Schwerpunktseminar Universität Essen.
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FOURIER–MUKAI TRANSFORMATION AND ABELIAN VARIETIES

Abstract. We would like to learn about the Fourier–Mukai transform [6] using Polishchuk's book [9]. Our concrete goal is to understand the proof of the Torelli theorem given by Beilinson and Polishchuk in [2].

We will try to cover the relevant parts of [9]. These are Chapters 8-11, 16, 17 and 19-21. Due to limited time we have to leave out many interesting topics, particularly the transcendental side of the theory will be ignored unfortunately.

The seminar is divided into four sessions, corresponding to Friday morning/afternoon and Saturday morning/afternoon. Each session consists of 2–3 talks of *strictly less* than 60 minutes. Following is a more detailed description of what will be covered.

1. Crash course abelian varieties

Participants are expected to familiarize themselves with some standard results on abelian varieties, as for example in Chapter 8 & 9. These will be discussed only briefly in the first talk. With some more detail we cover the construction of the dual abelian variety and Poincaré bundle, and the material of Chapter 10. You also might want to consult other sources, for example Mumford [7] or Milne [3, 4].

Talk 1 (Theorem of cube and dual abelian variety)**.** Recall rigidity lemma, theorem of the cube, theorem of the square, homogeneous line bundles (in particular Proposition 8.7 and Theorem 8.8) and criterion for ampleness. Seesaw, construction of the dual abelian variety and the Poincaré bundle (Chapter 8 & 9).

Talk 2 (Duality and biextensions)**.** Dual morphism, symmetric morphisms, (principal) polarizations. Cartier duality. The canonical map is an isomorphism (Corollary 10.2). When is a line bundle a pull-back (Theorem 10.5). Symmetric morphisms come from line bundles (Theorem 13.7) (Chapter 9 & 10).

2. The Fourier–Mukai transform

The Fourier–Mukai transform is put to good use in order to proof Theorem 9.4, saying that the dual abelian variety represents Pic^0. After that we discuss in detail the definition and basic properties of the Fourier–Mukai transform. This is essentially Chapter 11, maybe including bits from Chapter 14 or 15 as illustration at the end.

Talk 3 (The dual abelian variety represents Pic^0)**.** This talk is devoted to the proof of Theorem 9.4, using (a variant of) the Fourier–Mukai Transform, as described in 11.1 and 11.2.

Talk 4 (Basic properties of the Fourier–Mukai Transform)**.** These are covered in 11.3 and 11.4. and should all be discussed carefully.

3. Jacobians of curves

We recall basics about Jacobians of curves, their principal polarization, theta divisors and the Poincaré bundle. A more detailed study of properties of the g-fold symmetric product of a curve follows and leads the way to our main goal: a proof of the Torelli theorem. The material is covered in Chapters 16, 17 and 19. One might also look at [7, 8, 5]

Talk 5 (Construction of the Jacobian)**.** Chapter 16 explains how the Jacobian of a curve is obtained by gluing (translates of) an open piece of the g-fold symmetric product. We need that the d-fold symmetric product parameterizes effective divisors of degree d (Theorem 16.4). Construction of the Poincaré bundle and moduli property of the Jacobian.

Talk 6 (Theta divisors)**.** Principal polarization of Jacobians and determinantal description as explained in 17.1–17.3.

Talk 7 (Symmetric powers of Curves)**.** The locus over which the d-th symmetric product of a curve maps isomorphically to the Jacobian (Corollary 19.3). Description of the Picard group of the d-th symmetric power of a curve as in Theorem 19.7. Exercise 19.1 should be discussed as it will be needed later.

4. Torelli Theorem

We proceed to the proof of the Torelli Theorem as given in Chapter 20 and 21.

Talk 8 (Varieties of special divisors)**.** We consider the loci W_d^r of line bundles L of degree d on a curve such that $h^0(L) > r$. The singular locus of W_d^0 is W_d^1 (Corollary 20.3), there are dimension estimates for W_d^r (Theorem 20.4). From this we obtain the non-singularity of a Theta divisor in codimension 1 (Corollary 20.5). You can also consult [1].

Talk 9 (The Torelli theorem)**.** The Torelli theorem tells us how to recover a curve from its Jacobian and a theta divisor (Theorem 21.1), see also the original source [2].

References

[1] E. Arabello, M. Cornalba, P.a. Griffiths, J. Harris. *Geometry of Algebraic Curves, Vol. I.* Springer, New York, 1984.
[2] A. Beilinson and A. Polishchuk. Torelli theorem via Fourier–Mukai Transform. In *Moduli of Abelian Varieties*, volume 195 of *Progr. Math.*, pages 127–132. Birkhäuser, Basel, 2001.
[3] J. Milne. *Abelian Varieties.* Course notes available at `www.jmilne.org`.
[4] J. Milne. Abelian Varieties. In *Arithmetic Geometry*, ed. Cornell, Silverman, Chapter 5, Springer, 1998.
[5] J. Milne. Jacobian Varieties. In *Arithmetic Geometry*, ed. Cornell, Silverman, Chapter 7, Springer, 1998.
[6] S. Mukai. *Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves.* Nagoya Math. J. 81 (1981), 153–175.
[7] D. Mumford. *Abelian varieties.* Oxford University Press, London, 1970.
[8] D. Mumford. *Curves and their Jacobians.* The University of Michigan Press, Ann Arbor, Mich., 1975.
[9] A. Polishchuk. *Abelian varieties, Theta Functions and the Fourier Transform.* Cambridge University Press, Cambridge, 2003.