GENERALIZATION OF A CRITERION FOR SEMISTABLE VECTOR BUNDLES

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Abstract. It is known that a vector bundle E on a smooth projective curve Y defined over an algebraically closed field is semistable if and only if there is a vector bundle F on Y such that both $H^0(X,\, E\otimes F)$ and $H^1(X,\, E\otimes F)$ vanishes. We extend this criterion for semistability to vector bundles on curves defined over perfect fields. Let X be a geometrically irreducible smooth projective curve defined over a perfect field k, and let E be a vector bundle on X. We prove that E is semistable if and only if there is a vector bundle F on X such that $H^i(X,\, E\otimes F)\,=\,0$ for all i. We also give an explicit bound for the rank of F.

1. Introduction

A theorem due to Faltings says that a vector bundle E on a smooth projective curve Y defined over an algebraically closed field of characteristic zero is semistable if and only if there is a vector bundle F on Y such that both $H^i(X,\, E\otimes F)\,=\,0$ for all i [3, p. 514, Theorem 1.2]. It is is known that this criterion for semistability extends to vector bundles on smooth projective curves defined over an algebraically closed fields of positive characteristic. (See [5], [1] for related results.)

Our aim here is to investigate this criterion for curves defined over finite fields and more generally over perfect fields. We prove the following theorem.

Theorem 1.1. *Let X be a geometrically irreducible smooth projective curve defined over a perfect field k. A vector bundle E over X is semistable if and only if there is a vector bundle F over X such that $H^i(X,\, E\otimes F)\,=\,0$ for all i.*

We also produce an effective bound for the rank of F in Theorem 1.1. More precisely, given nonnegative integers g and r, and an integer d, there is an explicit integer $R(g,r,d)$ such that for any triple $(k\,,X\,,E)$, where

- k is a perfect field,
- X is a geometrically irreducible smooth projective curve of genus g defined over k, and
- E is a vector bundle over X of rank r and degree d,

the vector bundle E is semistable if and only if there is a vector bundle F over X of rank $R(g,r,d)$ such that $H^i(X,\, E\otimes F)\,=\,0$ for all i. (See Theorem 3.1.)

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Let X be a scheme defined over a field k, and let $D \subset X$ be an effective divisor. Then there is a geometric point of X that lies outside D. Corollary 2.5 bounds the degree of the field extension K/k such that $D(K) \subsetneqq X(K)$. This is used in the proof of Theorem 1.1.

2. Rational points outside a given hypersurface

Let k be any field. The algebraic closure of k will be denoted by $\overline{k}$.

Lemma 2.1. *Let $D \subset \mathbb{A}^n_k$ be an effective divisor defined over $\overline{k}$. Given any field extension K/k such K has more than $\deg(D)$ elements, there exists a K-rational point in $\mathbb{A}^n_K$ that lies outside D.*

Proof. One follows the proof of Proposition 1.3(a) in [6, p. 4] almost word for word simply replacing "*infinite field*" by "*field with more than* $\deg(D)$ *elements*": We assume that D is given by the polynomial $F \in \overline{k}[X_1, \ldots, X_n]$, and proceed by induction over n. For $n = 1$ it is the statement that a polynomial of degree d cannot have more than d zeros. Now assume that X_n occurs in F and write

$$F = \varphi_0 + \varphi_1 X_n + \ldots \varphi_t X_n^t ,$$

where $\varphi_i \in \overline{k}[X_1, \ldots, X_{n-1}]$ and $\varphi_t \neq 0$. Since t and $\deg(\varphi_t)$ are both at most $\deg(D)$, we conclude from the induction hypothesis the existence of a point $(x_1, \ldots, x_{n-1}) \in K^{n-1}$ such that $\varphi_t(x_1, \ldots, x_{n-1}) \neq 0$. Now the polynomial $X_n \longmapsto F(x_1, \ldots, x_{n-1}, X_n)$ has at most t zeros. $\square$

Lemma 2.2. *Let $D \subset \mathbb{P}^n_k$ be an effective divisor in projective space $\mathbb{P}^n$ defined over $\overline{k}$. Then for any extension K/k such that K has at least $\deg(D)$ elements, there is a K-rational point in $\mathbb{P}^n(K)$ that lies outside $D(K)$.*

Proof. Again we proceed by induction on n. The case $n = 1$ is obvious. For $n > 1$, we consider the pencil of hyperplanes defined over K passing through a codimension two linear subspace. Since there are more than $\deg(D)$ of these hyperplanes, the union of all these hyperplanes cannot be contained in D. Thus, there exists a hyperplane $H \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$ that intersects D properly. Now the proof is completed by the induction hypothesis. $\square$

Let $\mathrm{Grass}(m, n)$ be the Grassmannian of m–dimensional linear subspaces of k^n. Let

$$\iota : \mathrm{Grass}(m, n) \longrightarrow \mathbb{P} := \mathbb{P}^{\binom{n}{m}-1} \tag{1}$$

be the Plücker embedding. By an hypersurface of degree d on $\mathrm{Grass}(m, n)$ we will mean one from the complete linear system $|\iota^*\mathcal{O}_{\mathbb{P}}(d)|$.

Lemma 2.3. *Let $D \subset \mathrm{Grass}(m, n)$ be a hypersurface in the Grassmannian. If a field extension K/k has more than $m \cdot \deg(D)$ elements, then there is a K–rational point of $\mathrm{Grass}(m, n)(K)$ that is not contained in $D(K)$.*

Proof. We consider the dense open cell in the Grassmannian given by the open immersion $j : \mathbb{A}^{m\cdot(n-m)} \longrightarrow \mathrm{Grass}(m,n)$ defined by

$$(a_{i,j})_{i=1,\dots,m\;\; j:=m+1,\dots,n} \longmapsto \mathrm{span}\begin{pmatrix} 1 & 0 & \cdots & 0 & a_{1,m+1} & a_{1,m+2} & \cdots & a_{1,n} \\ 0 & 1 & \cdots & 0 & a_{2,m+1} & a_{2,m+2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & a_{m,m+1} & a_{m,m+2} & \cdots & a_{m,n} \end{pmatrix}$$

The Plücker embedding (see (1)) restricted to $\mathbb{A}^{m\cdot(n-m)}$ is given by the $m \times m$–minors of degree at most m of the above matrix. Therefore, j^*D is a divisor of degree $m \cdot \deg(D)$. Now the proof is completed using Lemma 2.1. $\square$

Proposition 2.4. *Let $\mathcal{O}_X(H)$ be a globally generated ample line bundle on a projective scheme X of dimension n defined over k. Let $D \subset X$ be an effective divisor $D \subset X$. Let K_1/k be a field extension that has more than* $\max\{(n+1)H^n, D.H^{n-1}-1\}$ *elements. Then there exists a field extension K_2/K_1 with $[K_2 : K_1] \leq H^n$, such that there is a K_2–rational point of $X(K_2)$ that does not lie in $D(K_2)$.*

Proof. We consider the short exact sequence of vector bundles

$$0 \longrightarrow W \longrightarrow H^0(L) \otimes \mathcal{O}_X \longrightarrow L \longrightarrow 0$$

over X. Let $\mathrm{Grass}_X(n+1, W)$ be the Grassmann bundle over X parameterizing all $(n+1)$–dimensional subspaces in the fibers of W. We have

$$\begin{aligned} \dim \mathrm{Grass}_X(n+1,W) &= \dim(X) + (h^0(L) - n - 2)(n+1) \\ &= \dim(\mathrm{Grass}(n+1, H^0(L))) - 1\,. \end{aligned}$$

We will show that the degree of the hypersurface $\mathrm{Grass}_X(n+1, W)$ in $\mathrm{Grass}(n+1, H^0(L))$ is H^n. To prove this, take any subspace $U \subset H^0(X, L)$ of dimension $n+2$. The $(n+1)$-dimensional subspaces of U form a projective line $\mathbb{P}^1_k$ in $\mathrm{Grass}(n+1, H^0(L))$. The degree of the restriction of $\iota^*\mathcal{O}_{\mathbb{P}}(1)$ (see (1)) to this $\mathbb{P}^1_k$ is one. To compute the intersection number of the line with $\mathrm{Grass}_X(n+1, W)$ we may assume that $H^0(X,\, L) = U$. So it suffices to count the intersection of a line in $\mathbb{P}(U)$ with the divisor $X \subset \mathbb{P}(U)$. Thus, we conclude that the hypersurface $\mathrm{Grass}_X(n+1, W) \subset \mathrm{Grass}(n+1, H^0(L))$ is of degree H^n.

Now, using Lemma 2.3 and the assumption on K_1 we conclude that there exits a K–point in $\mathrm{Grass}(n+1, H^0(X, L))$ not lying in $\mathrm{Grass}_X(n+1, W)$. This yields a finite morphism $X \xrightarrow{\pi} \mathbb{P}^n$ defined over K_1. Now $\pi_*(D)$ is a divisor of degree $D.H^{-1}$ on $\mathbb{P}^n$. Our assumption on the number of elements in K_1 and Lemma 2.2 together imply that there is a K_1–rational point P in the complement of $\pi_*(D)$ in $\mathbb{P}^n$. The morphism $X_P \longrightarrow \mathrm{Spec}(K_1)$ is finite of degree H^n, and it is defined over K_1. Thus, we find at least one point in X_P defined over a field K_2 as in the statement of the proposition. This completes the proof of the proposition. $\square$

Proposition 2.4 has the following corollary.

Corollary 2.5. *Given positive integers n, α and β, define*

$$M(n, \alpha, \beta) := \alpha\lceil \log_2(\max\{(n+1)\alpha + 1, \beta\})\rceil\,.$$

For any quadruple $(k\,,X\,,H\,,D)$, where

- *k is a field,*
- *X is a projective scheme of dimension n defined over k,*
- *$H \subset X$ is a base point–free ample hypersurface with $H^n = \alpha$, and*
- *$D \subset H$ is an effective divisor with $D.H^{n-1} = \beta$,*

there is a field extension K/k of degree $[K : k] \leq M(n, \alpha, \beta)$ with the property that $X(K)$ has a K–rational point that does not lie in $D(K)$.

If we restrict ourselves only to infinite fields, then $M(n, \alpha, \beta)$ in Corollary 2.5 can be taken to be α. If we fix a prime p and restrict ourselves only to fields of characteristic p, then $M(n, \alpha, \beta)$ in Corollary 2.5 can be taken to be $\alpha\lceil \log_p(\max\{(n+1)\alpha+1, \beta\})\rceil$.

3. Semistability criterion over perfect fields

Theorem 3.1. *Let X be a geometrically irreducible smooth projective curve of genus g defined over a perfect field k. Fix a positive integer r and an integer d. Then there is an explicit positive integer R that depends only on r, d and g (in particular, R is independent of k) with the following property: A vector bundle E over X of rank r and degree d is semistable if and only if there is a vector bundle F over X of rank R such that $H^i(X,\, E \otimes F) = 0$ for all i.*

Proof. If E is not semistable, then clearly there is no F such that $H^i(X,\, E\otimes F) = 0$ for all i. Let E be a semistable vector bundle over X of rank r and degree d. We will construct R and F.

The moduli space of semistable vector bundles over X of rank r' and degree d' will be denoted by $\mathcal{U}_X(r', d')$.

Let $h := \gcd(r, d)$. Furthermore, we set $\overline{r} := \frac{r}{h}$, and $\overline{d} := \frac{d}{h}$. For any integer $n \geq 1$, consider the morphism $\mathcal{U}_X(n\overline{r}, n(\overline{r}(g-1)-\overline{d})) \longrightarrow \mathcal{U}_X(nr\overline{r}, nr\overline{r}(g-1))$ defined by $V \longmapsto V \otimes E$. Let Θ_E denote the pull back of the natural theta divisor in $\mathcal{U}_X(nr\overline{r}, nr\overline{r}(g-1))$ by this morphism. Therefore, $\Theta_E \times_k \overline{k}$ consists of all semistable vector bundles W over $X_{\overline{k}} = X \times_k \overline{k}$ of rank $n\overline{r}$ and degree $n(\overline{r}(g-1)-\overline{d})$ such that $H^0(X_{\overline{k}},\, W \otimes (E \otimes_k \overline{k})) \neq 0$. (We note that a vector bundle V' over X is semistable if and only if the vector bundle $V' \otimes_k \overline{k}$ over $X_{\overline{k}}$ is semistable; see [4, p. 222].) The subscheme Θ_E defined above is either an effective Cartier divisor in the complete linear system $|h\cdot\Theta|$ or it is the entire moduli space $\mathcal{U}_X(n\overline{r}, n(\overline{r}(g-1)-\overline{d}))$ (cf. [2, § 0.2.1]).

Popa showed that this is indeed a divisor in the linear system $|h\cdot\Theta|$ for all $n \geq \frac{r^2+1}{4}$ (see [7, p. 490, Theorem 5.3]). Let n be the smallest integer such that $n \geq \frac{r^2+1}{4}$. Consider the effective divisor $\Theta_E \subset \mathcal{U}_X(n\overline{r}, n(\overline{r}(g-1)-\overline{d}))$ for $n := \lceil\frac{r^2+1}{4}\rceil$. By Corollary 2.5, there exists an integer M and a field extension K/k such that Θ_E does not contain all K–rational points of $\mathcal{U}_X(n\overline{r}, n(\overline{r}(g-1)-\overline{d}))$.

The integer R in the statement of the theorem will be $n\overline{r}M!$.

Since Θ_E does not contain all K–rational points of $\mathcal{U}_X(n\overline{r}, n(\overline{r}(g-1)-\overline{d}))$, there exists a vector bundle F_1 of rank $n\overline{r}$ defined over $X_K = X \times_k K$, where K/k is some Galois extension of degree dividing $M!$, such that

$$H^0(X_K,\, (E \otimes_k K) \otimes F_1) \,=\, 0 \,=\, H^1(X_K,\, (E \otimes_k K) \otimes F_1)\,. \tag{2}$$

From (2) it follows that

$$H^0(X_K,\, (E \otimes_k K) \otimes \sigma^* F_1) \,=\, 0 \,=\, H^1(X_K,\, (E \otimes_k K) \otimes \sigma^* F_1) \tag{3}$$

for all $\sigma \in \mathrm{Gal}(K/k)$.

Now we consider the direct sum

$$F_2 \,:=\, \bigoplus_{\sigma \in \mathrm{Gal}(K/k)} \sigma^* F_1\,.$$

This F_2 is a vector bundle defined over X. From (3) it follows immediately that that $H^i(X,\, E \otimes F_2) \,=\, 0$ for all i. Also, the rank of F_2 clearly divides $n\overline{r}M!$. Finally we set $m \,:=\, \frac{n\overline{r}M!}{\mathrm{rk}(F_2)}$, and $F \,:=\, F_2^{\oplus m}$, and obtain the asserted vector bundle of rank $R = n\overline{r}M!$. $\square$

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