Blockseminar Essen/Düsseldorf am 22.1.05

Thema: Moduli of sheaves on K3-surfaces and derived categories

Throughout, we work only with projective $K3$-surfaces S over the complex numbers.

Talk 1: Moduli spaces of simple sheaves on K3-surfaces

Purpose: State and discuss Mukai's Theorems on the moduli space Spl_S.

Details: The two results in question are as follows ([2], Section 0). Theorem A: The moduli space Spl_S of simple $\mathcal{O}_S$-modules $\mathcal{E}$ with Mukai vector $v \in \tilde{H}(S, \mathbb{Z})$ is smooth, of dimension $(v, v)+2$. Theorem B: The moduli space Spl_S has a canonical symplectic structure.

Introduce the Mukai lattice $\tilde{H}(S, \mathbb{Z})$ with its bilinear form and Hodge structure, and discuss the Mukai vector $v = v(\mathcal{E})$ ([3], Section 1,2 or [1], Section 6.1). Remind us that $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E})$ is the tangent vector space to the deformation functor for simple sheaves ([1], Section 2.A.5). Compute the embedding dimension $(v, v) + 2$ of the moduli space Spl_S at a given point $[\mathcal{E}]$ with Riemann-Roch ([2], page 109). If time permits, explain Mukai's proof of Beauville's result that Hilb^n_S carries a symplectic structure ([2], Example 0.4).

Talk 2: Smoothness of the moduli space

Purpose: Prove Theorem A as in [2], Section 1.

Details: Remind us about the meaning of formal smoothness for the given deformation problem. Explain how Mukai replaces the lifting problem for the sheaf $\mathcal{E}_0$ by the seemingly more difficult lifting problem for extensions $X = (0 \to \mathcal{G}_0 \to \mathcal{O}_S^{\oplus n} \to \mathcal{E}_0 \to 0)$. Explain the obstruction in $\mathrm{Ext}^1(\mathcal{G}_0, \mathcal{E}_0)$, and how it embeds via trace maps into the obstruction group $H^2(S, \mathcal{O}_S)$ for lifting invertible sheaves.

Talk 3: Symplectic structures on the moduli space

Purpose: Prove Theorem B as in [2], Section 2.

Details: Introduce the relative Ext-sheaves $W^i = \mathcal{E}xt^i(\mathcal{E}, \mathcal{E})$ and their Yoneda pairing, and discuss their naturality. Explain how the Atiyah class identifies W^1 with the tangent bundle of Spl_S. Check skew-symmetry and nondegenerateness of the Yoneda pairing, and explain that W^2 is trivial.

Talk 4: Equivalences of derived categories for K3-surfaces

Purpose: State, discuss, and sketch a proof for Orlov's Theorem on the derived categories for K3-surfaces.

Details: Let S, S' be two projective K3-surfaces. The result in question is the following theorem: The derived categories $D(S), D(S')$ of coherent sheaves are equivalent as triangulated categories if and only if the lattice of transcendental cycle $T_S, T_{S'}$ are isomorphic as Hodge structures ([4], Section 3).

Compare with the Torelli-Theorem for K3-surfaces. Concentrate on [4], Section 3 and treat everything else as black box, in particular the result on the existence of kernels ([4], Theorem 2.18). Remind us that Picard group and dualizing sheaf depend only on the derived category. Explain how Mukai's moduli spaces of sheaves show up in the proof, but treat the results from [3] as black box.

Talk 5: Finding moduli spaces that are K3

Purpose: Explain Mukai's results about the existence and compactness of certain moduli spaces $\mathcal{M}_A(v)$ of stable sheaves on S, which appear in a crucial step in talk 5.

Details: Let S be a $K3$-surface, and $v = (r, l, s)$ be an element from the Mukai lattice $\tilde{H}(S, \mathbb{Z})$, with $(v, v) = 0$, $r > 1$, and l ample. Explain Mukai's arguments ([3], Theorem 5.4 and Proposition 4.3) that there exists a polarization A of S so that the moduli space $\mathcal{M}_A(v)$ of stable sheaves on S with Mukai vector v is nonempty and compact. Maybe just cite that this moduli space then must be a K3-surface.

References

[1] D. Huybrechts, M. Lehn: The geometry of moduli spaces of sheaves. Aspects of Mathematics E31. Vieweg, Braunschweig, 1997.

[2] S. Mukai: Symplectic structure of the moduli space of sheaves on an abelian or K3 surface. Invent. Math. 77 (1984), 101-116.

[3] S. Mukai: On the moduli space of bundles on K3 surfaces. I. In: M. Atiyah et al. (eds.), Vector bundles on algebraic varieties. Tata Inst. Fundam. Res. 11 (1987), 341–413.

[4] D. Orlov: Equivalences of derived categories and K3 surfaces. J. Math. Sci. 84 (1997), 1361–1381. Available at alg-geom/9606006.

Further reading:

[5] T. Bridgeland, A. Maciocia: Complex surfaces with equivalent derived categories. Math. Z. 236 (2001), 677–697.

[6] S. Hosono, B. Lian, K. Oguiso, S.-T. Yau: Fourier-Mukai partners of a K3 surface of Picard number one. In: D. Cutkosky (ed.) et al.: Vector bundles and representation theory, pp. 43-55. Contemp. Math. 322. Providence, American Mathematical Society, 2003.

[7] D. Huybrechts, P. Stellari: Equivalences of twisted K3 surfaces. Preprint, math.AG/0409030.

[8] S. Mukai: Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves. Nagoya Math. J. 81 (1981), 153–175.

[9] S. Mukai: Moduli of vector bundles on K3-surfaces and symplectic manifolds. Sugako Expositions 1 (1988), 138–174.

[10] S. Mukai: Duality of polarized $K3$ surfaces. In: K. Hulek (ed.) et al., New trends in algebraic geometry, pp. 311–326. London Math. Soc. Lecture Note Ser. 264. Cambridge Univ. Press, Cambridge, 1999.

[11] K. Oguiso: K3 surfaces via almost-primes. Math. Res. Lett. 9 (2002), 47–63.