GENERALIZATION OF A CRITERION FOR SEMISTABLE VECTOR BUNDLES

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Abstract. It is known that a vector bundle \( E \) on a smooth projective curve \( Y \) defined over an algebraically closed field is semistable if and only if there is a vector bundle \( F \) on \( Y \) such that both \( H^0(\mathcal{X}, E \otimes F) \) and \( H^1(\mathcal{X}, E \otimes F) \) vanishes. We extend this criterion for semistability to vector bundles on curves defined over perfect fields. Let \( X \) be a geometrically irreducible smooth projective curve defined over a perfect field \( k \), and let \( E \) be a vector bundle on \( X \). We prove that \( E \) is semistable if and only if there is a vector bundle \( F \) on \( X \) such that \( H^i(\mathcal{X}, E \otimes F) = 0 \) for all \( i \). We also give an explicit bound for the rank of \( F \).

1. Introduction

A theorem due to Faltings says that a vector bundle \( E \) on a smooth projective curve \( Y \) defined over an algebraically closed field of characteristic zero is semistable if and only if there is a vector bundle \( F \) on \( Y \) such that both \( H^i(\mathcal{X}, E \otimes F) = 0 \) for all \( i \) [3, p. 514, Theorem 1.2]. It is is known that this criterion for semistability extends to vector bundles on smooth projective curves defined over an algebraically closed fields of positive characteristic. (See [5], [1] for related results.)

Our aim here is to investigate this criterion for curves defined over finite fields and more generally over perfect fields. We prove the following theorem.

Theorem 1.1. Let \( X \) be a geometrically irreducible smooth projective curve defined over a perfect field \( k \). A vector bundle \( E \) over \( X \) is semistable if and only if there is a vector bundle \( F \) over \( X \) such that \( H^i(\mathcal{X}, E \otimes F) = 0 \) for all \( i \).

We also produce an effective bound for the rank of \( F \) in Theorem 1.1. More precisely, given nonnegative integers \( g \) and \( r \), and an integer \( d \), there is an explicit integer \( R(g, r, d) \) such that for any triple \((k, X, E)\), where

- \( k \) is a perfect field,
- \( X \) is a geometrically irreducible smooth projective curve of genus \( g \) defined over \( k \), and
- \( E \) is a vector bundle over \( X \) of rank \( r \) and degree \( d \),

the vector bundle \( E \) is semistable if and only if there is a vector bundle \( F \) over \( X \) of rank \( R(g, r, d) \) such that \( H^i(\mathcal{X}, E \otimes F) = 0 \) for all \( i \). (See Theorem 3.1.)

Date: April 25, 2008.
Let $X$ be a scheme defined over a field $k$, and let $D \subset X$ be an effective divisor. Then there is a geometric point of $X$ that lies outside $D$. Corollary 2.5 bounds the degree of the field extension $K/k$ such that $D(K) \subsetneq X(K)$. This is used in the proof of Theorem 1.1.

2. Rational points outside a given hypersurface

Let $k$ be any field. The algebraic closure of $k$ will be denoted by $\overline{k}$.

**Lemma 2.1.** Let $D \subset \mathbb{A}^n_k$ be an effective divisor defined over $k$. Given any field extension $K/k$ such $K$ has more than $\deg(D)$ elements, there exists a $K$-rational point in $\mathbb{A}^n_K$ that lies outside $D$.

**Proof.** One follows the proof of Proposition 1.3(a) in [6, p. 4] almost word for word simply replacing “infinite field” by “field with more than $\deg(D)$ elements”: We assume that $D$ is given by the polynomial $F \in \overline{k}[X_1, \ldots, X_n]$, and proceed by induction over $n$. For $n = 1$ it is the statement that a polynomial of degree $d$ cannot have more than $d$ zeros. Now assume that $X_n$ occurs in $F$ and write $F = \varphi_0 + \varphi_1 X_n + \ldots \varphi_t X_n^t$, where $\varphi_i \in \overline{k}[X_1, \ldots, X_{n-1}]$ and $\varphi_t \neq 0$. Since $t$ and $\deg(\varphi_t)$ are both at most $\deg(D)$, we conclude from the induction hypothesis the existence of a point $(x_1, \ldots, x_{n-1}) \in K^{n-1}$ such that $\varphi_t(x_1, \ldots, x_{n-1}) \neq 0$. Now the polynomial $X_n \mapsto F(x_1, \ldots, x_{n-1}, X_n)$ has at most $t$ zeros. □

**Lemma 2.2.** Let $D \subset \mathbb{P}^n_k$ be an effective divisor in projective space $\mathbb{P}^n_k$ defined over $k$. Then for any extension $K/k$ such that $K$ has at least $\deg(D)$ elements, there is a $K$-rational point in $\mathbb{P}^n(K)$ that lies outside $D(K)$.

**Proof.** Again we proceed by induction on $n$. The case $n = 1$ is obvious. For $n > 1$, we consider the pencil of hyperplanes defined over $K$ passing through a codimension two linear subspace. Since there are more than $\deg(D)$ of these hyperplanes, the union of all these hyperplanes cannot be contained in $D$. Thus, there exists a hyperplane $H \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$ that intersects $D$ properly. Now the proof is completed by the induction hypothesis. □

Let $\text{Grass}(m, n)$ be the Grassmannian of $m$-dimensional linear subspaces of $k^n$. Let

\[(1) \quad \iota : \text{Grass}(m, n) \longrightarrow \mathbb{P} := \mathbb{P}^{\binom{n}{m}-1}\]

be the Plücker embedding. By an hypersurface of degree $d$ on $\text{Grass}(m, n)$ we will mean one from the complete linear system $|\iota^*\mathcal{O}_\mathbb{P}(d)|$.

**Lemma 2.3.** Let $D \subset \text{Grass}(m, n)$ be a hypersurface in the Grassmannian. If a field extension $K/k$ has more than $m \cdot \deg(D)$ elements, then there is a $K$-rational point of $\text{Grass}(m, n)(K)$ that is not contained in $D(K)$.
Proof. We consider the dense open cell in the Grassmannian given by the open immersion \( j : \mathbb{A}^{m(n-m)} \rightarrow \text{Grass}(m, n) \) defined by
\[
(a_{i,j})_{i=1,\ldots,m \ j=m+1,\ldots,n} \rightarrow \text{span}\begin{pmatrix}
1 & 0 & \cdots & 0 & a_{1,m+1} & a_{1,m+2} & \cdots & a_{1,n} \\
0 & 1 & \cdots & 0 & a_{2,m+1} & a_{2,m+2} & \cdots & a_{2,n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & a_{m,m+1} & a_{m,m+2} & \cdots & a_{m,n}
\end{pmatrix}
\]
The Plücker embedding (see (1)) restricted to \( \mathbb{A}^{m(n-m)} \) is given by the \( m \times m \)-minors of degree at most \( m \) of the above matrix. Therefore, \( j^*D \) is a divisor of degree \( m \cdot \deg(D) \).

Now the proof is completed using Lemma 2.1. \( \square \)

**Proposition 2.4.** Let \( \mathcal{O}_X(H) \) be a globally generated ample line bundle on a projective scheme \( X \) of dimension \( n \) defined over \( k \). Let \( D \subset X \) be an effective divisor \( D \subset X \). Let \( K_1/k \) be a field extension that has more than \( \max\{ (n+1)H^n, D.H^{n-1} - 1 \} \) elements. Then there exists a field extension \( K_2/K_1 \) with \( [K_2 : K_1] \leq H^n \), such that there is a \( K_2 \)-rational point of \( X(K_2) \) that does not lie in \( D(K_2) \).

**Proof.** We consider the short exact sequence of vector bundles
\[
0 \rightarrow W \rightarrow H^0(L) \otimes \mathcal{O}_X \rightarrow L \rightarrow 0
\]
over \( X \). Let \( \text{Grass}_X(n+1, W) \) be the Grassmann bundle over \( X \) parameterizing all \((n+1)-\)dimensional subspaces in the fibers of \( W \). We have
\[
\dim \text{Grass}_X(n+1, W) = \dim(X) + (h^0(L) - n - 2)(n+1) = \dim(\text{Grass}(n+1, H^0(L))) - 1.
\]

We will show that the degree of the hypersurface \( \text{Grass}_X(n+1, W) \) in \( \text{Grass}(n+1, H^0(L)) \) is \( H^n \). To prove this, take any subspace \( U \subset H^0(X, L) \) of dimension \( n+2 \). The \((n+1)\)-dimensional subspaces of \( U \) form a projective line \( \mathbb{P}^1_k \) in \( \text{Grass}(n+1, H^0(L)) \). The degree of the restriction of \( \iota^*\mathcal{O}_X(1) \) (see (1)) to this \( \mathbb{P}^1_k \) is one. To compute the intersection number of the line with \( \text{Grass}_X(n+1, W) \) we may assume that \( H^0(X, L) = U \). So it suffices to count the intersection of a line in \( \mathbb{P}(U) \) with the divisor \( X \subset \mathbb{P}(U) \). Thus, we conclude that the hypersurface \( \text{Grass}_X(n+1, W) \subset \text{Grass}(n+1, H^0(L)) \) is of degree \( H^n \).

Now, using Lemma 2.3 and the assumption on \( K_1 \) we conclude that there exists a \( K \)-point in \( \text{Grass}(n+1, H^0(X, L)) \) not lying in \( \text{Grass}_X(n+1, W) \). This yields a finite morphism \( X \rightarrow \mathbb{P}^n \) defined over \( K_1 \). Now \( \pi_*(D) \) is a divisor of degree \( D.H^{-1} \) on \( \mathbb{P}^n \). Our assumption on the number of elements in \( K_1 \) and Lemma 2.2 together imply that there is a \( K_1 \)-rational point \( P \) in the complement of \( \pi_*(D) \) in \( \mathbb{P}^n \). The morphism \( X_P \rightarrow \text{Spec}(K_1) \) is finite of degree \( H^n \), and it is defined over \( K_1 \). Thus, we find at least one point in \( X_P \) defined over a field \( K_2 \) as in the statement of the proposition. This completes the proof of the proposition. \( \square \)

Proposition 2.4 has the following corollary.

**Corollary 2.5.** Given positive integers \( n, \alpha \) and \( \beta \), define
\[
M(n, \alpha, \beta) := \alpha \lceil \log_2(\max\{ (n+1)\alpha + 1, \beta \}) \rceil.
\]
For any quadruple \((k, X, H, D)\), where

- \(k\) is a field,
- \(X\) is a projective scheme of dimension \(n\) defined over \(k\),
- \(H \subset X\) is a base point-free ample hypersurface with \(H^n = \alpha\), and
- \(D \subset H\) is an effective divisor with \(D.H^{n-1} = \beta\),

there is a field extension \(K/k\) of degree \([K:k] \leq M(n, \alpha, \beta)\) with the property that \(X(K)\) has a \(K\)–rational point that does not lie in \(D(K)\).

If we restrict ourselves only to infinite fields, then \(M(n, \alpha, \beta)\) in Corollary 2.5 can be taken to be \(\alpha\). If we fix a prime \(p\) and restrict ourselves only to fields of characteristic \(p\), then \(M(n, \alpha, \beta)\) in Corollary 2.5 can be taken to be \(\alpha \lceil \log_p(\max\{(n+1)\alpha + 1, \beta\})\rceil\).

3. Semistability criterion over perfect fields

**Theorem 3.1.** Let \(X\) be a geometrically irreducible smooth projective curve of genus \(g\) defined over a perfect field \(k\). Fix a positive integer \(r\) and an integer \(d\). Then there is an explicit positive integer \(R\) that depends only on \(r\), \(d\) and \(g\) (in particular, \(R\) is independent of \(k\)) with the following property: A vector bundle \(E\) over \(X\) of rank \(r\) and degree \(d\) is semistable if and only if there is a vector bundle \(F\) over \(X\) of rank \(R\) such that \(H^i(X, E \otimes F) = 0\) for all \(i\).

**Proof.** If \(E\) is not semistable, then clearly there is no \(F\) such that \(H^i(X, E \otimes F) = 0\) for all \(i\). Let \(E\) be a semistable vector bundle over \(X\) of rank \(r\) and degree \(d\). We will construct \(R\) and \(F\).

The moduli space of semistable vector bundles over \(X\) of rank \(r'\) and degree \(d'\) will be denoted by \(\mathcal{U}_X(r', d')\).

Let \(h := \gcd(r, d)\). Furthermore, we set \(\tau := \frac{r}{h}\), and \(\overline{d} := \frac{d}{h}\). For any integer \(n \geq 1\), consider the morphism \(\mathcal{U}_X(n\tau, n(\tau(g-1) - \overline{d})) \to \mathcal{U}_X(nr\tau, nr\tau(g-1))\) defined by \(V \mapsto V \otimes E\). Let \(\Theta_E\) denote the pull back of the natural theta divisor in \(\mathcal{U}_X(nr\tau, nr\tau(g-1))\) by this morphism. Therefore, \(\Theta_E \times_k \overline{k}\) consists of all semistable vector bundles \(W\) over \(X_{\overline{k}} = X \times_k \overline{k}\) of rank \(n\tau\) and degree \(n(\tau(g-1) - \overline{d})\) such that \(H^0(X_{\overline{k}}, W \otimes (E \otimes_k \overline{k})) \neq 0\). (We note that a vector bundle \(V'\) over \(X\) is semistable if and only if the vector bundle \(V' \otimes_k \overline{k}\) over \(X_{\overline{k}}\) is semistable; see [4, p. 222].) The subscheme \(\Theta_E\) defined above is either an effective Cartier divisor in the complete linear system \(|h \cdot \Theta|\) or it is the entire moduli space \(\mathcal{U}_X(n\tau, n(\tau(g-1) - \overline{d}))\) (cf. [2, § 0.2.1]).

Popa showed that this is indeed a divisor in the linear system \(|h \cdot \Theta|\) for all \(n \geq \frac{\tau^2 + 1}{4}\) (see [7, p. 490, Theorem 5.3]). Let \(n\) be the smallest integer such that \(n \geq \frac{\tau^2 + 1}{4}\). Consider the effective divisor \(\Theta_E \subset \mathcal{U}_X(n\tau, n(\tau(g-1) - \overline{d}))\) for \(n := \lceil \frac{\tau^2 + 1}{4} \rceil\). By Corollary 2.5, there exists an integer \(M\) and a field extension \(K/k\) such that \(\Theta_E\) does not contain all \(K\)–rational points of \(\mathcal{U}_X(n\tau, n(\tau(g-1) - \overline{d}))\).

The integer \(R\) in the statement of the theorem will be \(n\tau M\).
Since $\Theta_E$ does not contain all $K$-rational points of $\mathcal{U}_X(n\tau, n(\tau(g-1)-d))$, there exists a vector bundle $F_1$ of rank $n\tau$ defined over $X_K = X \times_k K$, where $K/k$ is some Galois extension of degree dividing $M!$, such that
\begin{equation}
H^0(X_K, (E \otimes_k K) \otimes F_1) = 0 = H^1(X_K, (E \otimes_k K) \otimes F_1).
\end{equation}
From (2) it follows that
\begin{equation}
H^0(X_K, (E \otimes_k K) \otimes \sigma^* F_1) = 0 = H^1(X_K, (E \otimes_k K) \otimes \sigma^* F_1)
\end{equation}
for all $\sigma \in \text{Gal}(K/k)$.

Now we consider the direct sum
$$F_2 := \bigoplus_{\sigma \in \text{Gal}(K/k)} \sigma^* F_1.$$
This $F_2$ is a vector bundle defined over $X$. From (3) it follows immediately that that $H^i(X, E \otimes F_2) = 0$ for all $i$. Also, the rank of $F_2$ clearly divides $n\tau M!$. Finally we set $m := \frac{n\tau M!}{rk(F_2)}$, and $F := F_2 \oplus m$, and obtain the asserted vector bundle of rank $R = n\tau M!$.

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