DELIGNE-BEILINSON COHOMOLOGY

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In these notes we describe the Deligne cohomology of a complex manifold as well as Beilinson's algebraic cohomology theory of a quasi-projective complex manifold and some of its properties. In fact, most of the content of our manuscript can be found (in a more compressed form) in the first paragraph of Beilinson's article [3]. We tried to include all details needed, and we hope that our presentation is sufficiently "down to earth" to serve as an introduction to this theory.

We like to emphasize that credit for the ideas presented here should be given to A. Beilinson, S. Bloch, P. Deligne and some other mathematicians, whereas any possible inaccuracies and errors are due to us (and to our efforts to be as explicit as possible).

In §1 we recall the definition of the (analytic) Deligne cohomology and - following [4] - we give S. Bloch's definition of the regulator map for curves, hoping that the concrete description in this case may help to understand the more formal calculations of the following chapters.
In §2 we describe the Deligne-Beilinson $(D - \overline{B})$ complex on a good compactification of a quasiprojective (real or complex) manifold and the corresponding cohomology theory. The properties of the $D - \overline{B}$ - cohomology arising from abstract nonsense are discussed and some of the cohomology groups are determined. At the end of §2 we explain to some extent the description of the $D - \overline{B}$ - complex $\mathbb{R}(p)_{\mathcal{D}}$ by using real C^{∞} forms.
The formal definition of the $D - \overline{B}$ - cohomology using relative cohomology is explained in §4. This might be a more conceptional approach. However, we have tried to avoid using the relative cohomology as far as possible, although it forces us to use a rather artificial way of defining the product on the $D - \overline{B}$ - complex (3.3).
In §3 the definition and properties of the product are explained. We could not resist to include the calculations of all the compatibilities and homotopies needed.
Without giving all details, we sketch in §5 the usual extensions of

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the definitions of the cohomology theory to simplicial schemes of finite type over $\mathbb{C}$. At the end of this section one constructs a complex of sheaves in the Zarisky topology, which on open subvarieties describes the $\mathcal{D}-\bar{\mathfrak{b}}$ - cohomology.

In §6 we recall the definition and some properties of the cycle class in the De Rham cohomology (following [2],[9] and [1]). Especially we explain the behaviour of those classes with respect to the Hodge filtration. These constructions are needed in §7. There we first explain the relations between the Deligne cohomology of a projective manifold and the intermediate Jacobian of Griffiths. We reproduce Deligne's definition of the cycle class in the $\mathcal{D}-\bar{\mathfrak{b}}$ - cohomology ([10]) and we compare it to the Abel-Jacobi map. Our presentation is slightly different from the one given in [10]. Finally, in §8 we sketch the definition of Chern classes of vector bundles in the $\mathcal{D}-\bar{\mathfrak{b}}$ - cohomology. We do not consider in this note Beilinson's description of the $\mathcal{D}-\bar{\mathfrak{b}}$ cohomology as an extension of Hodge structures.

Notations and conventions:

Throughout these notes X is a complex analytic variety. Even if X happens to be algebraic, it is considered as an analytic variety, except if the index "Zar" is added. Correspondingly $\Omega_X^\bullet$ denotes the De Rham complex of holomorphic differential forms.

We use the notations of the derived category, whenever it is necessary of bounded complexes (even if it is sometimes not explicitly mentioned). A nice introduction can be found in [6] or [14]. In particular we constantly use the notation of a cone of a map $f : A^\bullet \longrightarrow B^\bullet$ of complexes. If the map just exists in the derived category we always replace $B^\bullet$ by an injective resolution.

$\mathbb{H}^\bullet$ is the hypercohomology functor from the derived category of $\mathbb{Z}$-sheaves to the derived category of abelian groups whereas $\mathbb{H}^q(A^\bullet)$ is the q-th cohomology of the complex $\mathbb{H}^\bullet(A^\bullet)$. If A is a subring of $\mathbb{R}$ we write

$$A(p) = (2i\pi)^p \cdot A \subseteq \mathbb{C} .$$

Of course, for the purpose of this volume, one needs the cohomology theory for real algebraic varieties. However, as explained in (2.1,II), this theory is obtained from the one for complex varieties by a quite simple procedure, "compatible with all the statements made in these notes".

§1 The Deligne cohomology

The dilogarithm function and the regulator map on a Riemann surface (after S. Bloch)

1.1. Following [5] we define the Deligne complex $\mathbb{Z}(p)_{\mathcal{D},an}$ on a complex analytic manifold X as

$$0 \longrightarrow \mathbb{Z}(p) \longrightarrow \mathcal{O}_X \longrightarrow \Omega^1_X \longrightarrow \cdots \longrightarrow \Omega^{p-1}_X \longrightarrow 0$$

(where $\mathbb{Z}(p)$ is in degree zero) and the Deligne cohomology as

$$H^q_{\mathcal{D},an}(X,\mathbb{Z}(p)) := \mathbb{H}^q(X,\mathbb{Z}(p)_{\mathcal{D},an}) \ .$$

For simplicity, in this paragraph, we drop the sub-script "an" and write $\mathbb{Z}(p)_{\mathcal{D}}$ and $H^q_{\mathcal{D}}$.

1.2. We define a multiplication

$$\cup : \mathbb{Z}(p)_{\mathcal{D}} \otimes \mathbb{Z}(p')_{\mathcal{D}} \longrightarrow \mathbb{Z}(p+p')_{\mathcal{D}}$$

by
$$x \cup y = \begin{cases} x\cdot y & \text{if } \deg x = 0 \\ x\wedge dy & \text{if } \deg x > 0 \text{ and } \deg y = p' \\ 0 & \text{otherwise .} \end{cases}$$

$\cup$ is a morphism of complexes. In fact, if we denote the differential in $\mathbb{Z}(p)_{\mathcal{D}}$ by d (where, of course, $d : \mathbb{Z}(p) \longrightarrow \mathcal{O}_X$ is the inclusion) and $\mu = \deg x$ and $\mu' = \deg y$, we have:

$$d(x\cup y) = \left\{\begin{matrix} x\cdot dy & \mu = 0,\ \mu' < p' \\ x\cdot dy = dx\wedge dy & \mu = 0,\ \mu' = p' \\ dx\wedge dy & \mu > 0,\ \mu' = p' \\ 0 & \text{otherwise} \end{matrix}\right\} = dx\cup y + (-1)^{\mu} x \cup dy \ .$$

It is quite easy to show that $\cup$ is associative.

1.3. Using the usual arguments from homological algebra, or by calculating the Čech-cohomology on a suitable cover we obtain a ring structure on $\bigoplus_{p,q} H^q_{\mathcal{D}}(X,\mathbb{Z}(p))$. In fact, the product is anticommutative, i.e. for $\alpha \in H^q_{\mathcal{D}}(X,\mathbb{Z}(p))$ and $\beta \in H^{q'}_{\mathcal{D}}(X,\mathbb{Z}(p'))$ $\alpha\cup\beta = (-1)^{qq'}\beta\cup\alpha$. This will be shown in (1.6) for $p = p' = q = q' = 1$ and in a more general

set up in §3. For the reader who wants to check the anticommutativity directly we just reveal that the homotopy between $x \cup y$ and $(-1)^{\mu \cdot \mu'} y \cup x$ is given by:

$$h(x \otimes y) = \begin{cases} 0 & \mu = 0 \text{ or } \mu' = 0 \\ (-1)^{\mu} x \wedge y & \text{otherwise} \end{cases}$$

1.4. <u>Examples for low values of p and q:</u>

i) $(p = 0)$ Obviously $\mathbb{Z}(0)_{\mathcal{D}} = \mathbb{Z}$ and $H^q_{\mathcal{D}}(X,\mathbb{Z}(0))$ is nothing but the singular cohomology $H^q(X,\mathbb{Z})$.

ii) $(p = 1,\ q = 1)$ If $\mathcal{O}^*_X$ denotes the sheaf of invertible holomorphic functions, $\mathbb{Z}(1)_{\mathcal{D}}$ is quasi-isomorphic to $\mathcal{O}^*_X[-1]$ via $x \longrightarrow \exp(x)$. For a suitable open cover $\{U_\alpha\}$ of X an element of $H^1_{\mathcal{D}}(X,\mathbb{Z}(1))$ is represented by a Čech-cocycle

$$(2i\pi\, m_{\alpha\beta}, F_\alpha) \in C^1(\mathbb{Z}(1)) \times C^0(\mathcal{O})$$

where the cocycle condition says

$$\delta(F_\alpha) := F_\beta - F_\alpha = 2i\pi\, m_{\alpha\beta}\,.$$

Hence $f_\alpha := \exp(F_\alpha)$ is the restriction of $f \in H^0(X,\mathcal{O}^*_X)$ and the isomorphism

$$H^1_{\mathcal{D}}(X,\mathbb{Z}(1)) \longrightarrow H^0(X,\mathcal{O}^*_X)$$

maps the cohomology class of the cocycle to f.

iii) $(p = 2,\ q = 2)$ The exponential $x \longmapsto \exp(\frac{x}{2i\pi})$ and multiplication with $-(2i\pi)^{-1}$ on Ω^1_X defines a quasi-isomorphism:

$$\mathbb{Z}(2)_{\mathcal{D}} \longrightarrow (\mathcal{O}^*_X \xrightarrow{d\log} \Omega^1_X)[-1].$$

Hence $\rho \in H^2_{\mathcal{D}}(X,\mathbb{Z}(2))$ can be described by a Čech-cocycle

$$((2i\pi)^2 \cdot n_{\alpha\beta\gamma}, H_{\alpha\beta}, \Omega_\alpha) \in C^2(\mathbb{Z}(2)) \times C^1(\mathcal{O}) \times C^0(\Omega^1)$$

with

$$(2i\pi)^2 n_{\alpha\beta\gamma} = \delta H_{\alpha\beta} \ , \ -dH_{\alpha\beta} = \delta\Omega_\alpha \ .$$

An element of $\mathbb{H}^1(X, \mathcal{O}_X^* \to \Omega_X^1)$ is represented by

$$(\xi_{\alpha\beta}, \omega_\alpha) \in \mathcal{C}^1(\mathcal{O}^*) \times \mathcal{C}^0(\Omega^1)$$

with $\delta\xi_{\alpha\beta} = 1$, $d \log \xi_{\alpha\beta} = \delta\omega_\alpha$.

The image of ρ under the isomorphism of the two cohomology groups is given by

$$\xi_{\alpha\beta} = \exp\left(\frac{1}{2i\pi} H_{\alpha\beta}\right) \quad \text{and} \quad \omega_\alpha = \frac{-1}{2i\pi} \Omega_\alpha \ .$$

iv) The multiplication

$$\cup \ : \ H^1_{\mathcal{D}}(X, \mathbb{Z}(1)) \times H^1_{\mathcal{D}}(X, \mathbb{Z}(1)) \longrightarrow H^2_{\mathcal{D}}(X, \mathbb{Z}(2))$$

$$(2i\pi \ m_{\alpha\beta}, F_\alpha), (2i\pi \ n_{\alpha\beta}, G_\alpha) \longmapsto ((2i\pi)^2 m_{\alpha\beta} n_{\beta\gamma}, 2i\pi \ m_{\beta\alpha} G_\beta, F_\alpha dG_\alpha)$$

can be written via the isomorphisms ii) and iii) as

$$\cup \ : \ H^0(X, \mathcal{O}_X^*) \times H^0(X, \mathcal{O}_X^*) \longrightarrow \mathbb{H}^1(X, \mathcal{O}_X^* \longrightarrow \Omega_X^1)$$

with $f \cup g = \left(g^{-m_{\alpha\beta}}, \frac{-1}{2i\pi} F_\alpha \cdot \frac{dg}{g}\right)$. Hence, for a Čech cover $\{U_\alpha\}$ such that $\log f|_{U_\alpha}$ is defined (and denoted by $\log_\alpha f$) one can describe $f \cup g$ by the cocylce $(\xi_{\alpha\beta}, \omega_\alpha)$ with

$$\xi_{\alpha\beta} = g^{\frac{1}{2i\pi}(\log_\alpha f - \log_\beta f)} \quad \text{and} \quad \omega_\alpha = \frac{-1}{2i\pi} \log_\alpha f \ \frac{dg}{g} \ .$$

1.5. P. Deligne (see [3], 1.3) interprets $\mathbb{H}^1(X, \mathcal{O}_X^* \longrightarrow \Omega_X^1)$ as the group of rank one bundles ξ with holomorphic connection ∇ identifying (ξ, ∇) with the class of the Čech-cocycle $(\xi_{\alpha\beta}, \omega_\alpha)$, where $\xi|_{U_\alpha} = \mathcal{O}_{U_\alpha} \cdot e_\alpha$, $e_\beta = \xi_{\alpha\beta} \cdot e_\alpha$ and $\nabla e_\alpha = \omega_\alpha \cdot e_\alpha$.

By definition $\delta\xi_{\alpha\beta} = 1$ and the Leibniz rule

$$\nabla e_\beta = \omega_\beta \ \xi_{\alpha\beta} \ e_\alpha = \xi_{\alpha\beta} \ \omega_\alpha \ e_\alpha + d\xi_{\alpha\beta} \ e_\alpha$$

implies $\delta\omega_\alpha = d \log \xi_{\alpha\beta}$. The group structure corresponds to the $\otimes$-product of bundles with connection and $\mathcal{O}_X$ equipped with the usual

differential d is the unit. On the other hand each Čech-cocycle comes from a pair (ξ,∇). We have $(\xi,\nabla) = (\mathcal{O}_X,d)$ if and only if ξ has a non-trivial flat section, locally described by $\mu_\alpha \cdot e_\alpha$ with $\mu_\beta \cdot \xi_{\alpha\beta} = \mu_\alpha$ and $0 = \nabla\mu_\alpha \cdot e_\alpha = \mu_\alpha \omega_\alpha e_\alpha + d\mu_\alpha \cdot e_\alpha$. Hence $(\xi,\nabla) = (\mathcal{O}_X,d)$ if and only if $\xi_{\alpha\beta} = \mu_\alpha/\mu_\beta$ and $\omega_\alpha = d\log\mu_\alpha$, that is, if $(\xi_{\alpha\beta},\omega_\alpha)$ is $(\delta,d)(\lambda_\alpha)$ for $\lambda_\alpha = 1/\mu_\alpha$.

If one looks at the exact sequence

$$\mathbb{H}^1(X,\mathcal{O}_X^* \longrightarrow \Omega_X^1) \longrightarrow H^1(X,\mathcal{O}_X^*) \longrightarrow H^1(X,\Omega_X^1)$$

one finds the well known fact that a rank one bundle with trivial first De Rham Chern class has a holomorphic connection.

From now on we will identify the cohomology classes in $\mathbb{H}^1(X,\mathcal{O}_X^* \to \Omega_X^1)$ with the isomorphic-classes of bundles with connection. The product $f \cup g$ in (1.4, iv) defines for two functions $f,g \in H^0(X,\mathcal{O}_X^*)$ a rank one bundle with connection, which we call $r(f,g)$.

Lemma 1.6. (see [4])

a) $r(f,g) \otimes r(g,f) = (\mathcal{O}_X,d)$ for $f,g \in H^0(X,\mathcal{O}_X^*)$.

b) $r(1-g,g) = (\mathcal{O}_X,d)$ if $g,1-g \in H^0(X,\mathcal{O}_X^*)$.

Proof. We choose a Čech-cover such that $\log_\alpha g$, $\log_\alpha f$ (or $\log_\alpha(1-g)$ in part b)) are defined.

a) Then $r(f,g) \otimes r(g,f)$ is represented by

$$\xi_{\alpha\beta} = g^{\frac{1}{2i\pi}(\log_\alpha f - \log_\beta f)} \cdot f^{\frac{1}{2i\pi}(\log_\alpha g - \log_\beta g)} ,$$

$$\omega_\alpha = \frac{-1}{2i\pi}(\log_\alpha f \, \frac{dg}{g} + \log_\alpha g \, \frac{df}{f}) .$$

A flat section is given by

$$\lambda_\alpha = \exp(-\frac{1}{2i\pi} \log_\alpha f \cdot \log_\alpha g) .$$

b) To obtain a flat section one has to find λ_α satisfying

$$\lambda_\beta/\lambda_\alpha = g^{\frac{1}{2i\pi}(\log_\alpha(1-g) - \log_\beta(1-g))}$$

and $\quad \dfrac{d\lambda_\alpha}{\lambda_\alpha} = \dfrac{-1}{2i\pi} \log_\alpha(1-g)\, \dfrac{dg}{g} \quad .$

The second differential equation leads to the solution

$$\lambda_\alpha = \exp\left(-\frac{1}{2i\pi} \int \log_\alpha(1-g)\frac{dg}{g}\right)$$

(S. Bloch's <u>dilogarithm</u> <u>function</u>).

Since $\log_\alpha(1-g) - \log_\beta(1-g)$ is constant on the components of $U_{\alpha\beta}$ one has

$$\begin{aligned}\lambda_\beta/\lambda_\alpha &= \exp\left(\frac{1}{2i\pi} \int (\log_\alpha(1-g) - \log_\beta(1-g))\frac{dg}{g}\right)\\ &= \exp\left(\frac{1}{2i\pi} (\log_\alpha(1-g) - \log_\beta(1-g))\log g\right) .\end{aligned}$$

1.8. From now on, we consider a compact Riemann surface Y, a finite set of points S and $j : X = Y - S \longrightarrow Y$. We define $\mathcal{O}^*_Y(*S)$ to be the sheaf of meromorphic functions, holomorphic and invertible on X and $\Omega^1_Y(\log S)$ to be the sheaf of meromorphic differential forms, holomorphic on X and of logarithmic growth at S. If $f,g \in H^0(Y,\mathcal{O}^*_Y(*S))$ the cocycle of $r(f,g)$ is by (1.4, iv) in fact a cocycle in

$$\mathcal{O}^*_Y(*S) \longrightarrow \Omega^1_Y(\log S).$$

For $x \in S$ let $\mathrm{ord}_x : \mathcal{O}^*_Y(*S) \longrightarrow \mathbb{Z}_x$ denote the order of a zero or pole and let $\mathrm{res}_x : \Omega^1_Y(\log S) \longrightarrow \mathbb{C}_x$ denote the Cauchy-Poincaré residue. We have $\mathrm{res}_x d\log = \mathrm{ord}_x$,

$$\text{kernel } \Big(\coprod_{x\in S} \mathrm{ord}_x\Big) = \mathcal{O}^*_Y \quad \text{and} \quad \text{kernel } \Big(\coprod_{x\in S} \mathrm{res}_x\Big) = \Omega^1_Y .$$

Altogether we obtain a distinguished triangle (see [6] or (2.2) for this notation)

$$(1.9) \qquad \begin{tikzcd} \mathbb{C}^*_Y \arrow[r, "q.i."] & (\mathcal{O}^*_Y \xrightarrow{d\log} \Omega^1_Y) \arrow[r] & (\mathcal{O}^*_Y(*S) \xrightarrow{d\log} \Omega^1_Y(\log S)) \arrow[d, "(\mathrm{ord},\mathrm{res})"] \\ & & \Big(\coprod_{x\in S} \mathbb{Z}_x \hookrightarrow \coprod_{x\in S} \mathbb{C}_x\Big) \arrow[ul, "{[1]}"] \arrow[r, "q.i."', "\exp(2i\pi)"] & \coprod_{x\in S} \mathbb{C}^*_x[-1] \end{tikzcd}$$

The components of the induced map

$$\mathbb{H}^1(Y,\mathcal{O}^*_Y(*S) \longrightarrow \Omega^1_Y(\log S)) \longrightarrow \mathbb{H}^1(Y,\coprod_{x\in S} \mathbb{C}^*_x[-1]) = \coprod_{x\in S} \mathbb{C}^*_x$$

are denoted by ∂_x. If on a $\check{\text{C}}$ech cover $\{U_\alpha\}$ of Y $(\xi_{\alpha\beta}, \omega_\alpha)$ represents an element ρ of the left hand side, then $\partial_x(\rho) = \exp(2i\pi \cdot \mathrm{res}_x \omega_\alpha)$ for any α with $x \in U_\alpha$.

Lemma 1.10.

a) The natural map

$$\varphi : (\mathcal{O}_Y^*(*S) \longrightarrow \Omega_Y^1(\log S)) \longrightarrow Rj_*(\mathcal{O}_X^* \longrightarrow \Omega_X^1)$$

is a quasi-isomorphism.

b) $\partial_x \cdot r = T_x$ where T_x is the "tame-symbol"

$$T_x(f,g) = [(-1)^{\mathrm{ord}_x f \cdot \mathrm{ord}_x g} \cdot g^{\mathrm{ord}_x f} \cdot f^{-\mathrm{ord}_x g}](x)$$

Proof.

a) φ induces a morphism of the triangle (1.9) into the triangle

$$\begin{tikzcd} \mathbb{C}_Y^* \arrow[r] & Rj_*\mathbb{C}_X^* \arrow[r, "q.i."] \arrow[dl] & Rj_*(\mathcal{O}_X^* \longrightarrow \Omega_X^1) \\ \coprod_{x \in S} \mathbb{C}_x^*[-1] \arrow[u, "{[1]}"] & & \end{tikzcd}$$

being an isomorphism at two corners.

b) $\partial_x \circ r$ and T_x are multiplicative in both arguments. As for $\partial_x \circ r$ one has $T_x(f,g) \cdot T_x(g,f) = 1$. If both, f and g are units one has $\partial_x \cdot r(f,g) = 1$ and $T_x(f,g) = 1$. From the definition of T_x one obtains $T_x(1-g,g) = 1$. If t is a local parameter at x we can write $f = u \cdot t^\nu$ and $g = v \cdot t^\mu$ for local units u and v. By multiplicativity and (1.6,a) the proof of b) is reduced to

α) f a unit and $g = t$
β) $f = g = t$,

where we may assume that all poles and zeroes of t are in S. Since

$$r(t,t) \otimes r(\tfrac{1}{t-1}, t) = r(\tfrac{t}{t-1}, t) = r(\tfrac{t-1}{t}, \tfrac{1}{t}) = (\mathcal{O}_X, d)$$

(by (1.6,b)) and since the same holds for T_x we have $\partial_x r(t,t) = \partial_x r(t-1,t)$ and $T_x(t,t) = T_x(t-1,t)$. Hence case β) follows from α). The explicit description of $r(f,g)$ in 1.4, iv)

tells us that for a suitable cover of $Y - S$ $\omega_\alpha = \frac{1}{2i\pi} \log_\alpha f \frac{dt}{t}$ and $\mathrm{res}_x \omega_\alpha = \frac{1}{2i\pi} \log_\alpha f(x)$. Therefore $\partial_x r(f,t) = 1/f(x) = T_x(f,t)$.

1.11. By Matsumoto's description of K_2 of a field one has

$$K_2(\mathbb{C}(Y)) = \mathbb{C}(Y)^* \otimes_{\mathbb{Z}} \mathbb{C}(Y)^*/\langle g \otimes (1-g), g \in \mathbb{C}(Y) - \{0,1\}\rangle .$$

On the other hand, r induces a map

$$\mathbb{C}^*(Y) \otimes_{\mathbb{Z}} \mathbb{C}(Y)^* = \varinjlim_{S \subset Y} H^0(Y-S, \mathcal{O}_Y^*(*S)) \otimes_{\mathbb{Z}} H^0(Y-S, \mathcal{O}_Y^*(*S)) \longrightarrow \varinjlim_{S \subset Y} \mathbb{H}^1(Y-S, \mathcal{O}_Y^*(*S) \to \Omega_Y^1(\log S))$$

whose kernel contains all $g \otimes (1-g)$ (1.6,b). Therefore r factors over

$K_2(\mathbb{C}(Y))$.

From (1.10,a) we have a commutative diagram

$$\begin{tikzcd}
 & K_2(Y) \arrow[r] & K_2(\mathbb{C}(Y)) \arrow[rr, "\coprod T_x"] \arrow[d, "r"] & & \coprod_{x \in X} \mathbb{C}_x^* \arrow[d] \\
0 \arrow[r] & \mathbb{H}^1(Y, \mathcal{O}_Y^* \to \Omega_Y^1) \arrow[r] & \varinjlim_{S \subseteq Y} \mathbb{H}^1(Y, \mathcal{O}_Y^*(*S) \to \Omega_Y^1(\log S)) \arrow[rr, "\coprod \partial_x"] & & \coprod_{x \in X} \mathbb{C}_x^*
\end{tikzcd}$$

where the first line is the exact sequence obtained from the Gersten-Quillen resolution (we just need that this is a complex, which is easier to prove) and the second line is the exact sequence of the triangle (1.9). Therefore we obtain

Theorem 1.12 (Bloch, [4]) r induces a map

$$r : K_2(Y) \longrightarrow H^1(Y, \mathbb{C}^*) = \mathbb{H}^1(Y, \mathcal{O}_Y^* \longrightarrow \Omega_Y^1)$$

(called the regulator map).

Remarks 1.13.

The description due to S. Bloch of the regulator map may serve as an introduction to the constructions of §2. There we will define complexes $F^\cdot(p)$ such that on an open Riemann surface

$$X = Y - S \qquad \mathbb{H}^1(X, F^\cdot(1)) = H^0(Y, \mathcal{O}^*_X(*S)) \tag{2.12}$$

and such that

$$\mathbb{H}^2(X, F^\cdot(2)) = \mathbb{H}^1(X, \mathcal{O}^*_X \longrightarrow \Omega^1_X) = \mathbb{H}^1(Y, \mathcal{O}^*_Y(*S) \longrightarrow \Omega^1_Y(\log S)).$$

It will be even possible to realize $F^\cdot(p)$ as a complex of sheaves in the Zariski-topology, whereas for any algebraic manifold and $q \geqq 1$

$$\mathbb{H}^q_{Zar}(X, \mathbb{Z}(p)_\mathcal{D}) = \mathbb{H}^{q-1}_{Zar}(X, \Omega^{\cdot < p}_X).$$

The reason why this construction is not necessary in the case of a curve is just that the target group of the regulator map is

$$\mathbb{H}^2(X, F^\cdot(2)) \quad \text{and that} \quad 2 > \dim_\mathbb{C} Y. \tag{2.13}$$

§2 <u>The Deligne-Beilinson complex</u>

In this section we want to generalize the definition of the Deligne cohomology in several respects. In particular we want to explain A. Beilinson's "theory with logarithmic growth along the boundary" which - using GAGA - can be viewed as an algebraic version of the Deligne cohomology (see [3]).

For the applications to higher regulators described in this volume, A. Beilinson uses cohomology theories for real algebraic manifolds. The difference between the complex algebraic and the real algebraic theory only comes in when one calculates examples or when one tries to determine the image of the $D-\overline{b}$-cohomology in the Hodge filtration of the De Rham cohomology. Hence, as long as it is not stated otherwise, the definitions and results hold in either of the following situations:

2.1. I. X is an algebraic variety over $\mathbb{C}$ considered with the classical topology and $\mathcal{O}_X$ denotes the sheaf of holomorphic functions. $\mathbb{H}^\cdot$ is the hypercohomology viewed as a functor from the derived category (of complexes) of $\mathbb{Z}$-sheaves on X to the derived category of abelian groups and - for a complex $F^\cdot$ of sheaves - $\mathbb{H}^q(X, F^\cdot)$ is the q-th cohomology of the complex $\mathbb{H}^\cdot(X, F^\cdot)$, as usual calculated by Čech-cohomology or using injective resolutions.

II) X is an algebraic variety over $\mathbb{R}$. Then a sheaf (or a complex of sheaves) F on X is defined to be a pair (F,σ) consisting of a sheaf (or a complex) F on $X(\mathbb{C})$ and an involution σ compatible with the complex conjugation F_∞ on $X(\mathbb{C})$, i.e.: $\sigma : F \xrightarrow{\sim} F_{\infty *}F$. Of course, all morphisms and quasi-isomorphisms of complexes are supposed to be compatible with the involution chosen, $\langle\sigma\rangle \cong \mathbb{Z}/2$ operates on $\mathbb{H}^q(X(\mathbb{C}),F^\cdot)$ and on the complex $\mathbb{H}^\cdot(X(\mathbb{C}),F^\cdot)$ (in the derived category). If $H^\cdot(\langle\sigma\rangle,\ )$ denotes the group cohomology functor on the derived category of abelian groups with σ-action, we define $\mathbb{H}^\cdot(X,F^\cdot) = H^\cdot(\langle\sigma\rangle,\mathbb{H}^\cdot(X(\mathbb{C}),F^\cdot))$ and $\mathbb{H}^q(X,F^\cdot)$ as the q-th cohomology of this complex. In down to earth terms $\mathbb{H}^{p+q}(X,F^\cdot)$ is the abutment of a spectral sequence $H^p(\langle\sigma\rangle,\mathbb{H}^q(X(\mathbb{C}),F^\cdot))$ and, if $F^\cdot$ is a complex of sheaves over $\mathbb{Q}$, $\mathbb{H}^q(X,F^\cdot)$ are the invariants $\mathbb{H}^q(X(\mathbb{C}),F^\cdot)^\sigma$.

Examples:

On the constant sheaf $\mathbb{C}$ on $X(\mathbb{C})$, there are two possible involutions: $F_\infty:\mathbb{C} \longrightarrow F_{\infty *}\mathbb{C} = \mathbb{C}$ acting on $\mathbb{C}$ as identity and $\sigma:\mathbb{C} \longrightarrow F_{\infty *}\mathbb{C}$ acting as complex conjugation. <u>We always assume that the sheaf $\mathbb{C}$ on X is the pair $(\mathbb{C},\sigma)$.</u> Correspondingly, if $S^\cdot_{X(\mathbb{C})}$ denotes the complex of $\mathbb{R}$-valued C^∞ forms the involution chosen on $A^\cdot_{X(\mathbb{C})} = S^\cdot_{X(\mathbb{C})}\otimes_{\mathbb{R}}\mathbb{C}$ is the one induced by σ on the second factor. Restricting this to the subcomplex $\Omega^\cdot_{X(\mathbb{C})}$ of holomorphic forms we obtain the involution operating on the coefficients of a differential form by conjugation. On the algebraic differential forms this corresponds to the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ induced by base change from $\mathbb{R}$ to $\mathbb{C}$ on the algebraic Kähler differentials. Denoting all those involutions by σ we remark that σ respects the Hodge decompostion of $H^k(X(\mathbb{C}),\mathbb{C})$ i.e.: $\sigma(H^{k-p,p}) = H^{k-p,p}$.

2.2. Let $u:A^\cdot \longrightarrow B^\cdot$ be a morphism of complexes of sheaves on X. The cone of u is the complex

$$\mathrm{Cone}(A^\cdot \xrightarrow{u} B^\cdot) = C^\cdot_u := A^\cdot[1] \oplus B^\cdot$$

with the differentials

$$A^{q+1} \oplus B^q \xrightarrow{\delta} A^{q+2} \oplus B^{q+1}$$

$$(a,b) \overset{\delta}{\longmapsto} (-d(a),u(a)+d(b)).$$

The natural inclusion $B^q \longrightarrow C_u^q$ and the projection complete the triangle

$$\begin{tikzcd} A^\bullet \arrow[rr] & & B^\bullet \arrow[dl] \\ & C_u^\bullet \arrow[ul, "{[1]}"] & \end{tikzcd}$$

(where $B^\bullet \hookrightarrow C_u^\bullet \twoheadrightarrow A^\bullet[1]$ is exact).

An arbitrary triangle in the derived category is distinguished, if it is the image of one of those just constructed. If one applies a derived functor to a distinguished triangle one obtains a distinguished triangle, For example if

$$\begin{tikzcd} A^\bullet \arrow[rr] & & B^\bullet \arrow[dl] \\ & C^\bullet \arrow[ul, "{[1]}"] & \end{tikzcd}$$

is distinguished, then

$$\begin{tikzcd} \mathbb{H}^\bullet(A^\bullet) \arrow[rr] & & \mathbb{H}^\bullet(B^\bullet) \arrow[dl] \\ & \mathbb{H}^\bullet(C^\bullet) \arrow[ul, "{[1]}"] & \end{tikzcd}$$

(where $\mathbb{H}^\bullet$ denotes the hypercohomology functor in the derived category)

is distinguished and - regarding the cohomology of the complexes $\mathbb{H}^\bullet(A^\bullet)$, $\mathbb{H}^\bullet(B^\bullet)$ and $\mathbb{H}^\bullet(C^\bullet)$ - one obtains the long exact sequence

$$\cdots \longrightarrow \mathbb{H}^q(A^\bullet) \longrightarrow \mathbb{H}^q(B^\bullet) \longrightarrow \mathbb{H}^q(C^\bullet) \longrightarrow \mathbb{H}^{q+1}(A^\bullet) \longrightarrow \cdots$$

(see [6] or [14] for a nice introduction).

<u>Lemma 2.3.</u> Let $u_1 : A_1^\bullet \longrightarrow B^\bullet$ and $u_2 : A_2^\bullet \longrightarrow B^\bullet$ be two morphisms of complexes and $C^\bullet = \mathrm{Cone}\,(A_1^\bullet \oplus A_2^\bullet \xrightarrow{u_1-u_2} B^\bullet)[-1]$. Then

$$\begin{aligned} C^\bullet &= \mathrm{Cone}(A_1^\bullet \xrightarrow{u_1} \mathrm{Cone}\ (A_2^\bullet \xrightarrow{-u_2} B^\bullet))[-1] \\ &= \mathrm{Cone}(A_2^\bullet \xrightarrow{-u_2} \mathrm{Cone}\ (A_1^\bullet \xrightarrow{u_1} B^\bullet))[-1]\ . \end{aligned}$$

<u>Proof.</u> All three complexes are equal to $A_1^\bullet \oplus A_2^\bullet \oplus B^\bullet[-1]$ with the differential $\delta[-1] = -\delta$, i.e.:

$$(a_1,a_2,b) \longmapsto -(-d(a_1),-d(a_2),u_1(a_1) - u_2(a_2) + d(b))\ .$$

Corollary 2.4. Using the notations from 2.3. we have three long exact sequences:

a) $$\longrightarrow \mathbb{H}^q(C^\bullet) \longrightarrow \mathbb{H}^q(A_1^\bullet) \oplus \mathbb{H}^q(A_2^\bullet) \longrightarrow \mathbb{H}^q(B^\bullet) \longrightarrow \mathbb{H}^{q+1}(C^\bullet) \longrightarrow$$

b) $$\longrightarrow \mathbb{H}^q(C^\bullet) \longrightarrow \mathbb{H}^q(A_1^\bullet) \longrightarrow \mathbb{H}^q(\mathrm{Cone}(A_2^\bullet \xrightarrow{-u_2} B^\bullet)) \longrightarrow \mathbb{H}^{q+1}(C^\bullet) \longrightarrow$$

c) $$\longrightarrow \mathbb{H}^q(C^\bullet) \longrightarrow \mathbb{H}^q(A_2^\bullet) \longrightarrow \mathbb{H}^q(\mathrm{Cone}(A_1^\bullet \xrightarrow{u_1} B^\bullet)) \longrightarrow \mathbb{H}^{q+1}(C^\bullet) \longrightarrow .$$

2.5 Let X be a n-dimensional algebraic manifold (over $\mathbb{C}$ over $\mathbb{R}$). A "good compactification" of X is a proper algebraic manifold $\overline{X}$ with an embedding $j : X \longrightarrow \overline{X}$ such that $D = \overline{X} - X$ is a normal crossing divisor (i.e.: locally in the analytic topology D has smooth components intersecting transversally).

Let $\Omega^\bullet_{\overline{X}}$ (log D) be the De Rham complex of meromorphic forms on $\overline{X}$, holomorphic on X and with at most logarithmic poles along D. We have a filtration of $\Omega^\bullet_{\overline{X}}$ (log D) by subcomplexes

$$F^p_D = (0 \longrightarrow \Omega^p_{\overline{X}}(\log D) \longrightarrow \Omega^{p+1}_{\overline{X}}(\log D) \longrightarrow \dots \longrightarrow \Omega^n_{\overline{X}}(\log D)).$$

The properties of logarithmic forms needed are (see [7]):

a) Since j is affine $Rj_*\Omega^\bullet_X = j_*\Omega^\bullet_X$. There are quasi-isomorphisms:

$$Rj_*\mathbb{C} \longrightarrow Rj_*\Omega^\bullet_X = j_*\Omega^\bullet_X \longleftarrow \Omega^\bullet_{\overline{X}}(\log D)$$

and hence

$$H^q(X,\mathbb{C}) = \mathbb{H}^q(\overline{X},\Omega^\bullet_{\overline{X}}(\log D)).$$

b) The natural maps

$$\tau_p : \mathbb{H}^q(\overline{X},F^{p+1}_D) \longrightarrow \mathbb{H}^q(\overline{X},F^p_D) \quad \text{and}$$

$$\tau : \mathbb{H}^q(\overline{X},F^p_D) \longrightarrow \mathbb{H}^q(\overline{X},\Omega^\bullet_{\overline{X}}(\log D)) \text{ are injective.}$$

$H^q(X,\mathbb{C})$ carries a mixed Hodge structure, and the Hodge filtration $F^pH^q(X,\mathbb{C})$ is given by $\mathrm{Im}(\tau)$. Moreover the cokernel $H^q(X,\mathbb{C})/F^pH^q(X,\mathbb{C})$ of τ is the same as $\mathbb{H}^q(\overline{X},\Omega^{<p}_{\overline{X}}(\log D))$ where $\Omega^{<p}_{\overline{X}}(\log D)$ denotes the complex

$$0 \longrightarrow \mathcal{O}_{\overline{X}} \longrightarrow \Omega^1_{\overline{X}}(\log D) \longrightarrow \dots \longrightarrow \Omega^{p-1}_{\overline{X}}(\log D) \longrightarrow 0.$$

The cokernel of τ_p is $H^{q-p}(\overline{X},\Omega^p_{\overline{X}}(\log D))$.

c) By GAGA, $\mathbb{H}^q(\overline{X},F^p_D)$ can be calculated using the corresponding complex of algebraic differential forms in the Zariski topology.

d) $\mathbb{H}^q(\overline{X},F^p_D)$ is independent of the good compactification chosen.

Definition 2.6. Let A be a subring of $\mathbb{R}$ and $A(p) = (2i\pi)^p \cdot A \subseteq \mathbb{C}$. The Deligne-Beilinson complex (D - $\bar{B}$ - complex) of $(\overline{X},X)$ is $A(p)_{\mathcal{D}} = A(p)_{\mathcal{D},\overline{X}} = \text{Cone}\ (Rj_*A(p) \oplus F^p_D \xrightarrow{\varepsilon-\iota} Rj_*\Omega^{\cdot}_X)[-1]$ where ε and ι are the natural maps and where $Rj_*\Omega^{\cdot}_X$ is represented in such a way that both maps exist (for example by the direct image of an injective resolution of $\Omega^{\cdot}_X$).

If $f : Y \longrightarrow X$ is a morphism of algebraic manifolds we can choose good compactifications $\overline{Y}$ and $\overline{X}$ such that f extends to $\overline{f} : \overline{Y} \longrightarrow \overline{X}$. Thereby we obtain a morphism $\overline{f}^*: A(p)_{\mathcal{D},\overline{X}} \longrightarrow \overline{f}_*A(p)_{\mathcal{D},\overline{Y}}$.

2.7. Other descriptions: By (2.3) we may write as well:

$$A(p)_{\mathcal{D}} = \text{Cone}\ (F^p_D \longrightarrow Rj_*(\text{Cone}\ (A(p) \longrightarrow \Omega^{\cdot}_X)))[-1] \text{ or}$$

$$A(p)_{\mathcal{D}} = \text{Cone}\ (Rj_*A(p) \longrightarrow \text{Cone}\ (F^p_D \longrightarrow Rj_*\Omega^{\cdot}_X))[-1] .$$

Using the second description one sees immediately that $\mathbb{Z}(p)_{\mathcal{D}|X}$ is quasi-isomorphic to the complex $\mathbb{Z}(p)_{\mathcal{D},an}$ defined in (1.1). A quasi-isomorphism $\alpha : \mathbb{Z}(p)_{\mathcal{D},an} \longrightarrow \mathbb{Z}(p)_{\mathcal{D}|X}$ is given by

$$\begin{array}{ccccccccccccc}
\mathbb{Z}(p) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \dots & \longrightarrow & \Omega^{p-2}_X & \longrightarrow & \Omega^{p-1}_X & \longrightarrow & 0 & & \\
\downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_1} & & & & \downarrow{\scriptstyle \alpha_{p-1}} & & \downarrow{\scriptstyle \alpha_p} & & \downarrow & & \\
\mathbb{Z}(p) & \xrightarrow[-\varepsilon]{} & \mathcal{O}_X & \xrightarrow[-\delta_1]{} & \dots & \longrightarrow & \Omega^{p-2}_X & \xrightarrow[-\delta_{p-1}]{} & \Omega^p_X \oplus \Omega^{p-1}_X & \xrightarrow[-\delta_p]{} & \Omega^{p+1}_X \oplus \Omega^p_X & & \dots
\end{array}$$

for $\alpha_p(\omega) = (d\omega,\omega)\cdot(-1)^p$ and $\alpha_i(\omega) = (-1)^i\cdot\omega$. The proof follows easily since $\delta_{p-1}(\eta) = (0,d\eta)$ and $\delta_p(\psi,\eta) = (-d\psi,-\psi+d\eta)$.

Lemma 2.8. $\mathbb{H}^q(\overline{X},A(p)_{\mathcal{D}})$ is independent of the good compactification chosen.

Proof. $\mathbb{H}^{\cdot}(\overline{X},A(p)_{\mathcal{D}})$ is one edge of a distinguished triangle whose other two edges,

$$\mathbb{H}^{\cdot}(\overline{X},Rj_*A(p)) \oplus \mathbb{H}^{\cdot}(\overline{X},F_D^p) \quad \text{and} \quad \mathbb{H}^{\cdot}(\overline{X},Rj_*\Omega_X^{\cdot}),$$

remain quasi-isomorphic under τ^* for a morphism $\tau : \overline{X}' \longrightarrow \overline{X}$ between good compactifications of X.

Since each manifold over $\mathbb{C}$ allows a good compactification we can define:

<u>Definition 2.9.</u> Let X be an algebraic manifold (over $\mathbb{C}$ or $\mathbb{R}$). Then the <u>Deligne-Beilinson cohomology</u> (or $D-\overline{b}$ cohomology) is defined as

$$H^q_{\mathcal{D}}(X,A(p)) = \mathbb{H}^q(\overline{X},A(p)_{\mathcal{D}}) \ .$$

Keeping in mind that $\mathrm{Cone}(F_D^p \longrightarrow Rj_*\Omega_X^{\cdot})$ is quasi-isomorphic to $\Omega_X^{<p}(\log D)$ and that $\mathrm{Cone}\,(Rj_*A(p) \longrightarrow Rj_*\Omega_X^{\cdot}) = Rj_*\,\mathbb{C}/A(p)$ we can rewrite (2.4) as:

<u>Corollary 2.10.</u> There are long exact sequences

a) $$\longrightarrow H^q_{\mathcal{D}}(X,A(p)) \longrightarrow H^q(X,A(p)) \oplus F^pH^q(X,\mathbb{C}) \longrightarrow H^q(X,\mathbb{C}) \longrightarrow$$
$$\longrightarrow H^{q+1}_{\mathcal{D}}(X,A(p)) \longrightarrow \ldots$$

b) $$\longrightarrow H^q_{\mathcal{D}}(X,A(p)) \longrightarrow H^q(X,A(p)) \longrightarrow H^q(X,\mathbb{C})/F^p \longrightarrow H^{q+1}_{\mathcal{D}}(X,A(p)) \longrightarrow$$

c) $$\longrightarrow H^q_{\mathcal{D}}(X,A(p)) \longrightarrow F^pH^q(X,\mathbb{C}) \longmapsto H^q(X,\mathbb{C}/A(p)) \longrightarrow H^{q+1}_{\mathcal{D}}(X,A(p)) \longrightarrow$$

<u>Proposition 2.12.</u>

i) $H^q_{\mathcal{D}}(X,A(p)) = 0$ for $q \leq 0$ and $p \geq 1$

ii) $H^1_{\mathcal{D}}(X,A(1)) = \{f \in H^0(\overline{X},j_*\mathcal{O}_X/A(1))\,;\ df \in H^0(\overline{X},\Omega^1_{\overline{X}}(\log D))\}$

iii) Let $\mathcal{O}(X)^*_{alg}$ denote the group of algebraic invertible functions on X. Then there is a natural map

$$\rho : \mathcal{O}(X)^*_{alg} \longrightarrow H^1_{\mathcal{D}}(X,A(1)).$$

For $A = \mathbb{Z}$ the map ρ is an isomorphism.

Proof. Since $H^q(X,\mathbb{C}/A(p)) = 0$ for $q < 0$ and $F^pH^0(X,\mathbb{C}) = 0$ for $p \geq 1$ i) follows from (2.10,c).

ii) We have a morphism of complexes

$$\begin{array}{rcl} \widetilde{A(1)} & := & \mathrm{Cone}(F^1_D \longrightarrow j_*\mathrm{Cone}(A(1) \longrightarrow \Omega^\bullet_X))[-1] \\ & & \quad\ \downarrow \\ A(1)_{\mathcal{D}} & = & \mathrm{Cone}(F^1_D \longrightarrow Rj_*\mathrm{Cone}(A(1) \longrightarrow \Omega^\bullet_X))[-1] \end{array}$$

By (2.4,c) we obtain

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & H^0(\overline{X},j_*\mathbb{C}/A(1)) \arrow[r] \arrow[d, equal] & \mathbb{H}^1(\overline{X},\widetilde{A(1)}) \arrow[r] \arrow[d, "\eta"] & F^1H^1(X,\mathbb{C}) \arrow[r] \arrow[d, equal] & H^1(\overline{X},j_*\mathbb{C}/A(1)) \arrow[d, hook] \\ 0 \arrow[r] & H^0(X,\mathbb{C}/A(1)) \arrow[r] & H^1_{\mathcal{D}}(X,A(1)) \arrow[r] & F^1H^1(X,\mathbb{C}) \arrow[r] & H^1(X,\mathbb{C}/A(1)) \end{tikzcd}$$

and - using the five - Lemma we find η to be an isomorphism. $\widetilde{A(1)}$ is quasi isomorphic to

$$0 \longrightarrow \Omega^1_{\overline{X}}(\log D) \oplus j_*\mathcal{O}_X/A(1) \xrightarrow{\Delta} \Omega^2_{\overline{X}}(\log D) \oplus j_*\Omega^1_X \longrightarrow \ldots$$

$$(\omega,f) \longmapsto (+d\omega,+\omega-df)$$

and $H^1_{\mathcal{D}}(X,A(1))$ is given by $H^0(\ker \Delta)$.

iii) The inclusion $\mathbb{Z}(1) \longrightarrow A(1)$ induces $H^1_{\mathcal{D}}(X,\mathbb{Z}(1)) \longrightarrow H^1_{\mathcal{D}}(X,A(1))$ and we just have to consider $\mathbb{Z} = A$.
Since

$$\begin{tikzcd} \mathcal{O}_X/\mathbb{Z}(1) \arrow[rr, "d"] \arrow[dr, "\exp"'] & & \Omega^1_X \\ & \mathcal{O}^*_X \arrow[ur, "d\log"'] & \end{tikzcd}$$ commutes,

and since $\varphi \in H^0(\overline{X},j_*\mathcal{O}^*_X)$ is meromorphic along D if and only if $d\log\varphi \in H^0(\overline{X},\Omega^1_{\overline{X}}(\log D))$, we obtain from ii) that

$$H^1_{\mathcal{D}}(X,\mathbb{Z}(1)) = \{\varphi \in H^0(\overline{X},j_*\mathcal{O}^*_X);\ \varphi \text{ meromorphic along } D\}.$$

By GAGA, the meromorphic functions $\varinjlim_{\nu} H^0(\overline{X},\mathcal{O}_X(\nu\cdot D))$ are the same as the algebraic functions.

2.13. Remark. As in (1.1) one defines

$$H^q_{\mathcal{D},an}(X,A(p)) = \mathbb{H}^q(X,A(p) \longrightarrow \mathcal{O}_X \longrightarrow \dots \longrightarrow \Omega_X^{p-1})$$

which - by (2.7) - is the same as $\mathbb{H}^q(X,A(p)_{\mathcal{D}}|_X)$. One has the natural map

$$H^q_{\mathcal{D}}(X,A(p)) \longrightarrow H^q_{\mathcal{D},an}(X,A(p)).$$

This map is - of course - an isomorphism if X is compact, but also if $p > \dim X$, since in this case

$$A(p)_{\mathcal{D}} = \text{Cone }(Rj_*A(p) \to Rj_*\Omega^\cdot_X)[-1] = Rj_*(A(p)_{\mathcal{D}|X}) .$$

However, for example for $q = p = 1$ and $A = \mathbb{Z}$, we have just seen that

$$\mathcal{O}(X)^*_{alg} = H^1_{\mathcal{D}}(X,\mathbb{Z}(1)) \underset{\neq}{\hookrightarrow} H^1_{\mathcal{D},an}(X,\mathbb{Z}(1)) = H^0(X,\mathcal{O}^*_X).$$

2.14. <u>The "real" $D-\bar{b}$-cohomology</u>

Let $S^\cdot_X$ be the complex of $\mathbb{R}$-valued C^∞ forms over $X(\mathbb{C})$ and $A^\cdot_X$ be the complex of $\mathbb{C}$-valued C^∞ forms. Since $\mathbb{C} = \mathbb{R}(p) \oplus \mathbb{R}(p-1)$ for all p, one has maps

$$\pi_{p-1} : \Omega^\cdot_X \longrightarrow A^\cdot_X = S^\cdot_X \otimes_{\mathbb{R}} (\mathbb{R}(p) \oplus \mathbb{R}(p-1)) \longrightarrow S^\cdot_X(p-1) := S^\cdot_X \otimes_{\mathbb{R}} \mathbb{R}(p-1) .$$

In the derived category those are the same as the projections $\mathbb{C} \longrightarrow \mathbb{R}(p-1)$. Therefore we have quasi-isomorphisms

$$\text{Cone}(\mathbb{R}(p) \longrightarrow \Omega^\cdot_X) \longrightarrow S^\cdot_X(p-1) .$$

We denote the induced maps $F^p_D \longrightarrow j_*\Omega^\cdot_X \longrightarrow j_*S^\cdot_X(p-1)$ also by π_{p-1}. Since $Rj_*S^\cdot_X(p-1) = j_*S^\cdot_X(p-1)$, (2.7) implies:

<u>Lemma 2.15.</u> Let $\widetilde{\mathbb{R}(p)}_{\mathcal{D}} := \text{Cone }(F^p_D \xrightarrow{-\pi_{p-1}} j_*S^\cdot_X(p-1))[-1]$, and let $\rho_p : \mathbb{R}(p)_{\mathcal{D}} \longrightarrow \widetilde{\mathbb{R}(p)}_{\mathcal{D}}$ be the morphism given by $\rho_{p|F^p_D} = \mathrm{id}$, $\rho_{p|\mathbb{R}(p)} = 0$ and $\rho_{p|Rj_*\Omega^\cdot_X} = \pi_{p-1}$. Then ρ_p is a quasi-isomorphism.

<u>Corollary 2.16.</u>

a) For $q \leq p$ $H^q_{\mathcal{D}}(X,\mathbb{R}(p))$ is the q-th cohomology of the complex $H^0(X,\widetilde{\mathbb{R}(p)}_{\mathcal{D}})$.

b)

$$H^1_{\mathcal{D}}(X,\mathbb{R}(1)) = \{\eta \in H^0(\overline{X},j_*S^0_X);\ d\eta \text{ lies in } \mathrm{Im}(H^0(\overline{X},\Omega^1_{\overline{X}}(\log D)) \xrightarrow{\pi_0} H^0(\overline{X},j_*S^1_X))\}$$

$$= \{\eta \in H^0(\overline{X},j_*S^0_X);\ d_z\eta \in H^0(\overline{X},\Omega^1_{\overline{X}}(\log D))\}.$$

More precisely, if $d\eta = \pi_0(\varphi)$ then $d_z\eta = \frac{1}{2}\varphi$.

c) If $\dim X = 1$ then

$$H^2_{\mathcal{D}}(X,\mathbb{R}(2)) = H^1(X,\mathbb{R}(1)) .$$

Proof. $\widetilde{\mathbb{R}(p)}_{\mathcal{D}}$ is the complex

$$0 \to j_*S^0_X(p-1) \to \ldots \to j_*S^{p-2}_X(p-1) \to \Omega^p_{\overline{X}}(\log D) \oplus j_*S^{p-1}_X(p-1) \to \Omega^{p+1}_{\overline{X}}(\log D) \oplus j_*S^p_X(p-1) \to \ldots$$

where $j_*S^0_X(p-1)$ is in degree one. Since all the $j_*S^i_X(p-1)$ are acyclic one obtains a).
For $p = q = 1$ a) implies that $H^1_{\mathcal{D}}(X,\mathbb{R}(1))$ is the kernel of

$$H^0(\overline{X},\Omega^1_{\overline{X}}(\log D)) \oplus H^0(\overline{X},j_*S^0) \longrightarrow H^0(\overline{X},\Omega^2_{\overline{X}}(\log D)) \oplus H^0(\overline{X},j_*S^1_X)$$

$$(\varphi,\eta) \longmapsto (d\varphi, +\pi\varphi - d\eta) .$$

If $d\eta = \pi_0\varphi$ then $d_z\eta = \frac{1}{2}\varphi$ and $d\varphi = 0$, and we obtain the two descriptions of $H^1_{\mathcal{D}}(X,\mathbb{R}(1))$ given in b) .

c) is obvious since on a curve $F^2_D = 0$ and $\widetilde{\mathbb{R}(2)}_{\mathcal{D}} = j_*S^{\cdot}_X(1)[-1]$ is quasi-isomorphic to $Rj_*\mathbb{R}(1)[-1]$.

2.17. Remarks.

i) The isomorphism between the two explicit descriptions of $H^1_{\mathcal{D}}(X,\mathbb{R}(1))$ obtained in (2.12,ii) and (2.16,b) is given by $f \longmapsto \pi_0(f) = \eta$ and $df = 2d_z(\pi_0(f))$.

ii) Using the language of currents ([11], Chap. 3.1), one can rewrite (2.16,b) in a slightly different way. For example, if X is a curve and $S = \overline{X} - X$, we write $a_x = \mathrm{Res}_x(2d_z\eta) = \mathrm{Res}_x(d\varphi)$ for $x \in S$. Since $d_z\eta \in H^0(\overline{X},\Omega^1_{\overline{X}}(\log D))$, η has logarithmic poles and both, η and $d_z\eta$, are integrable. If $T_{d_z\eta}$ denotes the current associated to $d_z\eta$ the

generalized Cauchy formula implies (loc. cit.)

$$d_{\bar{z}} T_{d_z \eta} = 2i\pi \sum_{x \in S(\mathbb{C})} \mathrm{Res}_x(d_z\eta)\delta_x = i\pi \sum a_x \delta_x$$

where δ_x is the Dirac distribution. On the other hand, this equality implies that $d_z\eta$ has at most logarithmic poles. Hence

$$H^1_{\mathcal{D}}(X,\mathbb{R}(1)) = \{\eta \in H^0(X,S^0_X);\eta \text{ integrable and } d_{\bar{z}} T_{d_z \eta} = i\pi \sum_{x\in S(\mathbb{C})} a_x\delta_x\} .$$

§3 Products

The aim of this section is to extend the definition of the product given in (1.2) on $\mathbb{Z}(p)_{\mathcal{D},an}$ to the full $D - \bar{b}$ complex of a pair $(\bar{X},X)$, where, as in § 2, $\bar{X}$ is a good compactification of X (See [3]).

3.1. Example: We define

$$\cup_0 : A(p)_{\mathcal{D}|X} \otimes A(q)_{\mathcal{D}|X} \longrightarrow A(p+q)_{\mathcal{D}|X}$$

by

$$x \cup_0 y = \begin{cases} x\cdot y & \text{if } x \in A(p),\ y \in A(q) \\ x\cdot y & \text{if } x \in A(p),\ y \in \Omega^{\cdot}_X \\ x\wedge y & \text{if } x \in F^p,\ y\in F^q \\ x \wedge y & \text{if } x\in\Omega^{\cdot}_X,\ y \in F^q \\ 0 & \text{otherwise} \end{cases}$$

where x (and y) are supposed to be a local section of $A(p)$, F^p or $\Omega^{\cdot}_X$. For $A = \mathbb{Z}$ this product is compatible under the quasi-isomorphism α described in (2.7) with the product defined in (1.1), i.e.

$$\begin{tikzcd} \mathbb{Z}(p)_{\mathcal{D},an} \otimes \mathbb{Z}(q)_{\mathcal{D},an} \arrow[r, "\cup"] \arrow[d, "(\alpha\otimes\alpha)"] & \mathbb{Z}(p+q)_{\mathcal{D},an} \arrow[d, "\alpha"] \\ \mathbb{Z}(p)_{\mathcal{D}|X} \otimes \mathbb{Z}(q)_{\mathcal{D}|X} \arrow[r, "\cup_0"] & \mathbb{Z}(p+q)_{\mathcal{D}|X} \end{tikzcd}$$

is commutative.

Definition 3.2. Let $\alpha \in \mathbb{R}$. Then we define a product

$\cup_\alpha : A(p)_{\mathcal{D}} \otimes A(q)_{\mathcal{D}} \longrightarrow A(p+q)_{\mathcal{D}}$ by the following table:

	a_q	f_q	ω_q
a_p	$a_p \cdot a_q$	0	$(1-\alpha) a_p \cdot \omega_q$
f_p	0	$f_p \wedge f_q$	$(-1)^{\deg f_p} \cdot \alpha \cdot f_p \wedge \omega_q$
ω_p	$\alpha \cdot \omega_p \cdot a_q$	$(1-\alpha)\omega_p \wedge f_q$	0

representing elements of

	$A(q)$	F_D^q	$\Omega_X^\cdot$
$A(p)$	$A(p+q)$	0	$\Omega_X^\cdot$
F_D^p	0	F_D^{p+q}	$\Omega_X^\cdot$
$\Omega_X^\cdot$	$\Omega_X^\cdot$	$\Omega_X^\cdot$	0

concentrated in one degree.

3.3. To make sense out of this definition of a product one should interprete this table in the following way:
On X we have the products

$$A(p) \otimes A(q) \longrightarrow A(p+q)$$
$$A(p) \otimes \Omega_X^\cdot \longrightarrow \Omega_X^\cdot$$
$$\Omega_X^\cdot \otimes A(q) \longrightarrow \Omega_X^\cdot$$
$$j^*F_D^p \otimes \Omega_X^\cdot \longrightarrow \Omega_X^\cdot$$
$$\Omega_X^\cdot \otimes j^*F_D^q \longrightarrow \Omega_X^\cdot$$

as described above. They fit together to define a product

$$\cup_\alpha : \mathrm{Cone}(A(p) \oplus j^*F_D^p \longrightarrow \Omega_X^\cdot)[-1] \otimes \mathrm{Cone}(A(q) \oplus j^*F_D^q \longrightarrow \Omega_X^\cdot)[-1] \longrightarrow$$
$$\longrightarrow \mathrm{Cone}(A(p+q) \longrightarrow \Omega_X^\cdot)[-1].$$

One has to verify for elements γ and γ' of degree μ and μ' that
$$\delta(\gamma \cup_\alpha \gamma') = \delta\gamma \cup_\alpha \gamma' + (-1)^{\mu}\gamma \cup_\alpha \delta\gamma' .$$
Here again δ is as in (2.2) and $-\delta = \delta[-1]$ is the differential in the cone, shifted by -1. The left hand side is

$a_p \cdot a_q$	0	$(1-\alpha)a_p d\omega_q$
0		$(-1)^{\mu}\cdot\alpha\, df_p \wedge \omega_q +$ $(-1)^{2\mu}\alpha\, f_p \wedge d\omega_q$
$\alpha \cdot a_q d\omega_p$	$(1-\alpha)d\omega_p \wedge f_q +$ $(1-\alpha)(-1)^{\mu-1}\omega_p\, df_q$	0

whereas the right hand side is

$\alpha \cdot a_p \cdot a_q$ $+(1-\alpha)a_p \cdot a_q$	$(1-\alpha)a_p \cdot f_q$ $+(1-\alpha)a_p \cdot (-f_q)$	0 $+(1-\alpha)a_p d\omega_q$
$\alpha(-f_p) \cdot a_q$ $+(-1)^{2\mu}\alpha \cdot f_p \cdot a_q$		$(-1)^{\mu+1}\alpha(-df_p)\wedge\omega_q$ $+(-1)^{2\mu}\alpha f_q \wedge d\omega_p$
$\alpha \cdot a_q \cdot d\omega_p$ $+ 0$	$(1-\alpha)d\omega_p \wedge f_q$ $+(-1)^{\mu}(1-\alpha)\omega_p \wedge (-df_q)$	0

Here the entries live in

	$A(q)$	$(j_*F_D^q)^{\mu'}$	$\Omega_X^{\mu'-1}$
$A(p)$	$\mathcal{O}_X$	0	$\Omega_X^{\mu'}$
$(j_*F_D^p)^{\mu}$	0		$\Omega_X^{\mu+\mu'}$
$\Omega_X^{\mu-1}$	Ω_X^{μ}	$\Omega_X^{\mu+\mu'}$	0

Taking injective resolutions of F_D, $A(\ )$, $\Omega_X^{\cdot}$ we obtain a product

$$A(p)_{\mathcal{D}} \otimes A(q)_{\mathcal{D}} \longrightarrow Rj_*\mathrm{Cone}(A(p) \oplus j_*F_D^p \longrightarrow \Omega_X^{\cdot})[-1] \otimes Rj_*\mathrm{Cone}(A(q) \oplus j_*F_D^q \longrightarrow$$

$$\longrightarrow \Omega_X^{\cdot})[-1] \longrightarrow Rj_*\mathrm{Cone}(A(p+q) \longrightarrow \Omega_X^{\cdot})[-1] .$$

We complete this product to a product

$$A(p)_{\mathcal{D}} \otimes A(q)_{\mathcal{D}} \longrightarrow A(p+q)_{\mathcal{D}} = \mathrm{Cone}(F_D^{p+q} \longrightarrow Rj_*\mathrm{Cone}(A(p+q) \longrightarrow \Omega_X^{\cdot}))[-1]$$

by taking the usual wedge product $F_D^p \otimes F_D^q \longrightarrow F_D^{p+q}$. This is possible since - by the following computations - the wedge product commutes with the differentials in $A(\ )_{\mathcal{D}}$. One has

$$\delta(f_p \wedge f_q) = [-df_p \wedge f_q - (-1)^{\mu} f_p \wedge df_q, -f_p \wedge f_q]$$

in $F_D^{p+q} \oplus Rj_*\Omega_X^{\cdot}$ whereas $\delta f_p = [-df_p, -f_p] \in F_D^p \oplus Rj_*\Omega_X^{\cdot}$, $\delta f_p \cup_{\alpha} f_q = [-df_p \wedge f_q, -(1-\alpha) f_p \wedge f_q]$ and similarly

$$(-1)^{\mu} f_p \cup_{\alpha} \delta f_q = [-(-1)^{\mu} f_p \wedge df_q, -(-1)^{2\mu} \alpha f_p \wedge f_q].$$

3.4 Remark. The quite complicated description of the product is necessary, since at this stage, we tried to avoid the more formal language of sheaves on pairs of topological spaces. Nevertheless, the reader should compare the defintion with the definition of the tensor-product of those pairs, given in (4.5 - 4.8). From now on, we just work with the multiplication table (3.2) to verify the properties of the product, and we leave it to the reader to distinguish whether a given

expression lives on X or on $\overline{X}$.

Proposition 3.5.

a) $\cup_{1/2}$ is anti-commutative. More generally, if γ and γ' are concentrated in degree μ and μ' then

$$\gamma \cup_\alpha \gamma' = (\;1)^{\mu\cdot\mu'}\gamma' \cup_{(1-\alpha)} \gamma \,.$$

b) $\cup_0$ and $\cup_1$ are associative.

c) The element $(a_0 = 1,\ f_0 = 1)$ in $A(0)_{\mathcal{D}}$ is a left-identify for $\cup_0$ and a right-identity for $\cup_1$.

d) For $\alpha,\beta \in \mathbb{R}$ the products $\cup_\alpha$ and $\cup_\beta$ are homotopic.

Proof:

We choose elements γ and γ' living in $A(p)_{\mathcal{D}}$ and $A(q)_{\mathcal{D}}$ in degree μ and μ'.

a) is obvious from the definition. For example if $\gamma = f_p \in (j^*F_D^p)^\mu$ and $\gamma' = \omega_q \in \Omega_X^{\mu'-1}$, then

$$\gamma\cup_\alpha\gamma' = (-1)^\mu \alpha f_p \wedge \omega_q = (-1)^{\mu+\mu(\mu'-1)}\alpha\omega_q \wedge f_p = (-1)^{\mu\cdot\mu'}\gamma'\cup_{(1-\alpha)}\gamma \quad .$$

b) Let γ'' be an element of $A(r)_{\mathcal{D}}$. Using a) it is enough to consider $\cup_0$. If γ,γ' and γ'' represent all the three elements of $A(\ )$ or all the three elements of $F_D^{(\ )}$ the associativity is obvious. If two of the elements belong to $\Omega_X^{\cdot}$, then $(\gamma \cup_0\gamma')\cup_0\gamma'' = \gamma\cup_0(\gamma'\cup_0\gamma'') = 0$. The same holds if two of the elements are belonging to $A(\ )$ and one to $F_D^{(\ )}$ or one to $A(\ )$ and two to $F_D^{(\ )}$. Since $\alpha = 0$ both $\cup_0|_{\Omega_X^{\cdot}\otimes A(\ )}$ and $\cup_0|_{F_D^{(\ )}\otimes\Omega_X^{\cdot}}$ are zero. Hence the only cases left, where one of the two sides can be nonzero, are (a_p,a_q,ω_r), (a_p,ω_q,f_r) and (ω_p,f_q,f_r) and both, $(\gamma \cup_0 \gamma')\cup_0 \gamma''$ and $\gamma \cup_0 (\gamma' \cup_0 \gamma'')$ are $a_p\cdot a_q\cdot\omega_r$, $a_p\cdot\omega_q \wedge f_r$ and $\omega_p\wedge f_q\wedge f_r$ respectively.

c) Again it is enough to consider $\cup_0$ and $(1,1)\cup_0\gamma'$ is given by

	a_q	f_q	ω_q
$1 \in A(0)$	$1\cdot a_q$	0	$1\cdot\omega_q$
$1 \in (F_D^0)^0$	0	$1 \wedge f_q = f_q$	0

d) The homotopy between $\cup_\alpha$ and $\cup_\beta$ is given by

$$h : (A(p)_{\mathcal{D}} \otimes A(q)_{\mathcal{D}})^k \longrightarrow (A(p+q)_{\mathcal{D}})^{k-1}$$

$$h(\gamma \otimes \gamma') = \begin{cases} (-1)^\mu (\alpha-\beta)\gamma \wedge \gamma' & \text{if } \gamma \in \Omega_X^{\mu-1} \text{ and } \gamma' \in \Omega_X^{\mu'-1} \\ 0 & \text{otherwise} \end{cases}$$

where – as usual – γ and γ' are elements of degree μ and μ', each in $A(\ )$, $F_D^{(\ )}$ or $\Omega_X^{\cdot}$. We have to show that

$$\gamma \cup_\alpha \gamma - \gamma \cup_\beta \gamma' = (h\delta + \delta h)(\gamma \otimes \gamma') = h(\delta\gamma \otimes \gamma') + (-1)^\mu h(\gamma \otimes \delta\gamma') + \delta(h(\gamma \otimes \gamma')).$$

The left hand side is given by

0	0	$(\beta-\alpha)a_p \cdot \omega_q$
0	0	$(-1)^\mu(\alpha-\beta)f_p \wedge \omega_q$
$(\alpha-\beta)\omega_p \cdot a_q$	$(\beta-\alpha)\omega_p \wedge f_q$	0

in the notation of (3.2).

For the right hand side we remark first that $h(\delta\gamma \otimes \gamma') = 0$ if $\gamma' \neq \omega_q$, $h(\gamma \otimes \delta\gamma') = 0$ if $\gamma \neq \omega_p$ and $\delta(h(\gamma \otimes \gamma')) = 0$ if $(\gamma, \gamma') \neq (\omega_p, \omega_q)$. We have

$$(h\delta + \delta h)(\omega_p \otimes \omega_q) = (-1)^{\mu+1}(\alpha-\beta)d\omega_p \wedge \omega_q + (-1)^{2\mu}(\alpha-\beta)\omega_p \wedge d\omega_q + \delta((-1)^\mu(\alpha-\beta)\omega_p \wedge \omega_q) = 0,$$

$$(h\delta + \delta h)(a_p \otimes \omega_q) = h(\delta a_p \otimes \omega_q) = (-1)(\alpha-\beta)a_p \wedge \omega_q \quad ,$$

$$(h\delta + \delta h)(f_p \otimes \omega_q) = (-1)^{\mu+1}(\alpha-\beta)(-f_p) \wedge \omega_q \quad ,$$

$$(h\delta + \delta h)(\omega_p \otimes a_p) = (-1)^\mu h(\omega_p \otimes \delta a_q) = (-1)^{2\mu}(\alpha-\beta)\omega_p \cdot a_p \quad \text{and}$$

$$(h\delta + \delta h)(\omega_p \otimes f_q) = (-1)^{2\mu}(\alpha - \beta)\omega_p \wedge (-f_q).$$

3.6. Let $\varepsilon_A : A(p)_{\mathcal{D}} \longrightarrow Rj_*A(p)$ and
$\varepsilon_F : A(p)_{\mathcal{D}} \longrightarrow F^p_D$ be the projections

$$\varepsilon_\Omega : A(p)_{\mathcal{D}} \xrightarrow{\varepsilon_A} Rj_*A(p) \xrightarrow{\varepsilon} Rj_*\Omega^\cdot_X \quad \text{and}$$

$$\varepsilon'_\Omega : A(p)_{\mathcal{D}} \xrightarrow{\varepsilon_F} F^p_D \xrightarrow{\iota} Rj_*\Omega^\cdot_X .$$

By definition of $A(p)_{\mathcal{D}}$ (2.6) $\varepsilon_\Omega - \varepsilon'_\Omega$ is the composition of two maps in a distinguished triangle and hence $\varepsilon_\Omega - \varepsilon'_\Omega$ is homotopic to the zero map. We define products

$$\cup_A : A(p)_{\mathcal{D}} \otimes Rj_*A(q) \xrightarrow{\varepsilon_A \otimes \mathrm{id}} Rj_*A(p) \otimes Rj_*A(q) \xrightarrow{\cdot} Rj_*A(p+q)$$

$$\cup_F : A(p)_{\mathcal{D}} \otimes F^q_D \xrightarrow{\varepsilon_F \otimes \mathrm{id}} F^p_D \otimes F^q_D \xrightarrow{\wedge} F^{p+q}_D$$

and

$$\cup_\Omega : A(p)_{\mathcal{D}} \otimes Rj_*\Omega^\cdot_X \xrightarrow{\varepsilon_\Omega \otimes \mathrm{id}} Rj_*\Omega^\cdot_X \otimes Rj_*\Omega^\cdot_X \xrightarrow{\wedge} Rj_*\Omega^\cdot_X .$$

Since $\cup_\Omega$ can - up to homotopy - also be defined as

$$A(p)_{\mathcal{D}} \otimes Rj_*\Omega^\cdot_X \xrightarrow{\varepsilon'_\Omega \otimes \mathrm{id}} Rj_*\Omega^\cdot_X \otimes Rj_*\Omega^\cdot_X \xrightarrow{\wedge} Rj_*\Omega^\cdot_X$$

the morphism

$$Rj_*A(q) \oplus F^q \xrightarrow{\varepsilon - \iota} Rj_*\Omega^\cdot_X$$

is compatible with $\cup_A, \cup_F$ and, up to homotopy, with $\cup_\Omega$.
Moreover $\varepsilon_A(\gamma \cup_0 \gamma') = \gamma \cup_A \varepsilon_A\gamma'$ and $\varepsilon_F(\gamma \cup_0 \gamma') = \gamma \cup_F \varepsilon_F\gamma'$ as one easily verifies using the multiplication table (3.2).

For the natural map $\eta : Rj_*\Omega^\cdot_X \longrightarrow A(q)_{\mathcal{D}}$ one has as well $\gamma \cup_0 \eta(\omega_q) = \eta(\gamma \cup_\Omega \omega_q)$. Altogether we obtain:

Proposition 3.7. In the triangle

$$\begin{tikzcd} Rj_*A(q) \oplus F^1_D \arrow[rr] & & Rj_*\Omega^\cdot_X \arrow[dl, "{[1]}"] \\ & A(q)_{\mathcal{D}} \arrow[ul] & \end{tikzcd}$$

the operations of $A(p)_{\mathcal{D}}$ defined by $\cup_A, \cup_F, \cup_\Omega$ and $\cup_0$ are compatible with the morphisms.

3.8. Since $A(p)_{\mathcal{D}}$ has a flat resolution (of finite length) over $\mathbb{Z}$ ([6] V,6) one has a map $A(p)_{\mathcal{D}} \otimes^L A(q)_{\mathcal{D}} \longrightarrow A(p)_{\mathcal{D}} \otimes A(q)_{\mathcal{D}}$. Therefore one has for all $\alpha \in \mathbb{R}$ a product

$$\cup_\alpha : A(p)_{\mathcal{D}} \otimes^L A(q)_{\mathcal{D}} \longrightarrow A(p+q)_{\mathcal{D}}$$

and - by the usual constructions from homological algebra a product on the hypercohomology. By (3.5,d) this product is independent of α.
(3.5) and (3.7) give immediately:

Theorem 3.9. $\cup_\alpha$ induces a product $\cup$, making $\bigoplus_{p,q} H^q_{\mathcal{D}}(X, A(p))$ into a bigraded ring with unit. For $\gamma \in H^q_{\mathcal{D}}(X, A(p))$ and $\gamma' \in H^{q'}_{\mathcal{D}}(X, A(p'))$ we have $\gamma \cup \gamma' = (-1)^{qq'} \gamma' \cup \gamma$. Moreover one has an operation of $\bigoplus_{p,q} H^q_{\mathcal{D}}(X, A(p))$ on $\bigoplus_{p,q} H^q(X, A(p))$, $\bigoplus_{p,q} F^p H^q(X, \mathbb{C})$ and $\bigoplus_q H^q(X, \mathbb{C})$ coming via $\varepsilon_A, \varepsilon_F$ and ε_Ω from the standard products. The exact sequence

$$\longrightarrow H^q_{\mathcal{D}}(X, A(p)) \longrightarrow H^q(X, A(p)) \oplus F^p H^q(X, \mathbb{C}) \longrightarrow H^q(X, \mathbb{C}) \longrightarrow H^{q+1}_{\mathcal{D}}(X, A(p))$$

is compatible with the operations.

3.10. The product on the "real" D - $\overline{\text{B}}$ cohomology

We return to the notations introduced in (2.14). On $\widetilde{\mathbb{R}}(p)_{\mathcal{D}} = \mathrm{Cone}(F^p_D \xrightarrow{-\pi_{p-1}} j_* S^\cdot_X(p-1))[-1]$ one defines a product

$$\widetilde{\cup} : \widetilde{\mathbb{R}(p)}_{\mathcal{D}} \otimes \widetilde{\mathbb{R}(q)}_{\mathcal{D}} \longrightarrow \widetilde{\mathbb{R}(p+q)}_{\mathcal{D}}$$

given by

	f_q	s_q
f_p	$f_p \wedge f_q$	$(-1)^{\deg f_p} \pi_p f_p \wedge s_q$
s_p	$s_p \wedge \pi_q f_q$	0

Lemma 3.11.

a) $\widetilde{\cup}$ is a morphism of complexes.

b) $\rho_{p+q} \circ \cup_0$ is homotopic to $\widetilde{\cup} \circ (\rho_p \otimes \rho_q)$ (where $\rho_p : \mathbb{R}(p)_{\mathcal{D}} \longrightarrow \widetilde{\mathbb{R}(p)}_{\mathcal{D}}$ is the quasi-isomorphism given in (2.15)).

Proof: a) For γ and γ' of degree μ and μ'

$$\delta(\gamma \otimes \gamma') \in (F_D^{p+q})^{\mu+\mu'+1} \oplus j_* S_X^{\mu+\mu'}(p+q-1)$$

is (again the differentials are written as $\delta[-1] = -\delta$)

$[-df_p \wedge f_q - (-1)^{\mu} f_p \wedge df_q,\ -\pi_{p+q-1}(f_p \wedge f_q)]$	$[0, (-1)^{\mu}\pi_p df_p \wedge s_q + (-1)^{2\mu}\pi_p f_p \wedge ds_q]$
$[0, ds_p \wedge \pi_q f_q + (-1)^{\mu-1} s_p \wedge \pi_q df_q]$	0

whereas

$\delta\gamma \cup \gamma'$ is

$[-df_p \wedge f_q,\ -\pi_{p-1} f_p \wedge \pi_q f_q]$	$[0, -(-1)^{\mu+1}\pi_p df_p \wedge s_q]$
$[0, ds_p \wedge \pi_q f_q]$	0

and $(-1)^{\mu}\gamma \cup \delta\gamma'$ is

$[-(-1)^{\mu} f_p \wedge df_q, -(-1)^{2\mu}\pi_p f_p \wedge \pi_{q-1} f_q]$	$[0, (-1)^{2\mu}\pi_p f_p \wedge ds_q]$
$[0, (-1)^{\mu} s_p \wedge (-\pi_q df_q)]$	0

Since

$$\pi_{p+q-1}(f_p \wedge f_q) = \pi_{p+q-1}((\pi_{p-1} f_p + \pi_p f_p) \wedge (\pi_q f_q + \pi_{q-1} f_q)) = \pi_{p-1} f_p \wedge \pi_q f_q + \pi_p f_p \wedge \pi_{q-1} f_q$$

we obtain a).

b) The homotopy is

$$h(\gamma \otimes \gamma') = \begin{cases} (-1)^{\mu} \pi_p \gamma \wedge \pi_{q-1} \gamma' & \text{if } \gamma \in \Omega_X^{\mu-1} \text{ and } \gamma' \in \Omega_X^{\mu'-1} \\ 0 & \text{otherwise .} \end{cases}$$

We have to verify that

$$\rho_p \gamma \ \tilde{\cup}\ \rho_q \gamma' - \rho_{p+q}(\gamma \cup_0 \gamma') = h(\delta\gamma \otimes \gamma') + (-1)^{\mu} h(\gamma \otimes \delta\gamma') + \delta(h(\gamma \otimes \gamma')).$$

The left hand side is (see (3.2))

0	0	$-a_p \cdot \pi_{q-1}\omega_q$
0	0	$(-1)^{\mu} \pi_p f_p \wedge \pi_{q-1}\omega_q$
0	$\pi_{p-1}\omega_p \wedge \pi_q f_q -$ $-\pi_{p+q-1}(\omega_p \wedge f_q)$	0

As in the proof of (3.5d) all the terms occuring on the right hand side are evidently zero except

$$(h\delta+\delta h)(\omega_p \otimes \omega_q) = (-1)^{\mu+1}(\pi_p d\omega_p \wedge \pi_{q-1}\omega_q) + (-1)^{2\mu}(\pi_p\omega_p \wedge \pi_{q-1} d\omega_q) + (-1)^{\mu}(d(\pi_p\omega_p \wedge \pi_{q-1}\omega_q)) = 0,$$

$$(h\delta+\delta h)(a_p \otimes \omega_q) = h(\delta a_p \otimes \omega_q) = (-1) a_p \cdot \pi_{q-1}\omega_q$$

$$(h\delta + \delta h)(f_p \otimes \omega_q) = (-1)^{\mu+1} \pi_p(-f_p) \wedge \pi_{q-1}\omega_q$$

$$(h\delta + \delta h)(\omega_p \otimes a_q) = (-1)^{\mu} h(\omega_p \otimes \delta a_q) = (-1)^{2\mu} \pi_p\omega_p \cdot \pi_{q-1} a_q = 0$$

$$(h\delta+\delta h)(\omega_p \otimes f_q) = (-1)^{2\mu} \pi_p\omega_p \wedge \pi_{q-1}(-f_q) = -\pi_{p+q-1}(\omega_p \wedge f_q) + \pi_{p-1}\omega_p \wedge \pi_q f_q .$$

Example 3.12.

Let $[\varphi, \eta]$ and $[\varphi', \eta']$ represent two elements of $\mathbb{H}^1(\overline{X}, \widetilde{\mathbb{R}(1)}_{\mathcal{D}})$. Then

$$[\varphi, \eta] \ \tilde{\cup}\ [\varphi', \eta'] = [\varphi \wedge \varphi', \eta \wedge \pi_1\varphi' - \pi_1\varphi \wedge \eta'].$$

As we have seen in (2.16)

$$H^1_{\mathcal{D}}(X,\mathbb{R}(1)) = \{\eta \in H^0(\overline{X}, j_*S^0_X);\ d_z\eta \in H^0(\overline{X},\Omega^1_{\overline{X}}(\log D))\}$$

where φ corresponds to $2d_z\eta$. Hence the product of two elements η and η' is given by

$$[4\cdot d_z\eta \wedge d_z\eta', 2\cdot(\eta\cdot\pi_1 d_z\eta' - \eta'\cdot\pi_1 d_z\eta)]$$

in $\mathbb{H}^2(\overline{X},\widetilde{\mathbb{R}(2)}_{\mathcal{D}}) = H^2_{\mathcal{D}}(X,\mathbb{R}(2))$.

In particular, if $\dim X = 1$ and therefore $F^2_D = 0$, $\eta \cup \eta'$ is represented by $2\cdot\eta\cdot\pi_1 d_z\eta' - 2\cdot\eta'\cdot\pi_1 d_z\eta$ in $H^0(X,S^1_X(1))/dH^0(X,S^0_X(1)) = H^1(X,\mathbb{R}(1))$.

§ 4 Relative cohomology

In [3] the $D-\overline{b}$ - cohomology is defined using relative cohomology. This approach, giving $\mathbb{H}^\cdot(\overline{X},A(p)_{\mathcal{D}})$ as a derived functor on the category of sheaves on pairs of topological spaces, applied to $(F^p_D, Rj_*\mathrm{Cone}(A(p)\to\Omega^\cdot_X))$, will be needed in § 5 to define a $D-\overline{b}$ - complex on X in the Zariski-topology. One also defines a tensor product on this derived category, to obtain the product for the $D-\overline{b}$ - complexes in the Zariski-topology. In fact, using this tensor product one can simplify the definition (3.2) and clarify the constructions described in (3.3).

4.1. Let $j : T \longrightarrow \overline{T}$ be a continuous morphism of topological spaces. A sheaf on $(\overline{T},T)$ is a triple $F_{\overline{T},T} := (\overline{F}, F, \varphi)$ where $\overline{F}$ is a sheaf on $\overline{T}$, F is a sheaf on T and $\varphi : \overline{F} \longrightarrow j_*F$ a morphism of sheaves. Correspondingly a morphism $\alpha : F_{\overline{T},T} \longrightarrow F'_{\overline{T},T}$ is a pair of morphisms $\overline{\alpha} : \overline{F} \longrightarrow \overline{F}'$, $\alpha : F \longrightarrow F'$ such that $\alpha\varphi = \varphi'\overline{\alpha}$.

4.2. Let $Sh(\overline{T},T)$ denote the category of sheaves on $(\overline{T},T)$. It is easy to see that $Sh(\overline{T},T)$ has enough injectives. For example: if $\overline{I}$ and I are injective sheaves on $\overline{T}$ and T respectively, the triple $J_{\overline{T},T} = (\overline{J} = \overline{I} \oplus j_*I, J = I, pr_2)$ is injective in $Sh(\overline{T},T)$. If $F_{\overline{T},T}$ is any sheaf we can find $\overline{I}, I$ such that $\tau : \overline{F} \hookrightarrow \overline{I}$ and $\rho : F \hookrightarrow I$. Then $(\tau \oplus \rho\circ\varphi, \rho)$ defines an inclusion $F_{\overline{T},T} \hookrightarrow J_{\overline{T},T}$. Therefore each sheaf has a resolution by those "special injective sheaves".

4.3. Consider the functor

$$\Gamma^0 : Sh(\overline{T},T) \longrightarrow Ab \quad \text{defined as}$$

$$\Gamma^0(F_{\overline{T},T}) = \mathrm{Ker}(H^0(\overline{T},\overline{F}) \xrightarrow{\varphi} H^0(T,F)).$$

Obviously Γ^0 is left exact. If $D^+(\overline{T},T)$ is the derived category of complexes of sheaves in $Sh(\overline{T},T)$, bounded below, we define

$$R\Gamma^\cdot : D^+(\overline{T},T) \longrightarrow D^+(Ab)$$

to be the derived functor of Γ^0 .

Proposition 4.4.

a) If $F^\cdot_{\overline{T},T} = (\overline{F}^\cdot, F^\cdot, \varphi^\cdot)$ is a complex of sheaves on $(\overline{T},T)$ then

$$R\Gamma^\cdot(F^\cdot_{\overline{T},T}) = \mathrm{Cone}(\mathbb{H}^\cdot(\overline{T},\overline{F}^\cdot) \xrightarrow{R\varphi^\cdot_*} \mathbb{H}^\cdot(T,F^\cdot))[-1].$$

b) If $\overline{X}$ is a good compactification of the algebraic manifold X and if $A(p)_{\mathcal{D},\overline{X},X}$ denotes the complex $(F^p_D, \mathrm{Cone}(A(p) \xrightarrow{\varepsilon} \Omega^\cdot_X), -\iota)$ on $(\overline{X},X)$ then $H^q_{\mathcal{D}}(X,A(p))$ is the q-th cohomology of

$$R\Gamma^\cdot(A(p)_{\mathcal{D},\overline{X},X}).$$

Proof. It is enough to verify a) for the special injective sheaf $J_{\overline{T},T}$ defined in (3.2). On the right hand side of the equality we have the cone of

$$\mathbb{H}^\cdot(\overline{T},\overline{J}) = H^0(\overline{T},\overline{I}) \oplus H^0(T,I) \xrightarrow{-pr_2} \mathbb{H}^\cdot(T,J) = H^0(T,I),$$

which is quasi-isomorphic to $H^0(\overline{T},\overline{I})$. On the other hand $R\Gamma^\cdot(J_{\overline{T},T}) = R\Gamma^0(J_{\overline{T},T}) = H^0(\overline{T},\overline{I})$ as well.

b) By (2.7) $H^q_{\mathcal{D}}(X,A(p))$ is the q-th cohomology of

$$\mathbb{H}^\cdot(\overline{X},\mathrm{Cone}(F^p_D \xrightarrow{-1} Rj_*\mathrm{Cone}(A(p) \longrightarrow \Omega^\cdot_X))[-1]) = \mathrm{Cone}(\mathbb{H}^\cdot(\overline{X},F^p_D) \longrightarrow \mathbb{H}^\cdot(X,\mathrm{Cone}(A(p) \to \Omega^\cdot_X)))[-1].$$

4.5. For two complexes of sheaves

$$F^\cdot_{\overline{T},T} = (\overline{F}^\cdot, F^\cdot, \varphi^\cdot) \quad \text{and} \quad G^\cdot_{\overline{T},T} = (\overline{G}^\cdot, G^\cdot, \psi^\cdot)$$

we define the tensor product $F^\cdot_{\overline{T},T} \otimes G^\cdot_{\overline{T},T}$ to be the complex $(\overline{E}^\cdot, E^\cdot, \eta^\cdot)$ with

$\overline{E}^\cdot = \overline{F}^\cdot \otimes \overline{G}^\cdot$ and (for $\rho^\cdot = \varphi^\cdot \otimes \mathrm{id} - \mathrm{id} \otimes \psi^\cdot$)

$$E^\cdot = \mathrm{Cone}((j^*\overline{F}^\cdot \otimes G^\cdot) \oplus (F^\cdot \otimes j^*\overline{G}^\cdot) \xrightarrow{\rho^\cdot} F^\cdot \otimes G^\cdot)[-1].$$

The connecting morphism $\eta^\cdot$ is - on the level of sheaves - defined by

$$\eta^i = (-1)^i \cdot [(\mathrm{id} \otimes \Psi^i) \oplus (\varphi^i \otimes \mathrm{id}) \oplus 0] : E^i \longrightarrow j_*(j^*\overline{F}^i \otimes G^i) \oplus j_*(F^i \otimes j^*\overline{G}^i) \oplus (j_* F^{i-1} \otimes j_* G^{i-1}) .$$

Since $\rho^i \cdot \eta^i$ is the zero map η^i commutes with the differentials and $\eta^\cdot$ is a morphism of complexes.

4.6. If $C(\overline{T},T)$ denotes the category of complexes of sheaves on $(\overline{T},T)$ and $K(\overline{T},T)$ the corresponding homotopy category we have thereby constructed a bifunctor $\otimes . C(\overline{T},T) \times C(\overline{T},T) \longrightarrow C(\overline{T},T)$. Since the $\otimes$ product respects homotopies it also defines the bifunctor $\otimes : K(\overline{T},T) \times K(\overline{T},T) \longrightarrow K(\overline{T},T)$. For a fixed complex $F_{\overline{T},T}$, $F_{\overline{T},T} \otimes$ respects triangles and if both $\overline{F}$ and F are flat $F_{\overline{T},T}\otimes$ maps exact complexes to exact ones. Hence $F_{\overline{T},T} \otimes$ respects quasi-isomorphisms in this case. $Sh(\overline{T},T)$ has enough flat sheaves (for example, if $\overline{P}$ on $\overline{T}$ and P on T are flat and $\overline{P} \xrightarrow{\overline{\rho}} \overline{F}$ and $P \xrightarrow{\rho} F$ both surjective, $(\overline{P}, j^*\overline{P} \oplus P, \mathrm{id} \oplus 0)$ maps surjectively to $(\overline{F},F,\varphi)$ via $(\overline{\rho}, \varphi \circ \overline{\rho} + \rho)$).

The standard machinery of derived categories and derived functors shows the existence of a left derived functor

$$\otimes^L : D^-(\overline{T},T) \times D^-(\overline{T},T) \longrightarrow D^-(\overline{T},T).$$

(see [14], for example).
From now on we assume that T and $\overline{T}$ have finite cohomological dimension. Then both $R\Gamma^\cdot$ and $\otimes^L$ are defined on the derived category of bounded complexes.

4.7. If $H^\cdot_{\overline{T},T} = (\overline{H}^\cdot, H^\cdot, \gamma^\cdot)$ is a third complex of sheaves, a pairing

$U : F^\cdot_{\overline{T},T} \otimes G^\cdot_{\overline{T},T} \longrightarrow H^\cdot_{\overline{T},T}$ (and - using flat resolutions as in 3.8 $F^\cdot_{\overline{T},T} \otimes^L G^\cdot_{\overline{T},T} \longrightarrow H^\cdot_{\overline{T},T}$) is given by a pair

$$U_{\overline{X}} : \overline{F}^\cdot \otimes \overline{G}^\cdot \longrightarrow \overline{H}^\cdot \quad \text{and}$$

$$U_X : \mathrm{Cone}((j^*\overline{F}^\cdot \otimes G^\cdot) \oplus (F^\cdot \otimes j^*\overline{G}^\cdot) \xrightarrow{\rho^\cdot} F^\cdot \otimes G^\cdot)[-1] \longrightarrow H^\cdot$$

compatible with $\eta^\cdot$ and $\gamma^\cdot$. Taking the special injective resolutions described in (4.2) one obtains from U a pairing

$$U : R\Gamma^\cdot(F^\cdot_{\overline{T},T}) \otimes R\Gamma^\cdot(G^\cdot_{\overline{T},T}) \longrightarrow R\Gamma^\cdot(H^\cdot_{\overline{T},T}), \quad \text{and}$$

$$U : R\Gamma^\cdot(F^\cdot_{\overline{T},T}) \otimes^L R\Gamma^\cdot(G^\cdot_{\overline{T},T}) \longrightarrow R\Gamma^\cdot(H^\cdot_{\overline{T},T}) .$$

4.8. If – as in (4.4,b) – we consider on $(\bar{X},X)$ the complexes

$$F^{\cdot}_{\bar{X},X} = A(p)_{\mathcal{D},\bar{X},X} \, , \; G^{\cdot}_{\bar{X},X} = A(q)_{\mathcal{D},\bar{X},X} \text{ and } H^{\cdot}_{\bar{X},X} = A(p+q)_{\mathcal{D},\bar{X},X} \, ,$$

the multiplication table (3.2) defines pairings

$$A(p)_{\mathcal{D},\bar{X},X} \otimes^L A(q)_{\mathcal{D},\bar{X},X} \longrightarrow A(p+q)_{\mathcal{D},\bar{X},X} \, .$$

In fact, the first calculation made in (3.3) shows that $\cup_{\alpha,X}$ is well defined and the second part of (3.3) shows at the same time that $\cup_{\alpha,\bar{X}}$ is a morphism of complexes and that $\cup_\alpha = (\cup_{\alpha,\bar{X}}, \cup_{\alpha,X})$ is compatible with the morphisms $\eta^\cdot$ from (4.5) and $\gamma^\cdot = -\iota$. Hence (3.2) defines a product

$$R\Gamma^\cdot(A(p)_{\mathcal{D},\bar{X},X}) \otimes^L R\Gamma^\cdot(A(q)_{\mathcal{D},\bar{X},X}) \longrightarrow R\Gamma^\cdot(A(p+q)_{\mathcal{D},\bar{X},X})$$

which – on the cohomology of the complexes – coincides with (3.9) and is independent of α .

§ 5 Extensions and complements

5.1 The definitions and properties of the D – $\bar{\text{D}}$ – cohomology given in §2 and §3 carry over to the case of separated simplicial schemes $Z.$ of finite type over $\mathbb{C}$:
As in [8], 8.3, we can find a diagram $\bar{X}. \xhookleftarrow{j} X. \xrightarrow{p} Z.$ where p satisfies cohomological descent and where $\bar{X}.$ is proper and smooth and $D. = \bar{X}. - X.$ is a normal crossing divisor. We define $H^q_{\mathcal{D}}(X., A(p))$ as the hypercohomology (in the sense of cohomology of simplicial schemes) of $\quad \mathrm{Cone}(Rj_*A(p) \oplus F^p_{D.} \longrightarrow Rj_*\Omega^{\cdot}_{X.})[-1]$. As in (2.8) one obtains the independence of $H^q_{\mathcal{D}}(X., A(p))$ of the compactification $\bar{X}.$.

5.2 If $\bar{X}'. \xhookleftarrow{j'} X'. \xrightarrow{p'} Z.$ is a second diagram and $(\bar{\tau},\tau) : (\bar{X}'., X'.) \longrightarrow (\bar{X}., X.)$ a morphism, compatible with p and p', one knows, that τ^* is an isomorphism on the cohomology with values in $A(p)$ and $\mathbb{C}$. Moreover (loc. cit.) $\bar{\tau}_*$ is an isomorphism on the F-filtration on the DeRham cohomology. By (2.10, a) $\tau^* : H^q_{\mathcal{D}}(X., A(p)) \longrightarrow H^q_{\mathcal{D}}(X'., A(p))$ is an isomorphism as well. Since two diagrams as in (5.1) are dominated by a third one (loc. cit.) we can define:

Definition 5.3. The $D-\bar{b}$ - cohomology of $Z.$ is

$$H^q_{\mathcal{D}}(Z.,A(p)) := H^q_{\mathcal{D}}(X.,A(p)).$$

Remarks 5.4. If $f: Y. \longrightarrow Z.$ is a morphism of simplicial schemes one has - choosing the smooth hypercoverings and compactifications in the right way - the obvious map

$$f^* : H^q_{\mathcal{D}}(Z.,A(p)) \longrightarrow H^q_{\mathcal{D}}(Y.,A(p)).$$

The exact sequences (2.10) exist as well for simplicial schemes, the definition and the properties of the product remain unchanged. As in (2.1,II) the $D-\bar{b}$ - cohomology exists as well for simplicial schemes over $\mathbb{R}$.

5.5. Sheafification of the Zariski topology

Theorem. Let X be a smooth algebraic manifold.

a) There exists a complex $A(p)_{\mathcal{D},Zar}$ of sheaves in the Zariski topology on X such that for all open subvarieties $X' \subset X$ one has

$$H^q_{\mathcal{D}}(X',A(p)) = \mathbb{H}^q(X'_{Zar},A(p)_{\mathcal{D},Zar}) .$$

b) We have natural morphisms

$$c_0 : A \longrightarrow A(0)_{\mathcal{D},Zar} \quad \text{and} \quad c_1 : \mathcal{O}^*_{X,Zar}[-1] \longrightarrow A(1)_{\mathcal{D},Zar} .$$

(c_1 induces on $X' \subset X$ the morphism ρ described in (2.12,iii)).

c) In the derived category of sheaves in the Zariski-topology we have a product

$$A(p)_{\mathcal{D},Zar} \otimes^L A(q)_{\mathcal{D},Zar} \longrightarrow A(p+q)_{\mathcal{D},Zar}$$

inducing on $X' \subset X$ the product defined in (3.9).

Proof. Let $\mathcal{V}$ be the category of complex algebraic manifolds (or real ones - in case 2.1,II). We denote by Π the category of pairs $(\bar{V},V)$, where $\bar{V}$ is a proper complex (or real) algebraic manifold and $V \subset \bar{V}$ the complement of a normal crossing divisor.

We define a sheaf $F_{*,*}$ on Π to be a collection of sheaves $F_{\overline{V},V} = (F_{\overline{V}}, F_V, \varphi_V)$ on $(\overline{V},V)$ (as in 4.1), together with a morphism $f^* : (F_{\overline{V}}, F_V, \varphi_V) \longrightarrow (f_* F_{\overline{U}}, f_* F_U, f_* \varphi_U)$ for each morphism $f : (\overline{U},U) \longrightarrow (\overline{V},V)$, satisfying $(f.g)^* = g^* \circ f^*$ and $id^* = id$. One denotes by $Sh(\Pi)$ the category of sheaves on Π. As in (4.2) one finds that $Sh(\Pi)$ has enough injectives. If $\sigma : \Pi \longrightarrow V$ is the "forget-functor" $\sigma((\overline{V},V)) = V$, one defines for $F_{*,*} \in Sh(\Pi)$ the direct image $\sigma_* F_{*,*}$ to be the Zariski sheaf on V associated to the presheaf

$$X \longmapsto \varinjlim_{(\overline{X},X)\in\sigma^{-1}(X)} \Gamma^0(F_{\overline{X},X}),$$

where Γ^0 is the functor described in (4.3), and where the limit is taken over the direct family $\sigma^{-1}(X)$ of all good compactifications of X. $\sigma_* : Sh(\Pi) \longrightarrow Sh(V)$ is left exact. Let $R\sigma_* : D^+(\Pi) \longrightarrow D^+(V)$ be the derived functor. Since

$$H^0(X, \sigma_*(F_{\overline{X},X})) = \varinjlim_{\sigma^{-1}(X)} \Gamma^0(F_{\overline{X},X})$$

one has for a complex $F^{\cdot}_{*,*}$ of sheaves on Π :

$$\mathbb{H}^q(X, R\sigma_* F_{*,*}) \xleftarrow{\sim} \varinjlim_{\sigma^{-1}(X)} R\Gamma^q(F_{\overline{X},X})$$

Let $A(p)_{\mathcal{D},*,*}$ be the complex of sheaves introduced in (4.4,b). Then we define $A(p)_{\mathcal{D},Zar} := R\sigma_* A(p)_{\mathcal{D},*,*}$. From (4.4) and (2.9) one obtains

$$\mathbb{H}^q(X_{Zar}, A(p)_{\mathcal{D},Zar}) = \varinjlim_{\sigma^{-1}(X)} R\Gamma^q(A(p)_{\mathcal{D},\overline{X},X}) = \varinjlim_{\sigma^{-1}(X)} \mathbb{H}^q(\overline{X}, A(p)_{\mathcal{D}}) = H^q_{\mathcal{D}}(X, A(p)).$$

b) Since $A(0)_{\mathcal{D}}$ is quasi-isomorphic to the constant sheaf A $H^0_{\mathcal{D}}(X', A(0)) = A$ for each connected open subvariety X' of X and we obtain c_0 . Similarly, by (2.12,i) we can describe $A(p)_{\mathcal{D},Zar}$ for $p > 0$ by a complex starting in degree 1 and (2.12,iii) gives on each open subvariety $X' \subset X$ the morphism

$$c_1 : \mathcal{O}(X')^*_{alg} = H^0(X'_{Zar}, \mathcal{O}^*_{X,Zar}) \longrightarrow \mathbb{H}^1(X'_{Zar}, A(1)_{\mathcal{D},Zar}) = Ker(H^0(X'_{Zar}, (A(1)_{\mathcal{D},Zar})^1) \xrightarrow{\rho} H^0(X'_{Zar}, (A(1)_{\mathcal{D},Zar})^2)).$$

c) By (4.8) the products $\cup_\alpha$ from (3.2) define products on the complexes $A(\ )_{\mathcal{D},\overline{X},X}$ for all $(\overline{X},X) \in \Pi$. The product

$$A(p)_{\mathcal{D},*,*} \otimes^L A(q)_{\mathcal{D},*,*} \longrightarrow A(p+q)_{\mathcal{D},*,*}$$

in the derived category gives

$$A(p)_{\mathcal{D},\mathrm{Zar}} \overset{L}{\otimes} A(q)_{\mathcal{D},\mathrm{Zar}} \longrightarrow R\sigma_*(A(p)_{\mathcal{D},*,*} \overset{L}{\otimes} A(q)_{\mathcal{D},*,*}) \to R\sigma_*(A(p+q)_{\mathcal{D},*,*}) = A(p+q)_{\mathcal{D},\mathrm{Zar}} .$$

§ 6 The cycle map in the De Rham cohomology

In [10] one finds the definition (due to P. Deligne) of the class of a cycle in the Deligne cohomology. Before describing this construction in a slightly modified way (§ 7) we recall some of the properties of the cycle class in the De Rham cohomology. Especially we will need that those cycle classes behave well with respect to the F-filtration (6.10). Since we do not know any reference we sketch a proof. We thank F. El Zein and J.L. Verdier for useful conversations on those topics.

6.1. Let Y be an algebraic manifold over $\mathbb{C}$ and $\eta \in Y$ be an irreducible subvariety of codimension p. We will frequently use some properties of the local cohomology with support in η (see for example [14]):

a) If $\mathcal{F}^\cdot$ is a complex of sheaves and $Y' \subseteq Y$ an open subvariety one has an exact sequence

$$\ldots \longrightarrow \mathbb{H}^p_{\eta-Y'}(Y,\mathcal{F}^\cdot) \longrightarrow \mathbb{H}^p_\eta(Y,\mathcal{F}^\cdot) \longrightarrow \mathbb{H}^p_{\eta\cap Y'}(Y',\mathcal{F}^\cdot|_{Y'}) \longrightarrow \mathbb{H}^{p+1}_{\eta-Y'}(Y,\mathcal{F}^\cdot) \to \ldots$$

b) If $\mathcal{F}$ is a locally free $\mathcal{O}_X$ sheaf and $j < p$ one has $H^j_\eta(Y,\mathcal{F}) = 0$.

c) Assume that $\eta - Y' \neq \eta$. Then b) applied to the cycle $\eta - Y'$ implies that

$$H^p_\eta(Y,\mathcal{F}^\cdot) \longrightarrow H^p_{\eta\cap Y'}(Y',\mathcal{F}^\cdot|_{Y'}) .$$

d) Let $\mathcal{F}^\cdot$ be a complex of locally free $\mathcal{O}_X$ sheaves with $\mathcal{F}^i = 0$ for $i < p$. Then $\mathbb{H}^j_\eta(X,\mathcal{F}^\cdot) = 0$ for $j < 2p$ and

$$\mathbb{H}^{2p}_\eta(X,\mathcal{F}^\cdot) \hookrightarrow H^p_\eta(X,\mathcal{F}^p) .$$

In fact, one has the spectral sequence associated to the "filtration bête"

$$E_1^{ij} = H_\eta^j(X, F^i) \Rightarrow \mathbb{H}_\eta^{i+j}(X, F^\cdot).$$

By b) $H_\eta^j(X, F^i) = 0$ for $j < p$ and - of course - for $i < p$. Hence $E_1^{ij} = 0$ for all $i + j < 2p$. For $i + j = 2p$ one obtains that $E_\infty^{pp} = \mathbb{H}_\eta^{2p}(X, F^\cdot)$ is embedded in $E_1^{pp} = H_\eta^p(X, F^p)$.

The example we have in mind is: If F^p denotes the F-filtration of $\Omega_X^\cdot$ (see 2.5) then one has an inclusion

$$\mathbb{H}_\eta^{2p}(X, F^p) \hookrightarrow H_\eta^p(X, \Omega_X^p).$$

<u>6.2.</u> Since η is smooth at the general point one can find divisors $D_1, \ldots, D_p$ on Y and an open affine subvariety Y' of Y such that $D_i' = D_i \cap Y'$ are non singular divisors intersecting transversally and such that

$$\eta' = \eta \cap Y' = \bigcap_{i=1}^{p} D_i' \quad .$$

$\{U_i' = Y' - D_i'\}_{i=1,\ldots,p}$ is a covering of $Y' - \eta'$. Let $c(\eta')$ be the element of $H^{p-1}(Y' - \eta', \Omega_{Y'-\eta'}^p)$ given by the Čech-cocycle

$$\frac{dt_1 \wedge \ldots \wedge dt_p}{t_1 \cdot \ldots \cdot t_p} \quad \text{on} \quad U_{1,\ldots,p}' = Y' - \bigcup_{i=1}^{p} D_i' \ ,$$

where t_i is the defining equation of D_i'. By (6.1,a) we have a map

$$H^{p-1}(Y' - \eta', \Omega_{Y'-\eta'}^p) \longrightarrow H_{\eta'}^p(Y', \Omega_{Y'}^p) \ ,$$

surjective since Y' is affine. We denote the image of $c(\eta')$ by $c_\Omega(Y', \eta')$. Moreover, by (6.1,c) we have an inclusion

$$\iota \ : \ H_\eta^p(Y, \Omega_Y^p) \hookrightarrow H_{\eta'}^p(Y', \Omega_{Y'}^p) \ .$$

<u>Theorem 6.3. ([2] and [9])</u>

There exists a <u>cycle class</u> $c_\Omega(\eta) = c_\Omega(Y, \eta)$ of η on Y, lying in $H_\eta^p(Y, \Omega_Y^p)$ such that

$$\iota(c_\Omega(Y,\eta)) = c_\Omega(Y',\eta') .$$

Remark 6.4. a) F. El Zein [9] shows in addition that $c_\Omega(\eta)$ can be defined by a cocycle in the closed differential forms $(\Omega^p_Y)^{cl}$. Therefore $c_\Omega(\eta)$ is the image of a class $c_F(\eta)$ in $\mathbb{H}^{2p}_\eta(Y,F^p)$, uniquely determined by (6.1,d).

b) The image of $c_F(\eta)$ in $\mathbb{H}^{2p}_{|\eta|}(Y,\Omega^\bullet_Y) \cong H^{2p}_{|\eta|}(Y,\mathbb{C})$ is denoted by $c_{\mathbb{C}}(\eta)$. Of course, one can also consider the fundamental class of η in $H^{2p}_{|\eta|}(Y,\mathbb{Z})$ or - after multiplication with $(2i\pi)^p$ - in $H^{2p}_{|\eta|}(Y,\mathbb{Z}(p))$. We denote it by $c_{\mathbb{Z}}(\eta)$. The image of $c_{\mathbb{Z}}(\eta)$ is again $c_{\mathbb{C}}(\eta)$. In fact, by the description of (6.2) and (6.3) it is enough to consider the case $p = 1$. For divisors the equality of the two classes easily follows from the definition of $c_{\mathbb{Z}}(\eta)$ (see [7]).

Remark 6.5. Let D be a normal crossing divisor on Y, containing η. Then the image of $c_\Omega(\eta)$ in $H^p_\eta(Y,\Omega^p_Y(\log D))$ is zero.

Proof. Keeping the notations from (6.2) it is enough to show that the image of $c(\eta')$ in $H^{p-1}(Y'-\eta',\Omega^p_{Y'-\eta'}(\log(Y'\cap D)))$ is zero. We may choose the divisors $D_1,\ldots,D_p$ such that $D = \bigcup_{i=1}^r D_i$ for some r. Then the cocycle

$$\frac{dt_1\wedge\ldots\wedge dt_p}{t_1\cdot\ldots\cdot t_p}$$

in $C^{p-1}(\Omega^p_{Y'}(\log(Y'\cap D)))$ extends to $U'_{r+1,\ldots,p} = Y' - \bigcup_{i=r+1}^{p} D'_i$ and $c(\eta') = 0$.

6.6. Let $f : \bar{X} \longrightarrow Y$ be a birational morphism, isomorphic over $X = Y - \eta$, such that $D = f^{-1}(\eta)$ is a normal crossing divisor. One has natural maps

$$H^p_\eta(Y,\Omega^p_Y) \xrightarrow{f^*} H^p_D(\bar{X},\Omega^p_{\bar{X}}) \xrightarrow{\alpha} H^p_D(\bar{X},\Omega^p_{\bar{X}}(\log D)).$$

Proposition. The image of $c_\Omega(\eta)$ in $H^p_D(\bar{X},\Omega^p_{\bar{X}}(\log D))$ is zero.

Proof. One would like to say that $f^*c_\Omega(\eta)$ is the sum of cycle classes of codimension p cycles and that (6.5) implies (6.6). However to get hold of $f^*c_\Omega(\eta)$ we have to use the description of cycle classes given by B. Angéniol and M. Lejeune-Jalabert [1]:

Let $M^\cdot$ be a perfect complex of $\mathcal{O}_X$ sheaves on Y. The first Atiyah class $\lambda^1_M \in \mathrm{Ext}^1(M^\cdot, \Omega^1_Y \otimes^L M^\cdot)$ is the obstruction for $M^\cdot$ to have a holomorphic connection. One defines the p-th Atiyah class $\lambda^p_{M^\cdot}$ as the p-th exterior power of $\lambda^1_{M^\cdot}$ in $\mathrm{Ext}^p(M^\cdot, \Omega^p_Y \otimes^L M^\cdot)$. If $M^\cdot$ is acyclic outside of a subvariety $Z \subseteq Y$ one uses the isomorphism

$$\mathrm{Ext}^p(M^\cdot, \Omega^p_Y \otimes^L M^\cdot) \simeq \varinjlim_m \mathrm{Ext}^p(\mathcal{O}_{Z_m} \otimes^L M^\cdot, \Omega^p_Y \otimes^L M^\cdot)$$

and the trace

$$\mathrm{Ext}^p(\mathcal{O}_{Z_m} \otimes^L M^\cdot, \Omega^p_Y \otimes^L M^\cdot) \longrightarrow \mathrm{Ext}^p(\mathcal{O}_{Z_m}, \Omega^p_Y)$$

to define the p-th Newton class ${}^Z\nu^p_{M^\cdot}$ in $\varinjlim_m \mathrm{Ext}^p(\mathcal{O}_{Z_m}, \Omega^p_Y) = H^p_Z(Y, \Omega^p_Y)$ (see [1], § II).

As shown in the proof of II, 2.5.3 (loc. cit.) $c_\Omega(\eta)$ is - up to a constant - the same as the p-th Newton class ${}^\eta\nu^p_{\mathcal{O}_\eta}$. By II, 4.2.1 $f^*\lambda^p_{\mathcal{O}_\eta} = \lambda^p_{Lf^*\mathcal{O}_\eta}$ in $\mathrm{Ext}^p(Lf^*\mathcal{O}_\eta, \Omega^p_X \otimes^L Lf^*\mathcal{O}_\eta)$. The trace is compatible with pullbacks ([13],V, 3.9.3) and one obtains $f^{*\,\eta}\nu^p_{\mathcal{O}_\eta} = {}^D\nu^p_{Lf^*\mathcal{O}_\eta}$. Therefore (6.6) follows from:

<u>Lemma 6.7.</u> Let $M^\cdot$ be a perfect complex of sheaves on $\bar{X}$, exact outside of D. Then $\alpha({}^D\nu^p_{M^\cdot}) = 0$ for

$$\alpha : H^p_D(\bar{X}, \Omega^p_X) \longrightarrow H^p_D(\bar{X}, \Omega^p_X(\log D)).$$

<u>Proof.</u> We denote by α as well the morphism

$$\mathrm{Ext}^p(M^\cdot, M^\cdot \otimes \Omega^p_X) \longrightarrow \mathrm{Ext}^p(M^\cdot, M^\cdot \otimes \Omega^p_X(\log D))$$

and we call $\alpha(\lambda^p_{M^\cdot})$ the logarithmic Atiyah class of $M^\cdot$.

<u>Case I:</u> Assume that $M^\cdot$ is quasi-isomorphic to a locally free sheaf on a smooth divisor $D' \subseteq D$. We may write $M^\cdot = (M^{-1} \xhookrightarrow{\varphi} M^0)$ for locally free $\mathcal{O}_X$ - modules M^{-1} and M^0. On a suitable Čech cover $\{U_i\}$ we have isomorphisms $M^r|_{U_i} \simeq \mathcal{O}^{\oplus\nu}_{U_i}$ and, if f_i is an equation for $D' \cap U_i$, $\varphi_i = \varphi|U_i$ can be given by a diagonal matrix with 1 and f_i in the diagonal. As in [1], II, 1.5 the logarithmic Atiyah class is represented by a Čech-cocycle of morphisms

$$\delta^p_r(i_0,\dots,i_k) : M^r|_{U_{i_0},\dots,i_k} \longrightarrow M^{r+p-k} \otimes \Omega^p_X(\log D)|_{U_{i_0},\dots,i_k}.$$

In our situation only $r=0$, $k=p$ and $r=-1$, $p \geq k \geq p-1$ may occur. We claim, that $\alpha(\lambda^p_{M^\cdot})$ can be represented by a cocycle $\delta^p_r(i_0,\ldots,i_p)$. Since $\alpha(\lambda^p_{M^\cdot})$ is obtained from $\alpha(\lambda^1_{M^\cdot})$ by exterior product (II, 1.4 loc.cit), it is enough to verify this for $p=1$. Since $M^\cdot|_{U_i}$ has a logarithmic connection for all i, $\alpha(\lambda^1_{M^\cdot|U_i})$ is zero. This means in particular that $\delta^1_{-1}(i)$ is for all i on U_i a coboundary in the corresponding complex. Hence we can change the whole cocycle to obtain the representation wanted.

(Explicitly, if we use the notations from II, 1.5 (loc. cit) $\delta^1_{-1}(i_0) = d(\varphi_{i_0}|_{U_{i_0}}) = B\cdot df$ where B is a diagonal matrix having only 1 or 0 in the diagonal. We have a morphism

$$B\frac{df}{f} : M^{-1}|_{U_{i_0}} \longrightarrow M^{-1} \otimes \Omega^1_X (\log D)|_{U_{i_0}}$$

and $\delta^1_{-1}(i_0) = \varphi_{i_0} \cdot B\frac{df}{f}$. If d' denotes the differential in the Čech complex we have to change the Čech cocycle $\delta^p_r(\underline{i})$ by $d'(B\frac{df}{f})$ to obtain the representation wanted.)

Since $M^\cdot$ is acyclic outside of D we may pass to the limit and obtain $\alpha(\lambda^p_{M^\cdot})$ as a Čech cocycle in

$$\mathrm{Hom}(M^\cdot|_{U_{i_0\ldots i_p}}, M^\cdot \otimes \Omega^p_X(\log D)_{U_{i_0\ldots i_p}}) = \varinjlim_m \mathrm{Hom}(\mathcal{O}_{D_m} \otimes M^\cdot|_{U_{i_0\ldots i_p}}, M^\cdot \otimes \Omega^p_X(\log D)|_{U_{i_0\ldots i_p}}).$$

By definition of the trace map in [13], V, 3.7, the trace map can be calculated on a Čech-covering. Hence $\alpha({}^D\nu^p_{M^\cdot})$ is represented by a collection of elements of

$$\varinjlim_m \mathrm{Hom}(\mathcal{O}_{D_m}|_{U_{i_0\ldots i_p}}, \Omega^p_X(\log D)|_{U_{i_0\ldots i_p}}).$$

Those groups however are zero.

To reduce the general case to case I, we need that the logarithmic Newton classes $\alpha({}^D\nu^p_{M^\cdot})$ with support in D are additive for exact sequences of perfect complexes, acyclic outside of D. In fact, the proof in [1], II, 4.3 uses just the additivity of the trace ([13],V, 3.7.7) and carries over to logarithmic Newton classes with support.

<u>Case II:</u> If $D' \subseteq D$ is smooth and $M^\cdot$ quasi-isomorphic to a $\mathcal{O}_{D'}$-module, we can take an $\mathcal{O}_{D'}$ locally free resolution N. Since N is bounded

and

$$(0 \to N^{-r+1} \to \dots \to N^{-s} \to 0) \to (0 \to N^{-r} \to \dots \to N^{-s} \to 0) \to (0 \to N^{-r} \to 0)$$

is exact, case II follows from case I.

Case III: If $M^{\cdot}$ is quasi-isomorphic to any $\mathcal{O}_{\overline{X}}$ - coherent sheaf F with support in D, we can filter F by $F_{\underline{m}} = F \otimes \mathcal{O}_{\overline{X}}(-\sum_{i=1}^{r} m_i D_i)$. For $\underline{m}' = (m_1,\dots,m_i+1,\dots,m_r)$ $F_{\underline{m}}/F_{\underline{m}'}$ is an $\mathcal{O}_{D_i}$-sheaf and we are in case II.

Case IV: If $M^{\cdot}$ is any perfect complex, acyclic outside of D, we use the surjection

$$(0 \to M^{-1} \to \dots \to M^{-s-1} \xrightarrow{\delta_{-s}} M^{-s} \to 0) \to (0 \to M^{-s}/\mathrm{Im}\ \delta_{-s} \to 0)$$

with kernel

$$(0 \to M^{-r} \to \dots \to M^{-s-1} \to \mathrm{Im}\ \delta_{-s} \to 0) \xleftarrow{\sim} (0 \to M^{-r} \to \dots \to M^{-s-2} \to \mathrm{Ker}\ \delta_{-s} \to 0)$$

to reduce the proof of (6.7) to case III.

6.8. The definitions of the cycle classes with values in Ω^p, F^p, $\mathbb{C}$ and $\mathbb{Z}$ are - as usual - extended to the group $Z^p(Y)$ of codimension p-cycles. For example, for $\eta = \Sigma\, \nu_i \eta_i \in Z^p(Y)$ one defines

$$c_\Omega(\eta) = \Sigma\, \nu_i \cdot \mathrm{mult}(\eta_i) \cdot c_\Omega((\eta_i)_{red})$$

in $H^p_{|\eta|}(Y,\Omega^p_Y)$, where $|\eta|$ is the support of η. If, keeping the notations from (6.6), $f : \overline{X} \longrightarrow Y$ is a birational morphism, isomorphic over $X = Y - |\eta|$ and such that $f^{-1}(|\eta|) = D$ is a normal crossing crossing divisor one obtains as well that $\alpha f^*(c_\Omega(\eta)) = 0$.

Remark 6.9. One can consider the statement corresponding to (6.6) for c_F instead of c_Ω : If $F^{\cdot}_D$ denotes the F-filtration of $\Omega^{\cdot}_{\overline{X}}(\log D)$ it would be nice to know that $c_F(\eta)$ is mapped to zero under

$$\mathbb{H}^{2p}_{|\eta|}(Y,F^p) \longrightarrow \mathbb{H}^{2p}_D(\overline{X},F^p_D).$$

Without this we still obtain:

Proposition 6.10. If Y is a complete algebraic manifold and if $c_F(\eta)$ lies in the kernel of $\mathbb{H}^{2p}_{|\eta|}(Y,F^p) \longrightarrow \mathbb{H}^{2p}(Y,F^p)$, then $c_F(\eta)$ lies in the image of the composed map

$$\tau : \mathbb{H}^{2p-1}(\bar{X},F^p_D) \longrightarrow \mathbb{H}^{2p-1}(X,F^p) \longrightarrow \mathbb{H}^{2p}_{|\eta|}(Y,F^p).$$

Proof. Under the assumption $c_\Omega(\eta)$ lies in the kernel of $H^p_{|\eta|}(Y,\Omega^p_Y) \longrightarrow H^p(Y,\Omega^p_Y)$ and by (6.1,a) in the image of $H^{p-1}(X,\Omega^p_X)$. (6.6) and the commutative diagram

$$\begin{tikzcd}
H^{p-1}(\bar{X},\Omega^p_{\bar{X}}(\log D)) \arrow[r] \arrow[dr, "\gamma"'] & H^{p-1}(X,\Omega^p_X) \arrow[r] \arrow[d, leftrightarrow, "="] & H^p_{|D|}(\bar{X},\Omega^p_{\bar{X}}(\log D)) \\
& H^{p-1}(X,\Omega^p_X) \arrow[r, "\iota"] & H^p_{|\eta|}(Y,\Omega^p_Y) \arrow[u, "\alpha\circ f^*"']
\end{tikzcd}$$

with exact first row implies that $c_\Omega(\eta)$ lies in the image of $\iota\circ\gamma$. One has a commutative diagram

$$\begin{tikzcd}
\mathbb{H}^{2p-1}(\bar{X},F^p_D) \arrow[r, "\beta"] \arrow[d, "\tau"'] & H^{p-1}(\bar{X},\Omega^p_{\bar{X}}(\log D)) \arrow[d, "\iota\cdot\gamma"] \\
\mathbb{H}^{2p}_{|\eta|}(Y,F^p) \arrow[r, "\beta'"'] & H^p_{|\eta|}(Y,\Omega^p_Y)
\end{tikzcd} .$$

β' is injective (6.2,d) and, since $\bar{X}$ is compact, β is surjective (2.5).

§ 7 The cycle map in the Deligne cohomology

7.1. Let Y be a complete algebraic manifold, η a codimension p cycle and $X = Y - |\eta|$. We define $H^i_{|\eta|}(Y,\mathbb{Z}(p)_{\mathcal{D}})$ as the hypercohomology group $\mathbb{H}^i_{|\eta|}(Y,\mathbb{Z}(p)_{\mathcal{D}})$. By definition of $\mathbb{Z}(p)_{\mathcal{D}}$ as a cone (2.6) we have an exact sequence (2.2)

$$\longrightarrow H^{2p-1}_{|\eta|}(Y,\mathbb{C}) \longrightarrow \mathbb{H}^{2p}_{|\eta|}(Y,\mathbb{Z}(p)_{\mathcal{D}}) \longrightarrow H^{2p}_{|\eta|}(Y,\mathbb{Z}(p)) \oplus \mathbb{H}^{2p}_{|\eta|}(Y,F^p) \xrightarrow{\theta} H^{2p}_{|\eta|}(Y,\mathbb{C}) \longrightarrow \ldots$$

Since $2p-1$ is smaller than the real codimension $H^{2p-1}_{|\eta|}(Y,\mathbb{C}) = 0$. Moreover, since θ is the difference of the two natural maps ε and ι, $\theta(c_{\mathbb{Z}}(\eta),c_F(\eta))$ is zero (see 6.4). Therefore we may regard

$(c_{\mathbb{Z}}(\eta), c_F(\eta))$ as an element of $\mathbb{H}^{2p}_{|\eta|}(Y, \mathbb{Z}(p)_{\mathcal{D}})$, and we call it $c_{\mathcal{D}}(\eta)$. By the forget morphism

$$\mathbb{H}^{2p}_{|\eta|}(Y, \mathbb{Z}(p)_{\mathcal{D}}) \longrightarrow H^{2p}_{\mathcal{D}}(Y, \mathbb{Z}(p))$$

we obtain the cycle class of η without support, called $\psi(\eta)$ in the sequel.

<u>Remark 7.2.</u> If Y is non compact and $\mathbb{Z}(p)_{\mathcal{D}}$ is the $D - \bar{b}$ - complex on a good compactification $\bar{Y}$, the same construction works with $\mathbb{H}^{2p}_{|\eta|}(\bar{Y}, F^p_{(\bar{Y}-Y)})$ instead of $\mathbb{H}^{2p}_{|\eta|}(Y, F^p)$. However, since the class $c_F(\bar{Y}, \bar{\eta})$ of the closure $\bar{\eta}$ of η is already defined as an element of $\mathbb{H}^{2p}_{|\bar{\eta}|}(\bar{Y}, F^p)$ we can as well compactify first and use at the very end the map

$$H^{2p}_{\mathcal{D}}(\bar{Y}, \mathbb{Z}(p)) \longrightarrow H^{2p}_{\mathcal{D}}(Y, \mathbb{Z}(p))$$

to get classes in the $D - \bar{b}$ - cohomology of Y.

<u>7.3.</u> Let η be a codimension p-cycle and η' a codimension q-cycle. If both intersect properly $\eta \cdot \eta'$ is a codimension $p+q$ cycle. The product $\cup : \mathbb{Z}(p)_{\mathcal{D}} \otimes^L \mathbb{Z}(q)_{\mathcal{D}} \longrightarrow \mathbb{Z}(p+q)_{\mathcal{D}}$ defined in (1.1) (see also §3) gives

$$\cup : \mathbb{H}^{2p}_{|\eta|}(Y, \mathbb{Z}(p)_{\mathcal{D}}) \otimes \mathbb{H}^{2q}_{|\eta'|}(Y, \mathbb{Z}(q)_{\mathcal{D}}) \longrightarrow \mathbb{H}^{2(p+q)}_{|\eta \cdot \eta'|}(Y, \mathbb{Z}(p+q)_{\mathcal{D}}).$$

<u>Proposition 7.4.</u> If η and η' intersect properly $c_{\mathcal{D}}(\eta) \cup c_{\mathcal{D}}(\eta') = c_{\mathcal{D}}(\eta \cdot \eta')$ and $\psi(\eta) \cup \psi(\eta') = \psi(\eta \cdot \eta')$.

<u>Proof.</u> The second equality follows from the first one. By (3.7) the cup product is compatible with the usual products on $H^{2\cdot}_{|\;|}(Y, \mathbb{Z}(.))$ and $H^{2\cdot}_{|\;|}(Y, F^{\cdot})$. Since $c_{\mathcal{D}}$ is uniquely determined by $c_{\mathbb{Z}}$ and c_F, the first equality follows from the corresponding ones for $c_{\mathbb{Z}}$ and c_F (see [9], for example).

The same argument proves:

<u>Proposition 7.5.</u> If $g : Y' \longrightarrow Y$ is a morphism and η a codimension p cycle such that $g^*\eta$ is of codimension p as well, then $g^* c_{\mathcal{D}}(\eta) = c_{\mathcal{D}}(g^*\eta)$ in $\mathbb{H}^{2p}_{|g^*\eta|}(Y', \mathbb{Z}(p)_{\mathcal{D}})$ and $g^*(\psi(\eta)) = \psi(g^*\eta)$ in $H^{2p}_{\mathcal{D}}(Y', \mathbb{Z}(p))$.

Proposition 7.6. Let η_1 and η_2 be two rationally equivalent codimension p cycles on Y. Then $\psi(\eta_1)=\psi(\eta_2)$.

Proof. By definition of rational equivalence there is a codimension p cycle ξ on $Y\times\mathbb{P}^1$ and $x_1,x_2\in\mathbb{P}^1$ such that $\eta_k=\iota_k^*(\xi)$ for $\iota_k : Y\cong Y\times\{x_k\}\hookrightarrow Y\times\mathbb{P}^1$. If τ is an isomorphism of $\mathbb{P}^1$ with $\tau(x_1)=x_2$, $\iota_1^*\cdot(\mathrm{id}\times\tau)^*(\xi)=\eta_2$. τ^* acts on $H^\cdot(\mathbb{P}^1,\mathbb{Z})$ as identity. Hence $(\mathrm{id}\times\tau)^*$ is the identity on $H^\cdot(Y\times\mathbb{P}^1,\mathbb{Z})$ and therefore on $\mathbb{H}^\cdot(Y\times\mathbb{P}^1,F^p)$ as well. By (2.10,a) $(\mathrm{id}\times\tau)^*$ is the identity on $H_{\mathcal{D}}^{2p}(Y\times\mathbb{P}^1,\mathbb{Z}(p))$ and

$$\psi(\eta_2)=\iota_1^*\cdot(\mathrm{id}\times\tau)^*(\psi(\xi)) = \iota_1^*(\psi(\xi)) = \psi(\eta_1).$$

Corollary 7.7. Let $CH^\cdot(Y)=\bigoplus_{p\geq 0} CH^p(Y)$ be the Chowring of Y, i.e.: $CH^p = Z^p(Y)/\text{rat.eq.}$ and $H_{\mathcal{D}}^\cdot(Y)=\bigoplus_{p\geq 0} H_{\mathcal{D}}^{2p}(Y,\mathbb{Z}(p))$. Then ψ defines a ring-homomorphism

$$\psi : CH^\cdot(Y) \longrightarrow H_{\mathcal{D}}^\cdot(Y).$$

Moreover, ψ is compatible with $g^* : CH(Y)\longrightarrow CH^\cdot(Y')$ for $g : Y'\longrightarrow Y$.

Proof. By (7.6) ψ factors over $CH^\cdot(Y)$. Using the moving Lemma it is enough to verify the compatibility of ψ with the product for cycles intersecting properly, and to verify the compatibility of ψ with g^* for cycles η with $\operatorname{codim}(\eta)=\operatorname{codim}(g^*\eta)$. This has been done in (7.4) and (7.5).

7.8. Griffith's intermediate Jacobian

Recall that Y is a complete algebraic manifold. By (2.5) the subgroup $F^pH^q(Y,\mathbb{C})$ of $H^q(Y,\mathbb{C})$ is isomorphic to $\mathbb{H}^q(Y,F^p)$ and the quotient group $H^q(Y,\mathbb{C})/F^p$ is isomorphic to $\mathbb{H}^q(Y,\Omega_Y^{<p})$. Since

$$F^pH^{2p-1}(Y,\mathbb{C})\cap\overline{F^pH^{2p-1}(Y,\mathbb{C})}=0$$

the image of $H^{2p-1}(Y,\mathbb{Z}(p))$ in $H^{2p-1}(Y,\mathbb{C})/F^p$ is a lattice and

$$J^p(Y) = H^{2p-1}(Y,\mathbb{C})/H^{2p-1}(Y,\mathbb{Z}(p))+F^pH^{2p-1}(Y,\mathbb{C}) = \mathbb{H}^{2p-1}(Y,\Omega_Y^{<p})/H^{2p-1}(Y,\mathbb{Z}(p))$$

is a complex torus, called the p-th intermediate Jacobian of Y. We denote by $Hg^p(Y)$ the Hodge cycles of Y, i.e.

$$Hg^p(Y) = \mathrm{Ker}(H^{2p}(Y,\mathbb{Z}(p)) \oplus \mathbb{H}^{2p}(Y,F^p) \xrightarrow{\varepsilon-\iota} H^{2p}(Y,\mathbb{C})).$$

This coincides with the usual definition, since $F^{p+1}H^{2p}(Y,\mathbb{C}) \cap \overline{F^{p+1}H^{2p}(Y,\mathbb{C})} = 0$ and therefore $\mathrm{Ker}(\varepsilon - \iota) = \varepsilon^{-1}(H^{p,p}) \cap H^{2p}(Y,\mathbb{Z}(p))$. The exact sequence (2.10.a) implies:

$$0 \longrightarrow J^p(Y) \longrightarrow H^{2p}_{\mathcal{D}}(Y,\mathbb{Z}(p)) \longrightarrow Hg^p(Y) \longrightarrow 0 \tag{7.9}$$

is exact.

By (3.7) the cup product respects the exact sequence (7.9). Hence $J^{\cdot}(Y) = \bigoplus_{p\geq 0} J^p(Y)$ is an ideal of the commutative ring $H^{\cdot}_{\mathcal{D}}(Y) = \bigoplus_{p\geq 0} H^{2p}_{\mathcal{D}}(Y,\mathbb{Z}(p))$.

Proposition 7.10.

$J^{\cdot}(Y)$ is an ideal of square zero.

Proof. An element of $J^p(X)$ is represented by an element of $\mathbb{H}^{2p-1}(Y,\Omega_Y^{<p})$ or, by Hodge theory, of $\bigoplus_{\substack{k+\ell=2p-1\\ \ell<p}} H^k(Y,\Omega_Y^\ell)$. The differential d is zero on $H^k(Y,\Omega_Y^\ell)$ and (7.10) follows from the definition of $\cup$ given in (1.2).

Let us return to the cycle map

$$\psi : Z^p(Y) \longrightarrow H^{2p}_{\mathcal{D}}(Y,\mathbb{Z}(p)).$$

By construction $\varepsilon \circ \psi : Z^p(Y) \longrightarrow H^{2p}(Y,\mathbb{Z}(p))$ and factors through $\iota \circ \psi : Z^p(Y) \longrightarrow H^{2p}(Y,F^p)$ are the usual cycle maps. Hence, if $H^p(Y)_h$ denotes the subgroup of cycles homologous to zero, $\varepsilon \circ \psi$ and $i \circ \psi$ are zero on $Z^p(Y)_h$. By (7.9) we obtain a lifting of $\psi|_{Z^p(Y)_h}$ to $\psi_0 : Z^p(Y)_h \longrightarrow J^p(Y)$. In fact, by (7.7) ψ_0 factors through

$$CH^p(Y)_h = Z^p(Y)_h/\mathrm{rat.eq.}$$

Theorem 7.11. ψ_0 is the Abel Jacobi map.

Instead of the original definition by currents we use (proposed in [10])

7.12. The description of the Abel-Jacobi map using mixed Hodge structures.

Let η be a codimension p cycle on Y. One has the exact sequence

$$0 \longrightarrow H^{2p-1}(Y,\mathbb{Z}(p)) \xrightarrow{\beta} H^{2p-1}(X,\mathbb{Z}(p)) \longrightarrow H^{2p}_{|\eta|}(Y,\mathbb{Z}(p)) \longrightarrow H^{2p}(Y,\mathbb{Z}(p)) \to ..$$

where $X = Y - |\eta|$. All the cohomology groups carry mixed Hodge structures and the morphisms in the exact sequence respect them. Since $H^{2p}_{|\eta|}(Y,\mathbb{Z}(p))$ is generated by the cycle classes of the components of η, and since those cycle classes are by construction (6.4) of type (p,p), the cokernel of β is of type (p,p) and β induces isomorphisms

$$H^{2p-1}(Y,\mathbb{C})/F^pH^{2p-1}(Y,\mathbb{C}) \xrightarrow{\simeq} H^{2p-1}(X,\mathbb{C})/F^pH^{2p-1}(X,\mathbb{C})$$

and

$$(7.13) \qquad J^p(Y) \xrightarrow{\sim} H^{2p-1}(X,\mathbb{C})/H^{2p-1}(Y,\mathbb{Z}(p)) + F^pH^{2p-1}(X,\mathbb{C}).$$

Regarding the exact sequence one finds that for $\eta \in Z^p(Y)_h$ the fundamental class $c_{\mathbb{Z}}(\eta)$ is the image of a class $\widetilde{c_{\mathbb{Z}}(\eta)} \in H^{2p-1}(X,\mathbb{Z}(p))$ uniquely determined up to $\mathrm{Im}(\beta)$. We denote by $\widetilde{c_{\mathbb{Z}}(\eta)}$ as well the image in $H^{2p-1}(X,\mathbb{C})$. By (7.13) $\widetilde{c_{\mathbb{Z}}(\eta)}$ defines an element $\psi_0'(\eta) \in J^p(Y)$.

<u>Definition 7.14.</u> (El Zein and Zucker, [10])

$\psi_0' : Z^p(Y)_h \longrightarrow J^p(Y)$ is called the <u>Abel Jacobi map</u> .

<u>Proof of 7.11.</u> Consider the commutative diagram of exact sequences:

$$\begin{tikzcd}
J^p(Y) \arrow[r, hook] & H^{2p}_{\mathcal{D}}(Y,\mathbb{Z}(p)) \arrow[r] & H^{2p}(Y,\mathbb{Z}(p)) \oplus \mathbb{H}^{2p}(Y,F^p) \arrow[r] & H^{2p}(Y,\mathbb{C}) \\
0 \arrow[u] \arrow[r] & \mathbb{H}^{2p}_{|\eta|}(Y,\mathbb{Z}(p)_{\mathcal{D}}) \arrow[u] \arrow[r] & H^{2p}_{|\eta|}(Y,\mathbb{Z}(p)) \oplus \mathbb{H}^{2p}_{|\eta|}(Y,F^p) \arrow[u, "\rho"] \arrow[r, "\theta"] & H^{2p}_{|\eta|}(Y,\mathbb{C}) \arrow[u] \\
 & & H^{2p-1}(X,\mathbb{Z}(p)) \oplus \mathbb{H}^{2p-1}(X,F^p) \arrow[u, "\rho'"] \arrow[r, "\theta'"] & H^{2p-1}(X,\mathbb{C}) \arrow[u] \\
 & & & H^{2p-1}(Y,\mathbb{C}) \arrow[u, hook, "\beta"]
\end{tikzcd}$$

For $\eta \in Z^p(Y)_h$ we have $\rho(c_{\mathbb{Z}}(\eta),\ c_F(\eta)) = 0$. Therefore $(c_{\mathbb{Z}}(\eta),\ c_F(\eta)) = \rho'(\widetilde{c_{\mathbb{Z}}(\eta)},\ \widetilde{c_F(\eta)})$ for some

$(\widetilde{c_{\mathbb{Z}}(\eta)},\ \widetilde{c_F(\eta)}) \in H^{2p-1}(X,\mathbb{Z}(p)) \oplus H^{2p-1}(X,F^p)$.

Since $\theta(c_{\mathbb{Z}}(\eta), c_F(\eta)) = 0$, $\theta'(\widetilde{c_{\mathbb{Z}}(\eta)}, \widetilde{c_F(\eta)})$ lies in $\beta(H^{2p-1}(Y,\mathbb{C}))$. By the snake-lemma one finds $\psi_0(\eta)$ to be the image of $\beta^{-1}\theta'(\widetilde{c_{\mathbb{Z}}(\eta)}, \widetilde{c_F(\eta)})$ in $J^p(Y)$. By (6.10) $\widetilde{c_F(\eta)}$ lies in the image of

$$\mathbb{H}^{2p-1}(\bar{X}, F^p_D) = F^p H^{2p-1}(X,\mathbb{C}) \longrightarrow \mathbb{H}^{2p-1}(X,F^p).$$

Therefore $\theta'(\widetilde{c_{\mathbb{Z}}(\eta)}, \widetilde{c_F(\eta)})$ and $\theta'(\widetilde{c_{\mathbb{Z}}(\eta)}, 0)$ define the same element in

$$H^{2p-1}(X,\mathbb{C})/F^pH^{2p-1}(X,\mathbb{C}) + H^{2p-1}(Y,\mathbb{Z}(p))$$

and by (7.13) $\psi_0(\eta) = \psi_0'(\eta)$.

Remarks. a) The construction of the Abel-Jacobi map and of the cycle map in the Deligne cohomology has been done in [10] and [3] in a slightly different way. At first glance it seems surprising that the proof of (7.11) in [10] or [3] does not need the statement like (6.10). However, the proof given there uses a description of the Abel-Jacobi map by currents, which is different from the one given in (7.12). If one assumes that both coincide , it proves (6.10) directly, without studying the pullback of cycles. On the other hand one can use (6.10) to show that ψ_0' is the same as the Abel-Jacobi map defined by currents.
b) A different treatment of these topics can also be found in the Chapter by U.Jannsen (§1.21-23) in this book.

§ 8 Chern classes in the Deligne-Beilinson cohomology

8.1. Let X be an algebraic manifold or - using § 5 - any simplicial scheme of finite type over $\mathbb{C}$. In this section we sketch two methods to define Chern classes

$$c_p(E) \in H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p))$$

for locally free $\mathcal{O}_X$-sheaves E (called bundles in the sequel) of rank r on X. They should depend just on the isomorphism class and satisfy

A) (Functionality) For any morphism

$$f : Y \longrightarrow X \text{ one has } f^*c_p(E) = c_p(f^*E).$$

B) (Compatibility with the Chern-classes in $H^{\cdot}(\ ,\mathbb{Z})$)
$c_p(E)$ is mapped under $\varepsilon : H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p)) \longrightarrow H^{2p}(X,\mathbb{Z}(p))$ to the usual Chern classes of E.

Of course we can as well consider Chern classes in $H^{2p}_{\mathcal{D}}(X,A(p))$ for any subring A of $\mathbb{R}$. However those are just the image of the classes in $H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p))$.

Proposition 8.2. The Chern classes are uniquely determined by conditions A and B.

Proof. The classifying space $BG = BGL_r(\mathbb{C})$ is a simplicial scheme of finite type over $\mathbb{C}$ and, as proved in [8] there are elements c_p of pure weight (p,p), such that

$$H^{\cdot}(BG,\mathbb{Z}) = \mathbb{Z}[c_1,\ldots,c_r] \ .$$

Therefore $H^{2p-1}(BG,\mathbb{C}) = 0$ and $\iota : \mathbb{H}^{2p}(BG,F^p) \longrightarrow H^{2p}(BG,\mathbb{C})$ is an isomorphism. By (2.10,a) or (2.10,b)

$$\varepsilon : H^{2p}_{\mathcal{D}}(BG,\mathbb{Z}(p)) \longrightarrow H^{2p}(BG,\mathbb{Z}(p))$$

is an isomorphism, and the Chern classes of the universal bundle E^{un}_r are uniquely determined by B. If E on X is any bundle of rank r one can take a hypercovering $\rho : Z_0 \longrightarrow X$ such that ρ^*E is trivial on each Z_ν. Then there is a morphism $f : Z. \longrightarrow BG$ such that $f^*E^{un}_r = \rho^*E$. By A the Chern classes of ρ^*E are uniquely determined. Since the $D - \bar{b}$ cohomology of $Z.$ and X are isomorphic (5.2) one obtains (8.2).

8.3. For a non-singular variety A.Grothendieck defines in [12] Chern classes $\tilde{c}_p(E)$ of vector bundles E in the Chow group $CH^p(X)$. Those are functional and, under the cycle map $c_{\mathbb{Z}}$, compatible with the Chern classes in $H^{2p}(X,\mathbb{Z}(p))$. Therefore $c_p(E) = \psi(\tilde{c}_p(E))$ defines Chern classes for vector bundles on X, satisfying A and B by (7.7) and (6.4). Of course, one has to use (5.1-3) to extend this definition to arbitrary simplicial schemes of finite type over $\mathbb{C}$.

8.4. A second construction of Chern classes is based on (2.12,iii) and the splitting principle:
Recall that for an algebraic manifold X we constructed an isomorphism

$$\rho : \mathcal{O}(X)^*_{alg} \longrightarrow H^1_{\mathcal{D}}(X,\mathbb{Z}(1)).$$

By (5.5,b) ρ induces a morphism (in the derived category) of complexes

of sheaves in the Zarisky topology

$$c_1 : \mathcal{O}_X^* \longrightarrow Z(1)_{\mathcal{D},Zar}[1] .$$

Taking hypercohomology of sheaves in the Zarisky topology this gives a map

$$c_1 : H^1(X,\mathcal{O}_X^*) \longrightarrow H^2_{\mathcal{D}}(X,\mathbb{Z}(1)) .$$

Since invertible sheaves correspond to elements of $H^1(X,\mathcal{O}_X^*)$ we can use c_1 to define the first Chern class of an invertible sheaf. The induced morphism

$$\mathcal{O}_X^* \longrightarrow \mathbb{Z}(1)_{\mathcal{D},Zar}[1] \longrightarrow \mathbb{Z}(1)[1]$$

is in the derived category the "edge" morphism of the exponential sequence. This shows that $\varepsilon(c_1(L))$ is the first Chern class of L in $H^2(X,\mathbb{Z}(1))$.

<u>Proposition 8.5.</u> Let E be a vector bundle of rank r on X, $\pi : \mathbb{P} = \mathbb{P}(E) \longrightarrow X$ the corresponding projective bundle and $\mathcal{O}_{\mathbb{P}}(1)$ the tautological invertible sheaf on $\mathbb{P}$. Then for all q,q'

$$H^q_{\mathcal{D}}(\mathbb{P},\mathbb{Z}(q')) \xleftarrow{\sim} \bigoplus_{p=0}^{r-1} \pi^* H^{q-2p}_{\mathcal{D}}(X,\mathbb{Z}(q'-p)) \cup c_1(\mathcal{O}_{\mathbb{P}}(1))^p .$$

<u>Proof.</u> As is well known, the same maps are isomorphisms for $H^\cdot(\ ,\mathbb{Z}(\cdot))$, $H^\cdot(\ ,\mathbb{C})$ and by [8] for $\mathbb{H}^\cdot(\ ,F^\cdot)$. By (3.9) the cup product is compatible with the exact sequence (2.10,a) and therefore (8.5) holds.

<u>8.6.</u> Now one can define Chern classes of rank r vector bundles in the way of Hirzebruch and Grothendieck:

$$\text{In } H^{2r}_{\mathcal{D}}(\mathbb{P},\mathbb{Z}(r)) = \bigoplus_{p=1}^{r} \pi^* H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p)) \cup c_1(\mathcal{O}_{\mathbb{P}}(1))^{r-p}$$

we have a relation

$$\sum_{p=0}^{r} (-1)^p \cdot \pi^*\gamma_p \cup c_1(\mathcal{O}_{\mathbb{P}}(1))^{r-p} = 0$$

with $\gamma_p \in H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p))$ and $\gamma_0 = 1$. We define $c_p(E) = \gamma_p$.

As in [12] one shows that the Chern classes obtained are functorial and additive. Since the usual Chern classes can be defined by the splitting principle as well, one obtains (8.1,B).

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