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*Inventiones mathematicae*


# Cohomology of local systems on the complement of hyperplanes

**Hélène Esnault[1], Vadim Schechtman[2,*], and Eckart Viehweg[1]**

[1] FB 6, Mathematik, Universität-GHS-Essen, W-4300 Essen 1, Federal Republic of Germany
[2] School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA

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## 1

Let $\{H_i\}_{i\in I}$ be a finite collection of (distinct) hyperplanes in the complex projective space $\mathbb{P}^n$. Define $H = \bigcup_{i\in I} H_i$; $U = \mathbb{P}^n - H$. We denote by $\Omega^p_U$ the sheaves of holomorphic forms on $U$ for $0 \leqq p \leqq n$. We set $\mathcal{O}_U := \Omega^0_U$.

For any $i, j \in I$ we have $H_i - H_j = \operatorname{div}(f_{ij})$ for some $f_{ij} \in \mathbb{C}(\mathbb{P}^n)$; Define $\eta_{ij} = d \log f_{ij} \in \Gamma(U, \Omega^1_U)$. For a given $r \in \mathbb{N} - \{0\}$ we choose matrices $P_i \in \operatorname{End} \mathbb{C}^r$, $i \in I$, such that $\sum_{i\in I} P_i = 0$.

Fix $j \in I$ and define

$$\omega = \sum_{i\in I} \eta_{ij} \otimes P_i \in \Gamma(U, \Omega^1_U) \otimes \operatorname{End} \mathbb{C}^r.$$

This form doesn't depend on the choice of $j$. (It could be written as "$\sum_{i\in I} d \log H_i \otimes P_i$".)

The form $\omega$ defines the connection $d + \omega$ on the trivial bundle $\mathcal{E} := \mathcal{O}^r_U$. We suppose that $(d + \omega)$ *is integrable* which is equivalent to the condition $\omega \wedge \omega = 0$ as $d\omega = 0$. Let $\Omega^\bullet_U(\mathcal{E}) = \Omega^\bullet_U \otimes_{\mathcal{O}_U} \mathcal{E}$ be the de Rham complex with the differential $d + \omega$.

On the other hand, one defines finite dimensional subspaces $A^p \subset \Gamma(U, \Omega^p_U(\mathcal{E})) = \Gamma(U, \Omega^p_U) \otimes_{\mathbb{C}} \mathbb{C}^r$ as the $\mathbb{C}$-linear subspaces generated by all forms $\eta_{i_1 j_1} \wedge \ldots \wedge \eta_{i_p j_p} \otimes v$, $v \in \mathbb{C}^r$. As one obviously has $(d + \omega)A^p = \omega A^p \subset A^{p+1}$, the exterior product by $\omega$ defines

$$A^\bullet \colon 0 \to A^0 \xrightarrow{\omega} A^1 \xrightarrow{\omega} \ldots \xrightarrow{\omega} A^n \to 0$$

as a subcomplex of $\Gamma(U, \Omega^\bullet_U(\mathcal{E}))$.

If $L \subset \mathbb{P}^n$ is a linear subspace, we write $I_L = \{i \in I \mid L \subset H_i\}$. For $0 \leqq j \leqq n - 2$ we define $\mathcal{L}_j$ to be the set of linear subspaces $L \subset \mathbb{P}^n$, $\dim L = j$, such that

**(Bad)** for all $i_0 \in I_L$ one has $L = \bigcap_{i \in I_L - \{i_0\}} H_i$.

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We set $\mathscr{L}_{n-1} = \{H_i\}_{i \in I}$, $\mathscr{L} = \bigcup_{j=0}^{n-1} \mathscr{L}_j$.

In this note we prove the

**Theorem.** *Suppose that*

**(Mon)**: *for all* $L \in \mathscr{L}$, *none of the eigenvalues of* $\sum_{i \in I_L} P_i$ *lies in* $\mathbb{N} - \{0\}$.

*Then the inclusion*

$$A^\bullet \subsetneq \Gamma(U, \Omega_U^\bullet(\mathscr{E}))$$

*is a quasiisomorphism.*

An obvious consequence is the following

**Corollary.** *Under the assumption of the theorem, one has*

$$H^p(U, \mathscr{S}) \cong H^p(A^\bullet) \quad \text{for } 0 \leqq p \leqq n$$

*where* $\mathscr{S}$ *is the local system of flat sections of* $(\mathscr{E}, d + \omega)$ *on* $U$.

The corollary answers positively a conjecture due to Aomoto [A, p. 7]. In fact, there a slightly stronger monodromy condition was required. A partial result was obtained in [S–V, 4.6], by different methods. If all $P_i = 0$ then the result is due to Brieskorn [B, Lemma 5].

The reader will realize that the proof is straightforward using the methods of [E–V] and relies on [D II] and [B].

## 2

**Lemma.** *Let* $\pi : X \to \mathbb{P}^n$ *be any blowing up of* $\mathbb{P}^n$ *with centers in* $H$ *such that* $X$ *is smooth and* $D = \pi^{-1}H$ *is a normal crossing divisor. Then* $H^q(X, \Omega_X^p(\log D)) = 0$ *for* $q > 0$ *and* $H^0(X, \Omega_X^p(\log D))$ *is generated by forms* $\pi^{-1}(\eta_{i_1 j_1} \wedge \ldots \wedge \eta_{i_p j_p})$.

*Proof.* If $(\Omega_X^\bullet(\log D), d)$ denotes the standard logarithmic de Rham complex one knows [D II] that the Hodge–Deligne spectral sequence

$$E_1^{pq} = H^q(X, \Omega_X^p(\log D)) \Rightarrow \mathbb{H}^{p+q}(X, (\Omega_X^\bullet(\log D), d)) = H^{p+q}(U, \mathbb{C})$$

degenerates in $E_1$. On the other hand,

$$\pi^{-1}(\eta_{i_1 j_1} \wedge \ldots \wedge \eta_{i_p j_p}) \in H^0(X, \Omega_X^p(\log D))$$

and by [B, Lemma 5], those forms generate $H^p(U, \mathbb{C})$ (actually even $H^p(U, (2\pi\sqrt{-1})^p\,\mathbb{Z})$). □

*Proof of the theorem.* Using the lemma it is enough to construct a blow up $\pi : X \to \mathbb{P}^n$, as above, such that

$$H^h(U, \mathscr{S}) = \mathbb{H}^h(X, (\Omega_X^\bullet(\log D) \otimes \mathbb{C}^r, d + \omega)) .$$

Let $\mathscr{L}_j$ be the set of linear subspaces considered above. One has the sequence

$$X = X_{n-1} \xrightarrow{\tau_{n-1}} X_{n-2} \xrightarrow{\tau_{n-2}} \cdots \to X_1 \xrightarrow{\tau_1} X_0 = \mathbb{P}^n$$

where $\tau_s : X_s \to X_{s-1}$ is the blow up along the proper transform of $\bigcup_{\mathcal{L}_{s-1}} L$ under $\tau_1 \circ \tau_2 \circ \ldots \circ \tau_{s-1}$.

**Claim.** *$X$ is non singular, $\tau^* H$ is a normal crossing divisor for $\tau = \tau_1 \circ \ldots \circ \tau_{n-1}$ and for all $s$ the proper transform of $\bigcup_{\mathcal{L}_{s-1}} L$ under $\tau_1 \circ \tau_2 \circ \ldots \circ \tau_{s-1}$ is the disjoint union of the smooth proper transforms of the $L \in \mathcal{L}_{s-1}$.*

*Proof.* It is sufficient to consider a neighbourhood of some $p \in \mathbb{P}^n$ and to assume that $p \in L$ for all $L \in \mathcal{L}$.

If $p \in \mathcal{L}_0$ i.e. if $p$ is the center of $\tau_1$, one considers

$$X_1 \subsetneq \mathbb{P}^{n-1} \times \mathbb{P}^n \quad \text{and} \quad pr_1 : X_1 \to \mathbb{P}^{n-1} .$$

Since each $L \in \mathcal{L}$ is a linear subspace of $\mathbb{P}^n$ containing $p$, we have $\tau^* H = pr_1^* H' + E$ where $E$ is the exceptional divisor of $\tau_1$ and $H'$ is a configuration of hyperplanes in $\mathbb{P}^{n-1}$. By induction on $n$ we can assume that the blow up $Y_{n-2}$ of $\mathbb{P}^{n-1}$ given by the same construction for $H'$ is nonsingular and the pullback of $H'$ is a normal crossing divisor. As $X_{n-1} = Y_{n-1} \times_{\mathbb{P}^{n-1}} X_1$ and $E$ is a cross section of $pr_1 : X_1 \to \mathbb{P}^{n-1}$, the pullback of $E$ to $X_{n-1}$ and the pullback of $H'$ to $X_{n-1}$ intersect transversally,

If $p \notin \mathcal{L}$, i.e. if $\tau_1$ is an isomorphism we can find a neighbourhood $Z$ of $p$ and a smooth morphism $\sigma : Z \to \mathbb{P}^{n-1}$ such that $H|z$ is the union of the pullback of some configuration $H'$ of hyperplanes in $\mathbb{P}^{n-1}$ and a divisor $H''$ in $Z$ which has relative normal crossings over $\mathbb{P}^{n-1}$. In fact, consider the smallest dimension $d_0$ with $\mathcal{L}_{d_0} \neq \emptyset$. As $\bigcup_{j=0}^{n-2} \mathcal{L}_j$ is stable under intersection, $\mathcal{L}_{d_0}$ contains just one element, say $L'$, and for $j > d_0$ each $L \in \mathcal{L}_j$ contains $L'$. (We still assume that $p$ lies on all $L \in \mathcal{L}$.) Hence, to obtain $\sigma : Z \to \mathbb{P}^{n-1}$ we can take the blow up of some point $q \in L'$, $q \neq p$, and take $Z$ to be the complement of the exceptional divisor.

Then we argue by induction on $n$. □

Let $E$ be an irreducible component of $\tau^* H$. If $E$ is the proper transform of some $H_i$, we assumed in (**Mon**) that none of the eigenvalues of the residues of $(d + \omega)$ along $E$ lies in $\mathbb{N} - \{0\}$. If $E$ is exceptional, $E$ is obtained by blowing up the proper transform of some $L$ in $\mathcal{L}_{s-1}$ in $X_{s-1}$. Let $\varphi_i$ be a local parameter of $H_i$ in a general point of $L$ for $i \in I_L$. Then $\varphi_i \circ \pi = t \cdot \psi_i$ where $\psi_i$ is a local equation of the proper transform of $H_i$ and $t$ is a local equation of $E$. The connection on $X$ is locally given by

$$d + \sum_{i \in I_L} P_i \frac{d\psi_i}{\psi_i} + \left( \sum_{i \in I_L} P_i \right) \frac{dt}{t} .$$

Hence the residue of the connection along $E$ is $\sum_{i \in I_L} P_i$. By assumption (**Mon**) none of the eigenvalues of $\sum_{i \in I_L} P_i$ lies in $\mathbb{N} - \{0\}$.

By Deligne [D, 3.13] one knows that $(\Omega_X^\bullet(\log D) \otimes \mathbb{C}^r, d + \omega)$ is quasiisomorphic to $(Rj_*(\Omega_U^\bullet \otimes \mathbb{C}^r), d + \omega)$ for $j : U \to X$. The theorem is proved. □

## 3

In this section we give a condition equivalent to (**Bad**) of $n^0 1$.

As in the proof of the theorem, for each linear subspace $L$ of $\mathbb{P}^n$ we can consider the blowup $X \overset{\tau}{\to} \mathbb{P}^n$ of $L$ and the projection $p: X \to \mathbb{P}^r$ where $r+1 = \operatorname{codim} L = n - \dim L$. The points of $\mathbb{P}_L := \mathbb{P}^r$ parametrize the $n - r = \dim L + 1$ dimensional linear subspaces of $\mathbb{P}^n$ which contain $L$. In particular, if $H_i$ is a hyperplane containing $L$ we find a hyperplane $H'_i$ in $\mathbb{P}^r$ whose inverse image $p^{-1}H'_i$ is the proper transform of $H_i$ under $\tau$.

**Proposition.** *Let $L$ be a $j$-dimensional linear subspace of $\mathbb{P}^n$, $j \leqq n-2$, with $L = \bigcap_{i \in I_L} H_i$. Let $r = n - j - 1$ and $p: X \to \mathbb{P}_L$ as above. Then the following conditions are equivalent*:

a) $L \in \mathscr{L}_j$.

b) *For each $i_0 \in I_L$ the arrangement $\bigcup_{i \in I_L - \{i_0\}} H'_i$ of hyperplanes in $\mathbb{P}_L$ contains a coordinate symplex.*

c) $\chi\left(\mathbb{P}_L - \bigcup_{i \in I_L} H'_i\right) \neq 0$ *where $\chi$ denotes the Euler characteristic.*

d) $\chi\left(\mathbb{P}_L - \bigcup_{i \in I_L} H'_i\right) \cdot (-1)^r > 0$.

*Proof.* Obviously $\bigcup_{i \in I_L - \{i_0\}} H'_i$ contains a coordinate symplex if and only if $\bigcap_{i \in I_L - \{i_0\}} H'_i = \varnothing$.

Since $\bigcap_{i \in I_L - \{i_0\}} H_i = L \cup \tau\left(p^{-1}\left(\bigcap_{i \in I_L - \{i_0\}} H'_i\right)\right)$ the conditions a) and b) are equivalent.

Assume that c) holds true. Let us prove b).

Assume that for some $i_0$, $\bigcup_{i \in I_L - \{i_0\}} H'_i$ does not contain a coordinate symplex or equivalently that $\bigcap_{i \in I_L - \{i_0\}} H'_i$ contains a point $p$. We can choose $p$ to be the zero-point of $\mathbb{A}^r = \mathbb{P}^r - H'_{i_0}$ and for $i \in I_L - \{i_0\}$, $H'_i|_{\mathbb{A}^r}$ is a linear subspace of $\mathbb{C}^r \simeq \mathbb{A}^r$. If $\mathbb{C}^r - \{0\} \overset{\sigma}{\to} \mathbb{P}^{r-1}$ is the natural map,

$$U = \mathbb{C}^r - \bigcup_{i \in I_L - \{i_0\}} H'_i|_{\mathbb{C}^r} = \mathbb{P}^r - \bigcup_{i \in I_L} H'_i$$

is a trivial $\mathbb{C}^*$-bundle over an open subset of $\mathbb{P}^{r-1}$ and $\chi(U) = 0$.

Evidently, d) $\Rightarrow$ c). To see that b) implies d), let us fix some $i_0 \in I_L$. For $i_1 \neq i_0$ we can find a coordinate symplex

$$H_{j_0}, \ldots, H_{j_r} \quad \text{with } j_0 \ldots j_r \in I_L - \{i_1\}.$$

Obviously, one of the $H_{j_\nu}$ can be replaced by $i_0$ and we may assume that $j_r = i_0$. In other terms, the arrangement

$$H''_i = H'_i|_{H'_{i_0}}, \quad i \in I_L - \{i_0\} \text{ of } \mathbb{P}^{r-1} \simeq H'_{i_0}$$

again satisfies the assumption b) and by induction we may assume that

$$\chi\left(H'_{i_0} - \bigcup_{i \in I_L - \{i_0\}} H'_i \cap H'_{i_0}\right) \cdot (-1)^{r-1} > 0 .$$

Since the Euler characteristic is additive we have

$$\chi\left(\mathbb{P}^r - \bigcup_{i \in I_L - \{i_0\}} H'_i\right) = \chi\left(H'_{i_0} - \bigcup_{i \in I_L - \{i_0\}} H'_i \cap H'_{i_0}\right) + \chi\left(\mathbb{P}^r - \bigcup_{i \in I_L} H'_i\right) .$$

If the arrangement $H'_i$, $i \in I_L - \{i_0\}$ does not satisfy the assumption b), the left hand side of the equation is zero and d) holds true. Otherwise, by induction on the number of hyperplanes we may assume that the left hand side lies in $(-1)^r \cdot (\mathbb{N} - \{0\})$ and again one obtains d). □

Thus, the monodromy condition (**Mon**) should be verified only for those subspaces $L$ for which $e(L) := \chi\left(\mathbb{P}^n_L - \bigcup_{i \in I_L} H'_i\right) \neq 0$. This should be compared with [S–V], Theorem 3.7.

## 4

When applied to the special arrangements studied in [S–V, part I], the theorem provides the subset $A \subset \mathbb{C}^*$ which is a union of a finite set of arithmetic progressions such that for all $\kappa^{-1} \notin A$ (where $\kappa$ is as in loc. cit.), the construction of [loc. cit. 7.2.8] gives the *complete set of solutions* of Knizhnik–Zamolodchikov equations.

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