On projective linear groups over finite fields as Galois groups over the rational numbers

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Abstract

Ideas from Khare's and Wintenberger's article on the proof of Serre's conjecture for odd conductors are used to establish that for a fixed prime l infinitely many of the groups $\mathrm{PSL}_2(\mathbb{F}_{l^r})$ (for r running) occur as Galois groups over the rationals such that the corresponding number fields are unramified outside a set consisting of l, the infinite place and only one other prime.

1 Introduction

The aim of this article is to prove the following theorem.

1.1 Theorem. *Let l be a prime and s a positive number. Then there exists a set T of rational primes of positive density such that for each $q \in T$ there exists a modular Galois representation*

$$\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$$

which is unramified outside $\{\infty, l, q\}$ and whose projective image is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_{l^r})$ for some $r > s$.

1.2 Corollary. *Let l be a prime. Then for infinitely many positive integers r the groups $\mathrm{PSL}_2(\mathbb{F}_{l^r})$ occur as a Galois group over the rationals.* □

Using $\mathrm{SL}_2(\mathbb{F}_{2^r}) \cong \mathrm{PSL}_2(\mathbb{F}_{2^r})$ one obtains the following reformulation for $l = 2$.

1.3 Corollary. *For infinitely many positive integers r the group $\mathrm{SL}_2(\mathbb{F}_{2^r})$ occurs as a Galois group over the rationals.* □

This contrasts with work by Dieulefait, Reverter and Vila ([D1],[D2], [RV] and [DV]) who proved that the groups $\mathrm{PSL}_2(\mathbb{F}_{l^r})$ and $\mathrm{PGL}_2(\mathbb{F}_{l^r})$ occur as Galois groups over $\mathbb{Q}$ for fixed (small) r and infinitely many primes l.

Dieulefait ([D3]) has recently obtained a different, rather elementary proof of Corollary 1.2 under the assumption $l \geq 5$ with a different ramification behaviour, namely $\{\infty, 2, 3, l\}$. His proof has the virtue of working with a family of modular forms that does not depend on l.

In the author's PhD thesis [W] some computational evidence on the statement of Corollary 1.3 was exhibited. More precisely, it was shown that all groups $\mathrm{SL}_2(\mathbb{F}_{2^r})$ occur as Galois groups over $\mathbb{Q}$ for $1 \leq r \leq 77$, extending results by Mestre (see [S], p. 53), by computing Hecke eigenforms of weight 2 for prime level over finite fields of characteristic 2. However, at that time all attempts to prove the corollary failed, since the author could not rule out theoretically that all Galois representations attached to modular forms with image contained in $\mathrm{SL}_2(\mathbb{F}_{2^r})$ for r bigger than some fixed bound and not in any $\mathrm{SL}_2(\mathbb{F}_{2^a})$ for $a \mid r$, $a \neq r$, have a dihedral image.

The present paper has undergone some developments since the first writing. However, the core of the proof has remained the same. It uses a procedure borrowed from the ground breaking paper [KW] by Khare and Wintenberger. The representations that we will construct in the proof are almost - in the terminology of [KW] - good-dihedral. The way we make these representation by level raising is adopted from a part of the proof of [KW], Theorem 3.4. Although the paper [KW] is not cited in the proof, this paper owes its mere existence to it.

Meanwhile, the results of the present article have been generalised to the groups $\mathrm{PSp}_{2n}(\mathbb{F}_{l^r})$ for arbitrary n by Khare, Larsen and Savin (see [KLS]). Their paper [KLS] also inspired a slight strengthening of the main result of the present article compared to an early version.

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Notations

We shall use the following notations. By $S_k(\Gamma_1(N))$ we mean the complex vector space of holomorphic cusp forms on $\Gamma_1(N)$ of weight k. The notation $S_k(N, \chi)$ is used for the vector space of holomorphic cusp forms of level N, weight k for the Dirichlet character χ. If χ is trivial, we write $S_k(N)$ for short. We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and for every prime p an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and we choose once and for all an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and a ring surjection $\overline{\mathbb{Z}}_p \to \overline{\mathbb{F}}_p$, subject to which the following constructions are made. By $G_{\mathbb{Q}}$ we denote the absolute Galois group of the rational numbers. For a rational prime q, we let D_q and I_q be the corresponding decomposition and inertia group of the prime q, respectively. Given an eigenform $f \in S_k(\Gamma_1(N))$ one can attach to it by work of Shimura and Deligne a Galois representation $\rho_{f,p} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ with well-known properties. Choosing a lattice, reducing and semi-simplifying, one also obtains a representation $\overline{\rho}_{f,p} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$. We denote the composition $G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p) \twoheadrightarrow \mathrm{PGL}_2(\overline{\mathbb{F}}_p)$ by $\overline{\rho}^{\mathrm{proj}}_{f,p}$. All Galois representations in this paper are continuous. By ζ_{p^r} we always mean a primitive p^r-th root of unity.

2 On Galois representations

The basic idea of the proof is to obtain the groups $\mathrm{PSL}_2(\mathbb{F}_{l^r})$ as the image of some $\overline{\rho}_{g,l}^{\mathrm{proj}}$. In order to determine the possible images of $\overline{\rho}_{g,l}^{\mathrm{proj}}$, we quote the following well-known group theoretic result due to Dickson (see [Hu], II.8.27).

2.1 Proposition. (Dickson) *Let l be a prime and H a finite subgroup of $\mathrm{PGL}_2(\overline{\mathbb{F}}_l)$. Then a conjugate of H is isomorphic to one of the following groups:*

- *finite subgroups of the upper triangular matrices,*
- $\mathrm{PSL}_2(\mathbb{F}_{l^r})$ *or* $\mathrm{PGL}_2(\mathbb{F}_{l^r})$ *for* $r \in \mathbb{N}$,
- *dihedral groups D_r for $r \in \mathbb{N}$ not divisible by l,*
- A_4, A_5 *or* S_4.

We next quote a result by Ribet showing that the images of $\overline{\rho}_{g,l}^{\mathrm{proj}}$ for a non-CM eigenform g are "almost always" $\mathrm{PSL}_2(\mathbb{F}_{l^r})$ or $\mathrm{PGL}_2(\mathbb{F}_{l^r})$ for some $r \in \mathbb{N}$.

2.2 Proposition. (Ribet) *Let $f = \sum_{n\geq 1} a_n q^n \in S_2(N, \chi)$ be a normalised eigenform of level N and some character χ which is not a CM-form.*

Then for almost all primes p, i.e. all but finitely many, the image of the representation

$$\overline{\rho}_{f,p}^{\mathrm{proj}} : G_{\mathbb{Q}} \to \mathrm{PGL}_2(\overline{\mathbb{F}}_p)$$

attached to f is equal to either $\mathrm{PGL}_2(\mathbb{F}_{p^r})$ or to $\mathrm{PSL}_2(\mathbb{F}_{p^r})$ for some $r \in \mathbb{N}$.

Proof. Reducing modulo p, Theorem 3.1 of [R1] gives, for almost all p, that the image of $\overline{\rho}_{f,p}$ contains $\mathrm{SL}_2(\mathbb{F}_p)$. This already proves that the projective image is of the form $\mathrm{PGL}_2(\mathbb{F}_{p^r})$ or $\mathrm{PSL}_2(\mathbb{F}_{p^r})$ by Proposition 2.1. □

In view of Ribet's result, there are two tasks to be solved for proving Theorem 1.1. The first task is to avoid the "exceptional primes", i.e. those for which the image is not as in the proposition. The second one is to obtain at the same time that the field $\mathbb{F}_{p^r}$ is "big".

We will now use this result by Ribet in order to establish the simple fact that for a given prime l there exists a modular form of level a power of l having only finitely many "exceptional primes". The following lemma can be easily verified using e.g. William Stein's modular symbols package which is part of MAGMA ([Magma]).

2.3 Lemma. *In any of the following spaces there exists a newform without CM: $S_2(2^7)$, $S_2(3^4)$, $S_2(5^3)$, $S_2(7^3)$, $S_2(13^2)$.* □

2.4 Proposition. *Let l be a prime. Put*

$$N := \begin{cases} 2^7, & \text{if } l = 2, \\ 3^4, & \text{if } l = 3, \\ 5^3, & \text{if } l = 5, \\ 7^3, & \text{if } l = 7, \\ 13^2, & \text{if } l = 13, \\ l, & \text{otherwise.} \end{cases}$$

Then there exists an eigenform $f \in S_2(N)$ such that for almost all primes p, i.e. for all but finitely many, the image of the attached Galois representation $\overline{\rho}_{f,p}$ is of the form $\mathrm{PGL}_2(\mathbb{F}_{p^r})$ or $\mathrm{PSL}_2(\mathbb{F}_{p^r})$ for some $r \in \mathbb{N}$.

Proof. If $l \in \{2, 3, 5, 7, 13\}$, we appeal to Lemma 2.3 to get an eigenform f without CM. If l is not in that list, then there is an eigenform in $S_2(l)$ and it is well-known that it does not have CM, since the level is square-free. Hence, Proposition 2.2 gives the claim. □

We will be able to find an eigenform with image as in the preceding proposition which is "big enough" by applying the following level raising result by Diamond and Taylor. It is a special case of Theorem A of [DT].

2.5 Theorem. (Diamond, Taylor) *Let $N \in \mathbb{N}$ and let $p > 3$ be a prime not dividing N. Let $f \in S_2(N, \chi)$ be a newform such that $\overline{\rho}_{f,p}$ is irreducible. Let, furthermore, $q \nmid N$ be a prime such that $q \equiv -1 \mod p$ and $\mathrm{tr}(\overline{\rho}_{f,p}(\mathrm{Frob}_q)) = 0$.*

Then there exists a newform $g \in S_2(Nq^2, \tilde{\chi})$ such that $\overline{\rho}_{g,p} \cong \overline{\rho}_{f,p}$. □

2.6 Corollary. *Assume the setting of Theorem 2.5. Then the following statements hold.*

(a) The mod p reductions of χ and $\tilde{\chi}$ are equal.

(b) The restriction of $\rho_{g,p}$ to I_q, the inertia group at q, is of the form $\left(\begin{smallmatrix} \psi & * \\ 0 & \psi^q \end{smallmatrix}\right)$ with a character $I_q \xrightarrow{\psi} \mathbb{Z}[\zeta_{p^r}]^\times \hookrightarrow \mathbb{Z}_p[\zeta_{p^r}]^\times$ of order p^r for some $r > 0$.

(c) For all primes $l \neq p, q$, the restriction of $\overline{\rho}_{g,l}$ to D_q, the decomposition group at q, is irreducible and the restriction of $\overline{\rho}_{g,l}$ to I_q is of the form $\left(\begin{smallmatrix} \psi & * \\ 0 & \psi^q \end{smallmatrix}\right)$ with the character $I_q \xrightarrow{\psi} \mathbb{Z}[\zeta_{p^r}]^\times \twoheadrightarrow \mathbb{F}_l(\zeta_{p^r})^\times$ of the same order p^r as in (b).

Proof. (a) This follows from $\overline{\rho}_{g,p} \cong \overline{\rho}_{f,p}$.

(b) As q^2 precisely divides the conductor of $\rho_{g,p}$, the ramification at q is tame and the restriction to I_q is of the form $\left(\begin{smallmatrix} \psi_1 & * \\ 0 & \psi_2 \end{smallmatrix}\right)$ for non-trivial characters $\psi_i : I_q \to \overline{\mathbb{Z}}_p^\times$. Their order must be a power of p, since their reductions mod p vanish by assumption. Due to $q \equiv -1 \mod p$, local class field theory tells us that ψ_i cannot extend to $G_{\mathbb{Q}_q}$, as their orders would divide $q - 1$. Hence, the image

of $\rho_{g,p}|_{D_q} : D_q \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p) \to \mathrm{PGL}_2(\overline{\mathbb{Q}}_p)$ is a finite dihedral group. Consequently, $\rho_{g,p}|_{D_q}$ is an unramified twist of the induced representation $\mathrm{Ind}_{D_q}^{G_K}(\psi)$ with K the unramified extension of degree 2 of $\mathbb{Q}_q$ and $\psi : G_K \to \mathbb{Z}[\zeta_{p^r}]$ a totally ramified character of order p^r for some $r > 0$. Conjugation by $\rho_{g,p}(\mathrm{Frob}_q)$ exchanges ψ_1 and ψ_2. It is well-known that this conjugation also raises to the q-th power. Thus, we find $\psi_2 = \psi_1^q$ and without loss of generality $\psi = \psi_1$.

(c) With the notations and the normalisation used in [CDT], p. 536, the local Langlands correspondence reads (for $l \neq q$)

$$(\rho_{g,l}|W_q)^{\mathrm{ss}} \cong \overline{\mathbb{Q}}_l \otimes_{\overline{\mathbb{Q}}} \mathrm{WD}(\pi_{g,q}),$$

if g corresponds to the automorphic representation $\pi_g = \otimes \pi_{g,p}$. In particular, knowing by (b) that $\rho_{g,p}|_{I_q}$ is $\left(\begin{smallmatrix} \psi & * \\ 0 & \psi^q \end{smallmatrix}\right)$, it follows that $\rho_{g,l}|_{I_q}$ is also of that form. As ψ is of p-power order, it cannot vanish under reduction mod l if $l \neq p$. Hence, $\overline{\rho}_{g,l}|_{I_q}$ is of the claimed form. The irreducibility follows as in (b). □

3 Proof and remarks

In this section we prove Theorem 1.1 and comment on possible generalisations.

3.1 Lemma. *Let l be a prime and $s \in \mathbb{N}$. Then there is a set of primes p of positive density such that p is 1 mod 4 and such that if the group $\mathrm{GL}_2(\mathbb{F}_{l^t})$ possesses an element of order p for some $t \in \mathbb{N}$, then $2 \mid t$ and $t > s$.*

Proof. We take the set of primes p such that p is split in $\mathbb{Q}(i)$ and inert in $\mathbb{Q}(\sqrt{l})$. By Chebotarev's density theorem this set has a positive density. The first condition imposed on p means that 4 divides the order of $\mathbb{F}_p^\times$ and the second one that the order of the 2-Sylow subgroup of $\mathbb{F}_p^\times$ divides the order of l in $\mathbb{F}_p^\times$. If the order of $g \in \mathrm{GL}_2(\mathbb{F}_{l^t})$ is p, then $\mathbb{F}_{l^{2t}}^\times$ contains an element of order p (an eigenvalue of g). Thus, $l^{2t} - 1$ is divisible by p, whence the order of l in $\mathbb{F}_p^\times$ divides $2t$ and consequently 2 divides t. The condition $t > s$ can be met by excluding finitely many p. □

Proof of Theorem 1.1. Let us fix the prime l and the number s from the statement of Theorem 1.1. We will now exhibit a modular form g such that $\overline{\rho}_{g,l}^{\mathrm{proj}}$ has image equal to $\mathrm{PSL}_2(\mathbb{F}_{l^t})$ with $t > s$. We start with the eigenform $f \in S_2(l^*)$ provided by Proposition 2.4. Next, we let p be any of the infinitely many primes from Lemma 3.1 such that the image of $\overline{\rho}_{f,p}^{\mathrm{proj}}$ is $\mathrm{PSL}_2(\mathbb{F}_{p^r})$ or $\mathrm{PGL}_2(\mathbb{F}_{p^r})$ with some r. We want to obtain g by level raising. The next lemma will yield a set of primes of positive density at which the level can be raised in a way suitable to us.

3.2 Lemma. *Under the above notations and assumptions, the set of primes q such that*

(i) $q \equiv -1 \mod p$,

(ii) *q splits in $\mathbb{Q}(i, \sqrt{l})$ and*

(iii) $\overline{\rho}_{f,p}^{\mathrm{proj}}(\mathrm{Frob}_q)$ *lies in the same conjugacy class as $\overline{\rho}_{f,p}^{\mathrm{proj}}(c)$, where c is any complex conjugation,*

has a positive density.

Proof. The proof is adapted from [KW], Lemma 8.2. Let $L := \mathbb{Q}(\zeta_p, i, \sqrt{l})$ and $K/\mathbb{Q}$ such that $G_K = \ker(\overline{\rho}_{f,p}^{\mathrm{proj}})$. Conditions (i) and (ii) must be imposed on the field L and Condition (iii) on K. We know that $\mathrm{Gal}(K/\mathbb{Q})$ is either $\mathrm{PSL}_2(\mathbb{F}_{p^r})$ or $\mathrm{PGL}_2(\mathbb{F}_{p^r})$. In the former case the lemma follows directly from Chebotarev's density theorem, as the intersection $L \cap K$ is $\mathbb{Q}$, since $\mathrm{PSL}_2(\mathbb{F}_{p^r})$ is a simple group. In the latter case the intersection $L \cap K = M$ is an extension of $\mathbb{Q}$ of degree 2. As $p \equiv 1 \mod 4$ by assumption, $\overline{\rho}_{f,p}^{\mathrm{proj}}(c)$ is in $\mathrm{Gal}(L/M) \cong \mathrm{PSL}_2(\mathbb{F}_{p^r})$, since $\det(\overline{\rho}_{f,p}(c)) = -1$ is a square mod p. Consequently, any q satisfying Condition (iii) is split in $M/\mathbb{Q}$. Again as $p \equiv 1 \mod 4$, complex conjugation fixes the quadratic subfield of $\mathbb{Q}(\zeta_p)$, whence any prime q satisfying Conditions (i) and (ii) is also split in $M/\mathbb{Q}$. Hence, we may again appeal to Chebotarev's density theorem, proving the lemma. □

To continue the proof of Theorem 1.1 we let T be the set of primes provided by Lemma 3.2. Let $q \in T$. Condition (iii) assures that $\overline{\rho}_{f,p}(\mathrm{Frob}_q)$ has trace 0, since it is a scalar multiple of the matrix representing complex conjugation, i.e. $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Hence, we are in the situation to apply Theorem 2.5 and Corollary 2.6. We, thus, get an eigenform $g \in S_2(l^* q^2, \chi)$ with χ a Dirichlet character of order a power of p (its reduction mod p is trivial) such that $\overline{\rho}_{g,l}|_{D_q}$ is irreducible and $\overline{\rho}_{g,l}|_{I_q}$ is of the form $\left(\begin{smallmatrix} \psi & * \\ 0 & \psi^q \end{smallmatrix}\right)$ with ψ a non-trivial character of order p^r. Hence, in particular, the image $\overline{\rho}_{g,l}^{\mathrm{proj}}(G_{\mathbb{Q}})$ contains an element of order p, as $\left(\begin{smallmatrix} \psi & * \\ 0 & \psi^q \end{smallmatrix}\right)$ cannot be scalar due to $p \nmid q - 1$.

We next show that $\overline{\rho}_{g,l}^{\mathrm{proj}}(G_{\mathbb{Q}})$ is not a dihedral group. It cannot be cyclic either because of irreducibility. If it were dihedral, then $\overline{\rho}_{g,l}^{\mathrm{proj}}$ would be a representation induced from a character of a quadratic extension $R/\mathbb{Q}$. If R were ramified at q, then $\overline{\rho}_{g,p}^{\mathrm{proj}}(I_q)$ would have even order, but it has order a power of p. So, R would be either $\mathbb{Q}(\sqrt{l})$ or $\mathbb{Q}(\sqrt{-l})$. As by Condition (ii) q would split in R, this implies that $\overline{\rho}_{g,p}^{\mathrm{proj}}|_{D_q}$ would be reducible, but it is irreducible. Consequently, $\overline{\rho}_{g,l}^{\mathrm{proj}}(G_{\mathbb{Q}})$ is either $\mathrm{PSL}_2(\mathbb{F}_{l^t})$ or $\mathrm{PGL}_2(\mathbb{F}_{l^t})$ for some t. By what we have seen above, $\mathrm{PGL}_2(\mathbb{F}_{l^t})$ then contains an element of order p. Hence, so does $\mathrm{GL}_2(\mathbb{F}_{l^t})$ and the assumptions on p imply $t > s$ and $2 \mid t$.

We know, moreover, that the determinant of $\overline{\rho}_{g,l}(\mathrm{Frob}_w)$ for any prime $w \nmid lq$ is $w^{k-1}\chi(w)$ for some fixed k (the Serre weight of $\overline{\rho}_{g,l}$). As $\chi(w)$ is of p-power order, $\mathbb{F}_{l^t}$ contains a square root of it. The square roots of elements of $\mathbb{F}_l^\times$ are all contained in $\mathbb{F}_{l^2}$ and thus also in $\mathbb{F}_{l^t}$, as t is even. Hence, $\overline{\rho}_{g,l}^{\mathrm{proj}}(\mathrm{Frob}_w) \in \mathrm{PSL}_2(\mathbb{F}_{l^t})$. As every conjugacy class contains a Frobenius element, the proof of Theorem 1.1 is finished. □

3.3 Remark. *One can develop the basic idea used here further, in particular, in order to try to establish an analogue of Theorem 1.1 such that the representations ramify at a given finite set of primes S and are unramified outside $S \cup \{\infty, l, q\}$.*

3.4 Remark. *It is desirable to remove the ramification at l. For that, one would need that $\overline{\rho}_{g,l}^{\mathrm{proj}}$ is unramified at l. This, however, seems difficult to establish.*

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