A_{i-r}^c \leq \dim_K A_i^c$. The strong Lefschetz property implies that the map l^r is injective, and hence f' is injective, as required. \square

Corollary 4.4. *With the assumption of Theorem 4.1, we have*

$$\text{reg } H_{P_+}^{n-1}(P/f_\lambda^r P)_j = -n - j + r + 1.$$

Proof. We consider the exact sequence of P_0 -modules

$$(7) \quad 0 \rightarrow H_{P_+}^{n-1}(R)_j \rightarrow \bigoplus_{|b|=-n-j+r} P_0(-r)z^b \xrightarrow{f^r} \bigoplus_{|b|=-n-j} P_0 z^b \rightarrow H_{P_+}^n(R)_j \rightarrow 0,$$

where $R = P/f_\lambda^r P$. It follows that $H_{P_+}^{n-1}(R)_j$ is the second syzygy module of $H_{P_+}^n(R)_j$. Let

$$\cdots \rightarrow \bigoplus_{j=1}^{t_2} P_0(-a_{1j}) \rightarrow \bigoplus_{j=1}^{t_1} P_0(-a_{0j}) \rightarrow H_{P_+}^{n-1}(R)_j \rightarrow 0$$

be the minimal graded free resolution of $H_{P_+}^{n-1}(R)_j$. We combine two above resolutions, and obtain a graded free resolution for $H_{P_+}^n(R)_j$ of the form

$$\cdots \rightarrow \bigoplus_{j=1}^{t_1} P_0(-a_{0j}) \xrightarrow{d_0} \bigoplus_{|b|=-n-j+r} P_0(-r)z^b \xrightarrow{f^r} \bigoplus_{|b|=-n-j} P_0 z^b \rightarrow H_{P_+}^n(R)_j \rightarrow 0.$$

We choose a basis element $h \in \bigoplus_{j=1}^{t_1} P_0(-a_{0j})$ of degree a_{0j} . Thus

$$d_0(h) = \sum_{|b|=-n-j+r} h_b z^b$$

where $h_b \in P_0$ with $\deg h_b = a_{0j} - r$. Because the free resolution is minimal, at least one $h_b \neq 0$, so that $r < a_{0j}$ and hence $r - 1 \leq a_{0j} - 2$. Thus we have

$$\operatorname{reg} H_{P_+}^n(R)_j = \max_{i,j} \{0, r - 1, a_{ij} - i - 2\} = \max_{i,j} \{a_{ij} - i - 2\}.$$

Theorem 4.1 implies that

$$\operatorname{reg} H_{P_+}^{n-1}(R)_j = \max_{i,j} \{a_{ij} - i\} = -n - j + r + 1.$$

□

Corollary 4.5. *With the assumption of the Theorem 4.1 the P_0 -module $H_{P_+}^{n-1}(P/f_\lambda^r P)_j$ has a linear resolution.*

Proof. Taking the k -th component of the exact sequence (7), we obtain the exact sequence of K -vector spaces

$$0 \rightarrow [H_{P_+}^{n-1}(R)_j]_k \rightarrow \bigoplus_{\substack{|a|=k-r \\ |b|=-n-j+r}} Kx^a z^b \xrightarrow{f^r} \bigoplus_{\substack{|a|=k \\ |b|=-n-j}} Kx^a z^b \rightarrow [H_{P_+}^n(R)_j]_k \rightarrow 0.$$

For k we distinguish several cases. Let $k = -n - j + r + 1$. One has

$$\dim_K V_{k-r, -n-j+r} > \dim_K V_{k, -n-j}.$$

This implies that

$$[H_{P_+}^{n-1}(R)_j]_k \neq 0 \quad \text{for all } k \geq -n - j + r + 1,$$

since $H_{P_+}^{n-1}(R)_j$ is torsion-free.

Let $k = -n - j + r$. Then $\dim_K V_{k-r, -n-j+r} = \dim_K V_{k, -n-j}$, so that $[H_{P_+}^{n-1}(R)_j]_k = 0$. Finally let $k < -n - j + r$. We claim that $\dim_K V_{k-r, -n-j+r} = \binom{n+k-r-1}{k-r} \binom{-j+r-1}{-n-j+r}$ is less than $\dim_K V_{k, -n-j} = \binom{n+k-1}{k} \binom{-j-1}{-n-j}$. Indeed,

$$\dim_K V_{k-r, -n-j+r} = \prod_{i=1}^r \frac{-j+r-i}{-n-j+r-i+1} \quad \text{and} \quad \dim_K V_{k, -n-j} = \prod_{i=1}^r \frac{n+k-i}{k-i+1}.$$

Since $\frac{-j+r-i}{-n-j+r-i+1} < \frac{n+k-i}{k-i+1}$ for all $i = 1, \dots, r$ if and only if $k(n-1) < (-n-j+r)(n-1)$, the claim is clear. Thus the regularity of $H_{P_+}^{n-1}(R)_j$ is equal to the least integer k such that $[H_{P_+}^{n-1}(R)_j]_k \neq 0$. This means that P_0 -module $H_{P_+}^{n-1}(R)_j$ has a linear resolution, and its resolution is the form

$$\cdots \rightarrow P_0^{\beta_3}(n+j-r-2) \rightarrow P_0^{\beta_2}(n+j-r-1) \rightarrow P_0^{\beta_1}(-r) \rightarrow P_0^{\beta_0} \rightarrow H_{P_+}^n(R)_j \rightarrow 0.$$

□

In the above resolution we know already the Betti numbers $\beta_0 = \binom{-j-1}{-n-j}$ and $\beta_1 = \binom{-j+r-1}{-n-j+r}$. Next we are going to compute the remaining Betti numbers and also the multiplicity of $H_{P_+}^n(R)_j$. For this we need to prove the following extension of the formula of Herzog and Kühn [2].

Proposition 4.6. *Let M be finitely generated graded Cohen-Macaulay P_0 -module. Let*

$$0 \rightarrow P_0^{\beta_s}(-d_s) \rightarrow \cdots \rightarrow P_0^{\beta_1}(-d_1) \rightarrow P_0^{\beta_0} \rightarrow M \rightarrow 0,$$

be the minimal graded free resolution of M where $s = \text{codim}(M)$. Then

$$\beta_i = (-1)^{i+1} \beta_0 \prod_{j \neq i} \frac{d_j}{(d_j - d_i)}$$

Proof. We consider the square matrix A of size s and the following $s \times 1$ matrices of X and Y :

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ d_1 & d_2 & \cdots & d_s \\ \vdots & \vdots & \vdots & \vdots \\ d_1^{s-1} & d_2^{s-1} & \cdots & d_s^{s-1} \end{pmatrix}, \quad X = \begin{pmatrix} -\beta_1 \\ \beta_2 \\ \vdots \\ (-1)^s \beta_s \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} -\beta_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

With similar arguments as in the proof of Lemma 1.1 in [7] one has

$$\sum_{i=1}^s (-1)^i \beta_i d_i^k = \begin{cases} 0 & \text{for } 1 \leq k < s, \\ (-1)^s s! e(M) & \text{for } k = s, \end{cases}$$

Note that $\sum_{i=1}^s (-1)^i \beta_i = \beta_0$. Thus we can conclude that $AX = Y$. Now we can apply Cramer's rule for the computation of β_i . We replace the i -th column of A by Y , then we expand the determinant $|A|$ of A along to the Y , we get $\beta_i = -\beta_0 |A'| / |A|$ where A' is the matrix

$$\begin{pmatrix} d_1 & \cdots & d_{i-1} & d_{i+1} & \cdots & d_s \\ d_1^2 & \cdots & d_{i-1}^2 & d_{i+1}^2 & \cdots & d_s^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_1^{s-1} & \cdots & d_{i-1}^{s-1} & d_{i+1}^{s-1} & \cdots & d_s^{s-1} \end{pmatrix},$$

of size $s - 1$. A is a Vandermonde matrix whose determinant is $\prod_{1 \leq j < i \leq s} (d_i - d_j)$. We also note that

$$|A'| = \prod_{j \neq i} d_j \prod_{\substack{1 \leq t < k \leq s \\ t \neq i}} (d_k - d_t),$$

so the desired formula follows.

Proposition 4.7. *With the assumption of Proposition 4.6, we have*

$$e(M) = \frac{\beta_0}{s!} \prod_{i=1}^s d_i.$$

Proof. We consider the square matrix

$$(8) \quad M = \begin{pmatrix} \beta_1 d_1 & \beta_2 d_2 & \cdots & \beta_{s-1} d_{s-1} & \beta_s d_s \\ \beta_1 d_1^2 & \beta_2 d_2^2 & \cdots & \beta_{s-1} d_{s-1}^2 & \beta_s d_s^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_1 d_1^s & \beta_2 d_2^s & \cdots & \beta_{s-1} d_{s-1}^s & \beta_s d_s^s \end{pmatrix}$$

of size s .

We will compute the determinant $|M|$ of M in two different ways. First we replace the last column of M by the alternating sum of all columns of M . The resulting matrix will be denoted by M' . It is clear that $|M| = (-1)^s |M'|$. Moreover, due to [7, Lemma 1.1], the last column of M' is the transpose of the vector $(0, \dots, 0, (-1)^s se(M))$. Thus if we expand M' with respect to the last column we get

$$|M| = (-1)^s |M'| = s! e(M) |N|$$

where N is the matrix

$$N = \begin{pmatrix} \beta_1 d_1 & \beta_2 d_2 & \cdots & \beta_{s-1} d_{s-1} \\ \beta_1 d_1^2 & \beta_2 d_2^2 & \cdots & \beta_{s-1} d_{s-1}^2 \\ \vdots & \vdots & \vdots & \vdots \\ \beta_1 d_1^{s-1} & \beta_2 d_2^{s-1} & \cdots & \beta_{s-1} d_{s-1}^{s-1} \end{pmatrix}$$

of size $s - 1$. Thus

$$(9) \quad |M| = s! e(M) \prod_{i=1}^{s-1} \beta_i \prod_{i=1}^{s-1} d_i |V(d_1, \dots, d_{s-1})|$$

where $V(d_1, \dots, d_{s-1})$ is the Vandermonde matrix of size $s - 1$ whose determinant is $\prod_{1 \leq j < i \leq s-1} (d_i - d_j)$. On the other hand, directly from (8) we get

$$(10) \quad |M| = \prod_{i=1}^s \beta_i \prod_{i=1}^s d_i |V(d_1, \dots, d_s)|$$

where $V(d_1, \dots, d_s)$ is the Vandermonde matrix of size s whose determinant is $\prod_{1 \leq j < i \leq s} (d_i - d_j)$. In view of (9) and (10) we get the desired formula. \square

Now we are able to compute all Betti numbers and the multiplicity of $H_{P_+}^n(R)_j$. We recall that its resolution is the form

$$0 \rightarrow P_0^{\beta_n}(j - r + 1) \rightarrow P_0^{\beta_{n-1}}(j - r + 2) \rightarrow \cdots \rightarrow P_0^{\beta_3}(n + j - r - 2) \rightarrow P_0^{\beta_2}(n + j - r - 1) \rightarrow P_0^{\beta_1}(-r) \rightarrow P_0^{\beta_0} \rightarrow H_{P_+}^n(R)_j \rightarrow 0,$$

where $\beta_0 = \binom{-j-1}{-n-j}$ and $\beta_1 = \binom{-j+r-1}{-n-j+r}$.

Corollary 4.8. *With the above notation we have*

$$\beta_i = \frac{(-1)^i r (n-1)! \beta_0 \beta_1}{(i-2)!(n-i)!(-n-j+r+i-1)(n+j-i+1)} \quad \text{for all } i \geq 2,$$

and

$$e(H_{P_+}^n(R)_j) = \frac{r(-j+r-1)!\beta_0}{n!(-n-j+r)!}.$$

Proof. The assertion follows from Proposition 4.7 and Proposition 4.6. \square

5. LINEAR BOUNDS FOR THE REGULARITY OF THE GRADED COMPONENTS OF LOCAL COHOMOLOGY FOR HYPERSURFACE

In this section for a bihomogenous polynomial $f \in P$ we want to give a linear bound for the function $f_{i,R}(j) = \text{reg } H_{P_+}^i(R)_j$ where $R = P/fP$. First we prove the following

Proposition 5.1. *Let R be the hypersurface ring P/fP where $f = \sum_{i=1}^n f_i y_i$ with $f_i \in P_0$. Suppose that $\deg f_i = d$ and that $I(f)$ is the \mathfrak{m} -primary. Then there exists an integer q such that for $j \ll 0$ we have*

- (a) $\text{reg } H_{P_+}^n(R)_j \leq (-n-j+1)d + q$, and
- (b) $\text{reg } H_{P_+}^{n-1}(R)_j \leq (-n-j+1)d + q + 2$.

Proof. (a) From the exact sequence $0 \rightarrow P(-d, -1) \xrightarrow{f} P \rightarrow R \rightarrow 0$, we get exact sequence P_0 -modules

$$(11) \quad \bigoplus_{|b|=-n-j+1} P_0(-d)z^b \xrightarrow{f} \bigoplus_{|b|=-n-j} P_0z^b \rightarrow H_{P_+}^n(R)_j \rightarrow 0.$$

We first assume that $f_i = x_i$. Theorem 4.1 implies that $\text{reg } H_{P_+}^n(R)_j = -n-j$. We set $k = -n-j$. Thus we can get the surjective map of K -vector spaces

$$\bigoplus_{\substack{|a|=k \\ |b|=k+1}} Kx^a z^b \longrightarrow \bigoplus_{\substack{|a|=k+1 \\ |b|=k}} Kx^a z^b.$$

Replacing x_i by f_i , we therefore get a surjective map

$$\begin{aligned} \bigoplus_{|b|=k+1} (I(f)^k)_{dk} z^b &= \bigoplus_{\substack{|a|=k \\ |b|=k+1}} K f_1^{a_1} \dots f_n^{a_n} z^b \xrightarrow{f} \\ & \bigoplus_{\substack{|a|=k+1 \\ |b|=k}} K f_1^{a_1} \dots f_n^{a_n} z^b = \bigoplus_{|b|=k} (I(f)^{k+1})_{d(k+1)} z^b. \end{aligned}$$

Since $I(f)$ is \mathfrak{m} -primary by [3, Theorem 2.4] there exists an integer q such that

$$\text{reg}(P_0/I(f)^{k+1}) = (k+1)d + q \quad \text{for } k \gg 0.$$

We set $l = (k+1)d + q$. Then for $l \gg 0$ we have

$$(P_0)_{l+1} = (I(f)^{k+1})_{l+1}.$$

We take the $(l+1)$ -th component of the exact sequence (11) and consider the following diagram

$$\begin{array}{ccccccc}
\bigoplus_{|b|=k+1} (P_0)_{l-d+1} z^b & \longrightarrow & \bigoplus_{|b|=k} (P_0)_{l+1} z^b & \longrightarrow & [H_{P_+}^n(R)]_{l+1} & \longrightarrow & 0 \\
\uparrow & & \parallel & & & & \\
\bigoplus_{|b|=k+1} (I(f)^k)_{l-d+1} z^b & \longrightarrow & \bigoplus_{|b|=k} (I(f)^{k+1})_{l+1} z^b & \longrightarrow & & & 0,
\end{array}$$

in which left-hand vertical homomorphism is inclusion. Thus we conclude that $[H_{P_+}^n(R)]_{l+1} = 0$, so that $\text{reg } H_{P_+}^n(R)_j \leq l = (k+1)d + q$, as required.

For the proof (b), we notice that the exact sequence of P_0 -modules of (11) breaks into two short exact sequence of P_0 -modules

$$0 \rightarrow K_j \rightarrow \bigoplus_{|b|=k} P_0 z^b \rightarrow H_{P_+}^n(R)_j \rightarrow 0,$$

$$0 \rightarrow H_{P_+}^{n-1}(R)_j \rightarrow \bigoplus_{|b|=k+1} P_0(-d)z^b \rightarrow K_j \rightarrow 0,$$

where $K_j = \text{Im } f$. We see from the first of these sequences that $\text{reg } K_j \leq \text{reg } H_{P_+}^n(R)_j + 1$. The second short exact sequence, together with part (a) this theorem and the fact that $d \leq \text{reg } K_j$ implies that

$$\text{reg } H_{P_+}^{n-1}(R)_j \leq \max\{d, \text{reg } K_j + 1\} = \text{reg } K_j + 1 \leq (-n - j + 1)d + q + 2,$$

as desired. \square

Proposition 5.2. *Let $\mathbb{N}_d^n = \{\beta \in \mathbb{N}^n : |\beta| = d\}$, $P_0 = K[\{x_\beta\}_{\beta \in \mathbb{N}_d^n}]$ and $P = P_0[y_1, \dots, y_n]$. Let $R = P/fP$ where $f = \sum_{|\beta|=d} x_\beta y^\beta$. Then*

$$\text{reg } H_{P_+}^n(R)_j \leq (-n - j + 1)d - 1$$

Proof. We set $P_+ = (y_1, \dots, y_n)$ and $P_0 = K[x_1, \dots, x_m]$ where $m = \binom{n+d-1}{d}$, as usual. From the exact sequence $0 \rightarrow P(-1, -d) \xrightarrow{f} P \rightarrow R \rightarrow 0$, we get the exact sequence of P_0 -modules

$$\bigoplus_{|b|=-n-j+d} P_0(-1)(y^b)^* \xrightarrow{f} \bigoplus_{|b|=-n-j} P_0(y^b)^* \rightarrow H_{P_+}^n(R)_j \rightarrow 0,$$

whose i -th graded component is

$$(12) \quad \bigoplus_{\substack{|a|=i-1 \\ |b|=-n-j+d}} Kx^a(y^b)^* \xrightarrow{f} \bigoplus_{\substack{|a|=i \\ |b|=-n-j}} Kx^a(y^b)^* \rightarrow H_{P_+}^n(R)_{(i,j)} \rightarrow 0.$$

Here $(y^b)^* = z^b$ in the notation of Section 1. Now we exchange the role of x and y : We may write $f = \sum_{|\beta|=d} y^\beta x_\beta$ and set $Q_+ = (x_1, \dots, x_m)$ and $Q_0 = K[y_1, \dots, y_n]$.

From the exact sequence $0 \rightarrow P(-d, -1) \xrightarrow{f} P \rightarrow R \rightarrow 0$, we get the exact sequence of P_0 -modules

$$\bigoplus_{|b|=-m-t+1} Q_0(-d)(x^b)^* \xrightarrow{f} \bigoplus_{|b|=-m-t} Q_0(x^b)^* \rightarrow H_{Q_+}^m(R)_t \rightarrow 0.$$

whose s -th graded component is

$$\bigoplus_{\substack{|a|=s-d \\ |b|=-m-t+1}} Ky^a(x^b)^* \xrightarrow{f} \bigoplus_{\substack{|a|=s \\ |b|=-m-t}} Ky^a(x^b)^* \rightarrow H_{Q_+}^m(R)_{(s,t)} \rightarrow 0.$$

Applying the functor $\text{Hom}_K(-, K)$ to the above exact sequence and due to the exact sequence (12) we have

$$\begin{aligned} 0 \rightarrow H_{Q_+}^m(R)_{(s,t)}^* \rightarrow & \bigoplus_{\substack{|a|=s \\ |b|=-m-t}} K(y^a)^* x^b \xrightarrow{f} \\ & \bigoplus_{\substack{|a|=s-d \\ |b|=-m-t+1}} K(y^a)^* x^b \rightarrow H_{P_+}^n(R)_{(-m-t+1, -n-s+d)} \rightarrow 0. \end{aligned}$$

Therefore

$$H_{Q_+}^m(R)_{(s,t)}^* \cong H_{P_+}^{n-1}(R)_{(-m-t+1, -n-s+d)}.$$

Thus we have

$$\begin{aligned} 0 \rightarrow (H_{P_+}^{n-1}(R)_{-n-s+d})_{-m-t+1} \rightarrow & \bigoplus_{\substack{|a|=s \\ |b|=-m-t}} K(y^a)^* x^b \xrightarrow{f} \\ & \bigoplus_{\substack{|a|=s-d \\ |b|=-m-t+1}} K(y^a)^* x^b \rightarrow (H_{P_+}^n(R)_{-n-s+d})_{-m-t+1} \rightarrow 0. \end{aligned}$$

We set $j = -n - s + d$. Proposition 5.1 implies that $\text{reg } H_{P_+}^n(R)_j \leq (-n - j + 1)d + q$ for some q . Since $I(f) = (y_1, \dots, y_n)^d$, thus $\text{reg}(P_0/I(f)^{k+1}) = (k+1)d - 1$. Hence in Proposition 5.1 we have $q = -1$. \square

Now the main result of this section is the following

Theorem 5.3. *Let $P = K[x_1, \dots, x_m, y_1, \dots, y_n]$, and $f \in P$ be a bihomogenous polynomial such that $I(f)$ is \mathfrak{m} -primary. Let $R = P/fP$. Then the regularity of $H_{P_+}^n(R)_j$ is linearly bounded.*

Proof. We may write $f = \sum_{|\beta|=d} f_\beta y^\beta$ and let $\deg f_\beta = c$. From the exact sequence $0 \rightarrow P(-c, -d) \xrightarrow{f} P \rightarrow R \rightarrow 0$, we get the exact sequence of P_0 -modules

$$\bigoplus_{|b|=-n-j+d} P_0(-c)z^b \xrightarrow{f} \bigoplus_{|b|=-n-j} P_0z^b \rightarrow H_{P_+}^n(R)_j \rightarrow 0,$$

We first assume that $f_\beta = x_\beta$. Proposition 5.2 implies that $\text{reg } H_{P_+}^n(R)_j \leq (-n - j + 1)d - 1$. We set $k = (-n - j + 1)d$. Thus we get the surjective map of K -vector spaces

$$\bigoplus_{\substack{|a|=k-1 \\ |b|=-n-j+d}} Kx^a z^b \longrightarrow \bigoplus_{\substack{|a|=k \\ |b|=-n-j}} Kx^a z^b.$$

We proceed as in the proof of Proposition 5.1, and we get $[H_{P_+}^n(R)_j]_{kd+q'+1} = 0$ for some q' . Therefore $\text{reg } H_{P_+}^n(R)_j \leq (-n - j + 1)d^2 + q'$. \square

Corollary 5.4. *With the assumption of Theorem 5.3, we have*

$$\operatorname{reg} H_{P_+}^{n-1}(R)_j \leq (-n - j + 1)d^2 + q' + 2.$$

Proof. For the proof one use the same argument as in the proof of Proposition 5.1(b).

□

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AHAD RAHIMI, FACHBEREICH MATHEMATIK UND INFORMATIK, UNIVERSITÄT DUISBURG-ESSEN, CAMPUS ESSEN, 45117 ESSEN, GERMANY

E-mail address: `ahad.rahimi@uni-essen.de`