Ample sheaves on moduli schemes

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In this note we take up methods from [3] and [14], III, to give some criteria for certain direct image sheaves to be ample. To give a flavour of the result obtained let us state:

Theorem 0.1 *Let $f : X \to Y$ be a flat projective Gorenstein morphism of complex reduced schemes whose fibres are irreducible normal varieties with at worst rational singularities. Assume that for some $N > 0$ the map $f^* f_* \omega^N_{X/Y} \to \omega^N_{X/Y}$ is surjective and that for some $\mu > 0$ the sheaf $det(f_*\omega^\mu_{X/Y})$ is ample on Y. Then for all $\eta \geq 2$ the sheaf $f_*\omega^\eta_{X/Y}$ is ample, whenever it is non zero.*

Similar statements, replacing "ample" by "maximal Kodaira-dimension" and allowing degenerate fibres, played some role in "Iitaka's program" (see [1] or [9] and the references given there).

The proof of (0.1) is more or less parallel to the proof of a similar result in [3], where we assumed Y to be a curve and allowed degenerate fibres. Therefore it is not surprising that (0.1) is effective again, i.e. that one can measure the ampleness of $f_*\omega^\eta_{X/Y}$ with $det(f_*\omega^\mu_{X/Y})$ and invariants of the fibres.

The interest - if ever - in results like (0.1) comes from the theory of moduli spaces. Let $\mathcal{C}_h$ denote the functor of families of compact complex canonically polarized manifolds with Hilbert-polynomial h and C_h the coarse moduli scheme. If $\nu > 0$ is choosen such that ω^ν_X is very ample for all $X \in \mathcal{C}_h(Spec(\mathbb{C}))$, and if $\lambda_\eta \in Pic(C_h) \otimes \mathbb{Q}$ denotes the "sheaf" corresponding to $det(f_*\omega^\eta_{X/Y})$ for $f : X \to Y \in \mathcal{C}_h(Y)$, then the ample sheaf on C_h obtained by the second author in [14], II was of the form $\lambda^a_\nu \otimes \lambda^b_{\mu\cdot\nu}$ for $\mu, a, b >> 0$. Adding (0.1) to the methods employed in [14], II, we will show in §4 that in fact λ_ν itself is ample.

A quite similar improvement of the description of ample sheaves on moduli spaces M_h will be obtained in §5 for the moduli functor $\mathcal{M}_h$ of pairs $(f : X \to Y, \mathcal{H})$, where f is smooth, $\omega_{X/Y}$ numerically effective along the fibres and $\mathcal{H}$ a polarization with Hilbert polynomial h, up to isomorphism. As we explained in [14], III, §1, this is not the moduli functor

$\mathcal{P}_h$ of polarized manifolds usually considered, at least if one allows the irregularity of the manifolds to be positive. $\mathcal{P}_h$ is a quotient of $\mathcal{M}_h$. In [16] the second author constructed coarse quasi-projective moduli spaces P_h for $\mathcal{P}_h$. As explained in (5.11) the natural morphism $M_h \to P_h$ is finite, if for all $(F, \mathcal{H}) \in \mathcal{M}_h(Spec(\mathbb{C}))$ one knows that $\omega_F^\delta = \mathcal{O}_F$ for some δ. Under this assumption (5.10) implies:

The coarse moduli space P_h of polarised manifolds F with $\omega_F^\delta = \mathcal{O}_F$ exists as a quasi-projective scheme. If $\gamma_\delta \in Pic(P_h) \otimes \mathbb{Q}$ denotes the "sheaf" corresponding to $f_*\omega_{X/Y}^\delta$ for

$$f : X \to Y \in \mathcal{P}_h(Y),$$

then γ_δ is ample.

The same result, for moduli of K3 surfaces, is due to Pjatetskij-Šapiro and Šafarevich [12].

The final version of [16] will contain a discussion of ample sheaves on P_h in the general case, building up on the results explained here.

This note does not really contain any substantially new ideas. We use the proof of (0.1) and of the ampleness of λ_ν to recall and clarify some of methods from [3] and [14], III.

In §1 we consider the relation between weak positivity and ampleness. §2 contains a discussion of the invariant $e(\Gamma)$ introduced in [3] to measure the singularities of a divisor Γ on some manifold X. As sketched in [14], III, we extend the properties of $e(\Gamma)$ to varieties X with rational Gorenstein singularities. In fact, the arguments in [14], III, were a little bit too sketchy at this point and one statement has to be corrected (see 2.12).

In §3 we prove (0.1) and some similar result needed for the polarized case. We include a short discussion about the "effectivity" of (0.1).

After the discussion of ample sheaves on the moduli spaces C_h and M_h in §4 and §5 we will sketch in §6 some generalizations of (0.1) to fibre spaces with degenerate fibres. However, the arguments in §6 should be considered as a guide line to the possible proofs, far from being complete.

We keep the conventions from [14], II. Especially one should have in mind, that all schemes are supposed to be seperated and of finite type over $\mathbb{C}$, that points should be $\mathbb{C}$-valued and that locally free sheaves should be of finite rank, which is the same for different components of the base.

1 Weak positivity and ampleness

Whereas in [14] we had to use the notation "weakly positive over some open set and with respect to a desingularization of a compactification", we will get along in this note with a much simpler set up, related to ampleness. Using this, the properties of weakly positive sheaves needed in the sequel can be deduced in a quite simple way.

Let Y be a reduced quasi-projective scheme, $\mathcal{H}$ be an ample invertible sheaf and $\mathcal{F}$ be a locally free sheaf on Y.

Definition 1.1 $\mathcal{F}$ is called *weakly positive over* Y if for all $a > 0$ one can find some $b > 0$ such that the natural map $H^\circ(Y, S^{a\cdot b}(\mathcal{F}) \otimes \mathcal{H}^b) \otimes_{\mathbb{C}} \mathcal{O}_Y \to S^{a\cdot b}(\mathcal{F}) \otimes \mathcal{H}^b$ is surjective.

If $\mathcal{F}$ is invertible and Y is compact then "weakly positive over Y" is equivalent to "numerically effective". More generally we have:

Lemma 1.2 *$\mathcal{F}$ is weakly positive over Y if and only if for all $\eta > 0$ the sheaf $S^\eta(\mathcal{F}) \otimes \mathcal{H}$ is ample.*

Before proving (1.2) let us recall some simple properties of ample sheaves.

Lemma 1.3 *The following conditions are equivalent:*
a) $\mathcal{F}$ is ample.
b) For some $\eta > 0$ the sheaf $S^\eta(\mathcal{F}) \otimes \mathcal{H}^{-1}$ is generated by global sections.
c) For some $\eta > 0$ the sheaf $S^\eta(\mathcal{F}) \otimes \mathcal{H}^{-1}$ is weakly positive over Y.

Proof. The equivalence of a) and b) is shown in [5], 2.5, and obviously b) implies c). If c) holds true, then

$$S^{2\cdot b} S^\eta(\mathcal{F}) \otimes \mathcal{H}^{-2\cdot b + b}$$

is generated by global sections, as well as the quotient sheaf

$$S^{2\cdot b\cdot\eta}(\mathcal{F}) \otimes \mathcal{H}^{-b}.$$

Hence $S^{2\cdot b\cdot\eta}(\mathcal{F})$ is ample as a quotient of an ample sheaf and by [5], 2.4, we are done.

The last condition in (1.3) motivates the following definition, which will be used in §3.

Definition 1.4 Let $\mathcal{F}$ and $\mathcal{A}$ be locally free sheaves on Y, $\mathcal{A}$ of rank 1. We write $\mathcal{F} \succeq \frac{b}{\eta} \cdot \mathcal{A}$ if $S^\eta(\mathcal{F}) \otimes \mathcal{A}^{-b}$ is weakly positive over Y.

If $\mathcal{A}$ is ample in (1.4), then the statement $\mathcal{F} \succeq \frac{1}{\eta} \cdot \mathcal{A}$ implies that $\mathcal{F}$ is ample and measures "how ample" $\mathcal{F}$ is compared to $\mathcal{A}$.

Lemma 1.5 *Let $\tau : Y' \to Y$ be a finite morphism such that $\mathcal{O}_Y \to \tau_* \mathcal{O}_{Y'}$ splits. Then $\mathcal{F}$ is ample on Y if and only if $\tau^* \mathcal{F}$ is ample on Y'.*

Proof. It follows directly from the definition of ampleness that $\tau^*\mathcal{F}$ is ample if $\mathcal{F}$ is ample.

Assume that $\tau^*\mathcal{F}$ is ample. Let us choose b such that $\tau_*\mathcal{O}_{Y'} \otimes \mathcal{H}^b$ is generated by global sections. For some $\eta >> 0$ the sheaf $S^\eta(\tau^*\mathcal{F}) \otimes \tau^*\mathcal{H}^{-b-1}$ is generated by global sections. Since τ is finite we have surjections

$$\bigoplus \tau_*\mathcal{O}_{Y'} \to \tau_*\mathcal{O}_{Y'} \otimes S^\eta(\mathcal{F}) \otimes \mathcal{H}^{-b-1} \to S^\eta(\mathcal{F}) \otimes \mathcal{H}^{-b-1}$$

and

$$\bigoplus(\tau_*\mathcal{O}_{Y'}) \otimes \mathcal{H}^b \to S^\eta(\mathcal{F}) \otimes \mathcal{H}^{-1}.$$

By the choice of b and by (1.3), $\mathcal{F}$ must be ample.

Of course the assumption made for τ in (1.5) holds true if Y is normal and τ is finite. Other examples frequently used here are:

Lemma 1.6 *Let $\mathcal{L}$ be an invertible sheaf on Y.*
a) If for some $N > 0$ and some effective divisor D one has $\mathcal{L}^N = \mathcal{O}_Y(D)$, then one can find some scheme Y', a Cartier divisor D' on Y' and a finite flat morphism $\tau : Y' \to Y$ such that $\tau^*D = N \cdot D'$ and such that $\mathcal{O}_Y \to \tau_*\mathcal{O}_{Y'}$ splits.
b) For all $N > 0$ one can find a finite flat morphsim $\tau : Y' \to Y$ of schemes and an invertible sheaf $\mathcal{L}'$ on Y' such that $\tau^*\mathcal{L} = \mathcal{L}'^N$ and such that $\mathcal{O}_Y \to \tau_*\mathcal{O}_{Y'}$ splits.

Proof. In a) we may take

$$Y' = Spec(\bigoplus_{i \geq 0} \mathcal{L}^{-i} / (\mathcal{L}^{-N} \hookrightarrow \mathcal{O}_Y)).$$

Since $\mathcal{L} = \mathcal{O}_Y(B - C)$ for effective divisors B and C it is enough to consider b) for $\mathcal{L} = \mathcal{O}_Y(B)$. For $\mathcal{H}$ ample and $\mu >> 0$, $\mathcal{H}^{\mu \cdot N}(-B)$ is generated by global sections. If H is the zero set of a general section we can apply a) to $(\mathcal{H}^\mu)^N = \mathcal{O}_Y(H + B)$.

Proof of 1.2 Let $\mathcal{F}$ be weakly positive and $\eta > 0$. We find some $\tau : Y' \to Y$, by (1.6,b), satisfying the assumption made in (1.5) with $\tau^*\mathcal{H} = \mathcal{H}'^\eta$ for some ample sheaf $\mathcal{H}'$. By definition weak positivity is compatible with pullback and $S^{2\cdot b}(\tau^*\mathcal{F}) \otimes \mathcal{H}'^b$ will be generated by global sections for some $b > 0$. Then $S^{2\cdot b}(\tau^*\mathcal{F}) \otimes \mathcal{H}'^{2\cdot b}$ is ample as well as $\tau^*\mathcal{F} \otimes \mathcal{H}'$ and $S^\eta(\tau^*\mathcal{F}) \otimes \tau^*\mathcal{H}$. By (1.5) we are done.

On the other hand, if $S^\eta(\mathcal{F}) \otimes \mathcal{H}$ is ample we can find some $b > 0$ such that $S^b S^\eta(\mathcal{F}) \otimes \mathcal{H}^b$ is globally generated as well as its quotient $S^{b\cdot\eta}(\mathcal{F}) \otimes \mathcal{H}^b$.

The close connection between ample and weakly positive sheaves can also be expressed using coverings.

Lemma 1.7 *The following conditions are equivalent:*
a) $\mathcal{F}$ is weakly positive over Y.
b) There exists some finite morphism $\tau : Y' \to Y$ such that $\tau^*\mathcal{F}$ is weakly positive over

Y' and such that $\mathcal{O}_Y \to \tau_*\mathcal{O}_{Y'}$ splits.
c) There exists some $\mu > 0$ such that for all flat finite morphisms $\tau : Y' \to Y$ and all ample sheaves $\mathcal{A}'$ on Y' the sheaf $\tau^*\mathcal{F} \otimes \mathcal{A}'^{\mu}$ is weakly positive over Y'.
d) There exists some $\mu > 0$ such that for all flat finite morphisms $\tau : Y' \to Y$ and all ample sheaves $\mathcal{A}'$ on Y' the sheaf $\tau^*\mathcal{F} \otimes \mathcal{A}'^{\mu}$ is ample.

Proof. The equivalence of a) and b) follows from (1.2) and (1.5). Obviously a) implies d) and d) implies c) for given $\mu > 0$. Assume that c) holds true and let $\eta > 0$ be given. Let $\tau : Y' \to Y$ be the finite morphism constructed in (1.6, b) with $\tau^*\mathcal{H} = \mathcal{H}'^{\eta\cdot\mu+\eta}$. Then $\tau^*\mathcal{F} \otimes \mathcal{H}'^{(\mu+1)}$ as well as $S^\eta(\tau^*\mathcal{F}) \otimes \mathcal{H}'^{\eta\cdot(\mu+1)}$ is ample and (1.5) implies the ampleness of $S^\eta(\mathcal{F}) \otimes \mathcal{H}$.

(1.7) allows to carry over properties of ample sheaves to weakly positive sheaves. Translating the corresponding statements in [5] one obtains as in [14], III, 2.4.:

Lemma 1.8 *a) Let $\mathcal{F}$ and $\mathcal{G}$ be locally free and weakly positive over Y. Then $\mathcal{F} \otimes \mathcal{G}$ and $\mathcal{F} \oplus \mathcal{G}$ are weakly positive over Y.*
b) Let $\mathcal{F}$ be locally free. Then $\mathcal{F}$ is weakly positive over Y, if and only if for some $r > 0$, $\otimes^r\mathcal{F}$ is weakly positive over Y.
c) Positive tensor bundles of sheaves, weakly positive over Y are weakly positive over Y.

The more general notation of weakly positive sheaves one has to use in order to prove the existence of quasi-projective moduli schemes will only appear in §6:

Definition 1.9 Let Y be a reduced scheme, $j : Y_\circ \to Y$ be an open dense subscheme and $\sigma : Y' \to Y$ be a desingularization. Let $\mathcal{H}$ be ample and invertible on Y, let $\mathcal{F}$ be a coherent sheaf on Y and $\mathcal{F}'$ be a coherent sheaf on Y'. Then we call $\mathcal{F}_0 = j^*\mathcal{F}$ *weakly positive over Y_0 with respect to* $(Y', \mathcal{F})$ if the hollowing holds true:
i) $\mathcal{F}_0$ is locally free
ii) For all $\nu > 0$ one has inclusions

$$S^\nu(\mathcal{F}_0) \to j^*\delta_* S^\nu(\mathcal{F}')$$

iii) For all $a > 0$ one finds some $b > 0$ such that the natural map

$$V_{a,b} \otimes_{\mathbb{C}} \mathcal{O}_{Y_0} \to S^{a\cdot b}(\mathcal{F}_0) \otimes j^*\mathcal{H}^b$$

is surjective where

$$V_{a,b} = H^0(Y, \delta_* S^{a\cdot b}(\mathcal{F}') \otimes \mathcal{H}^b) \cap H^0(Y_0, S^{a\cdot b}(\mathcal{F}_0) \otimes j^*\mathcal{H}^b).$$

Here the reader should keep in mind that $S^\eta(\mathcal{F}') := i_* S^\eta(i^*\mathcal{F}')$ and $det(\mathcal{F}') := i_* det(i^*\mathcal{F}')$ where $i : U' \to Y'$ is the largest open subscheme such that $i^*\mathcal{F}'$ is locally free.

In fact, (1.9) does not really coincide with the definition used in [14], II and III, since there we assumed Y to be compact. Nevertheless, the properties stated in [14], II, 2.4 and III, 2.4 which are similar to (1.7) and (1.8) remain true for "weakly positive with respect to". Again there is a close connection to ampleness:

Lemma 1.10 *Keeping the notations from (1.9) assume that both, $\mathcal{F}_0$ and $\mathcal{F}'$ are invertible. Then the following two conditions are equivalent:*
a) For some $\eta > 0$ the sheaf $\mathcal{F}_0^\eta \otimes j^*\mathcal{H}^{-1}$ is weakly positive with respect to

$$(Y', \mathcal{F}'^\eta \otimes \delta^*\mathcal{H}^{-1}).$$

b) The condition ii) in (1.9) holds true, and for $\mu >> 0$ the map

$$V_\mu \otimes_{\mathbb{C}} \mathcal{O}_{Y_0} \to \mathcal{F}_0^\mu$$

is surjective, where

$$V_\mu = H^0(Y, \delta_* \mathcal{F}'^\mu) \cap H^0(Y_0, \mathcal{F}_0^\mu),$$

and the induced morphism $\phi_0 : Y_0 \to \mathbb{P}(V_\mu)$ is an embedding.

Proof. Assume that a) holds true. Choosing $a = 2$ in (1.9) we find that

$$\mathcal{F}_0^{2\cdot\eta\cdot\beta} \otimes j^*\mathcal{H}^{-\beta}$$

is generated by sections which lie in $H^0(Y, \delta_*\mathcal{F}'^{2\cdot\eta\cdot\beta} \otimes \mathcal{H}^{-\beta})$ for some $\beta > 0$. We obtain a sujection

$$\oplus j^*\mathcal{H}^\beta \to \mathcal{F}_0^{2\cdot\eta\cdot\beta}.$$

For $\beta >> 0$, $j^*\mathcal{H}^\beta$ is globally generated by sections of $H^0(Y, \mathcal{H}^\beta)$ and those sections seperate points and tangent directions. Hence we obtain b).

Let us assume b). By [14], II, 2.4,a), we are allowed to blow up Y as long as the centers do not meet Y_0. Hence we can assume that ϕ_0 extends to a morphism $\phi : Y \to \mathbb{P}(V_\mu)$. Choosing μ big enough and replacing $\mathcal{F}'$ by some smaller invertible sheaf we can moreover assume that ϕ is an embedding and that $\mathcal{F}' = \delta^*\phi^*\mathcal{O}_{\mathbf{P}(V_\mu)}(1)$. Since a) is independent of the choice of $\mathcal{H}$ we can take $\mathcal{H} = \phi^*\mathcal{O}_{\mathbf{P}(V_\mu)}(1)$ and a) is trivial.

By (1.10) it makes sense to define

Definition 1.11 Let $\mathcal{F}, \mathcal{F}', Y, Y'$ and Y_0 be as in (1.9). Then we call $\mathcal{F}_0$ *ample with respect to* $(Y', \mathcal{F}')$ if condition (1.9, ii) holds true and if for some $\eta > 0$ and some ample invertible sheaf $\mathcal{H}$ on Y the sheaf $S^\eta(\mathcal{F}_0) \otimes j^*\mathcal{H}^{-1}$ is weakly positive with respect to $(Y', S^\eta(\mathcal{F}') \otimes \delta^*\mathcal{H}^{-1})$.

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2 Singularities of divisors

In this section we recall and clarify some properties of the invariant e introduced in [3], §2, in order to measure singularities of divisors. At the end of this section we correct a mistake from [14], III.

Definition 2.1 Let V be a normal Gorenstein variety with at worst rational singularities, let $\mathcal{M}$ be an invertible sheaf on V and Γ be an effective divisor with $\mathcal{M} = \mathcal{O}_Y(\Gamma)$. We define:
a) $\mathcal{C}(\Gamma, e) = coker(\tau_* \omega_{V'}(-[\frac{\Gamma'}{e}]) \to \omega_V)$ where $\tau : V' \to V$ is a desingularization of V such that $\Gamma' = \tau^*\Gamma$ is a normal crossing divisor.
b) $e(\Gamma) = min\{e \in \mathbb{N} - \{0\}; \mathcal{C}(\Gamma, e) = 0\}$.
c) $e(\mathcal{M}) = sup\{e(\Gamma); \Gamma$ zero divisor of $s \in H^0(V, \mathcal{M})\}$.

By [13], 2.3, the cokernel $\mathcal{C}(\Gamma, e)$ does not depend on the desingularization $\tau : V' \to V$ choosen.

Assumptions 2.2 Throughout this section $f : X \to Y$ will be a flat Gorenstein morphism whose fibres $X_p := f^{-1}(p)$ are all reduced and normal with at most rational singularities. Γ denotes an effective Cartier divisor on X.

Proposition 2.3 *Assume in addition that Y is smooth, that X_p is not contained in Γ and let Δ be an effective normal crossing divisor on Y. Let $\tau : X' \to X$ be a desingularization such that $\Gamma' = \tau^*\Gamma$ as well as $\Delta' = \tau^* f^* \Delta$ and $\Gamma' + \Delta'$ are normal crossing divisors. Then X_p has a Zarisky open neighbourhood U such that*

$$\tau_* \omega_{X'}(-[\frac{\Gamma' + \Delta'}{e}]) \to \omega_X(-f^*[\frac{\Delta}{e}])$$

is surjective over U for $e \geq e(\Gamma|_{X_p})$.

Proof. We may assume that $[\frac{\Delta}{e}] = 0$. In fact, if $\Delta = \Delta_1 + e \cdot \Delta_2$ for effective divisors Δ_1 and Δ_2, then

$$f^*[\frac{\Delta}{e}] = f^*[\frac{\Delta_1}{e}] + f^*\Delta_2$$

and $\quad \frac{\Gamma'+\Delta'}{e}] = [\frac{\Gamma'+\tau^*f^*\Delta_1}{e} + \tau^* f^* \Delta_2] = [\frac{\Gamma'+\tau^*f^*\Delta_1}{e}] + \tau^* f^* \Delta_2.$

By projection formula we can replace Δ by Δ_1.

Let D be a smooth Cartier divisor containing p. If $p \in \Delta$, we choose D to be a component of Δ and we write α for the multiplicity of D in Δ. Then

$$f|_H : H := f^{-1}(D) \to D$$

fulfills again the assumptions made in (2.2). We can assume that the proper transform H' of H under τ is non singular and that H' intersects $\Gamma' + \Delta''$ transversally for

$$\Delta'' = \Delta' - \tau^* f^* \alpha \cdot D = \Delta' - \alpha \cdot \tau^* H.$$

By induction on $dim(Y)$ we may assume that

$$\tau_* \omega_{H'}(-[\frac{(\Gamma' + \Delta'')|_{H'}}{e}]) \to \omega_H(-f^*[\frac{(\Delta - \alpha \cdot D)|_D}{e}]) = \omega_H$$

is surjective over $W \cap H$ for some open neighbourhood W of X_p in X.

We have $0 \leq \alpha < e$ and

$$[\frac{\Gamma' + \Delta'}{e}] \leq [\frac{\Gamma' + \Delta''}{e}] + [\frac{\alpha \cdot \tau^* H}{e}] + (\tau^* H - H')_{red} \leq [\frac{\Gamma' + \Delta''}{e}] + (\tau^* H - H').$$

Therefore there is an inclusion

$$\omega_{X'}(-[\frac{\Gamma' + \Delta''}{e}] + H') \to \omega_{X'}(-[\frac{\Gamma' + \Delta'}{e}] + \tau^* H).$$

As in [3], 2.3, we consider the commutative diagram

$$\begin{array}{ccccc}
\tau_*\omega_{X'}(-[\frac{\Gamma'+\Delta''}{e}] + H') & \xrightarrow{\alpha} & \tau_*\omega_{H'}(-[\frac{(\Gamma'+\Delta'')|_{H'}}{e}]) & \xrightarrow{\beta} & \omega_H \\
\downarrow & & & & \| \\
\tau_*\omega_{X'}(-[\frac{\Gamma'+\Delta'}{e}]) \otimes \mathcal{O}_X(H) & \xrightarrow{\gamma} & \omega_X(H) & \longrightarrow & \omega_H
\end{array}$$

By [13], 2.3, α is surjective and hence $\beta \circ \alpha$ is surjective over $H \cap W$. Therefore we can find a neighbourhood U of X_p in W such that γ is surjective over U.

Corollary 2.4 *Keeping the notations from (2.2) we assume in addition that Y is a normal Gorenstein variety with at worst rational singularities and that X_p is not contained in Γ. Then X_p has a neighbourhood U with*

$$e(\Gamma|_U) \leq e(\Gamma|_{X_p}).$$

Proof. If Y is non singular this is nothing but (2.3) for $\Delta = 0$. In general let $\delta : Y' \to Y$ be a desingularization and

$$\begin{array}{ccc}
X' & \xrightarrow{\delta'} & X \\
{\scriptstyle f'}\downarrow & & \downarrow \\
Y' & \xrightarrow{\delta} & Y
\end{array}$$

be the fibre product. (2.3) applied to f' and all $p' \in \delta^{-1}(p)$ gives the existence of an open neighbourhood U' of $f'^{-1}\delta^{-1}(p) = \delta'^{-1}(X_p)$ with

$$e(\delta'^*\Gamma|_{U'}) \leq e(\delta'^*\Gamma|_{X'_{p'}}) = e(\Gamma|_{X_p}) = e.$$

Of course we can choose $U' = \delta'^{-1}(U)$ for an open neighbourhood U of X_p. If $\tau : X'' \to X'$ is a desingularization and $\Gamma'' = \tau^*\delta'^*\Gamma$ a normal crossing divisor then

$$\tau_*\omega_{X''}(-[\frac{\Gamma''}{e}]) \to \omega_{X'}$$

is an isomorphism over U' and

$$\delta'_* \tau_* \omega_{X''}(-[\frac{\Gamma''}{e}]) \to \delta'_* \omega_{X'}$$

is an isomorphism over U. By flat base change and projection formula we have

$$\delta'_* \omega_{X'} = \omega_{X/Y} \otimes \delta'_* f'^* \omega_{Y'} = \omega_{X/Y} \otimes f^* \delta_* \omega_{Y'} = \omega_X.$$

Proposition 2.5 *In addition to (2.2) assume that f is projective with connected fibres and that $e \geq e(\mathcal{O}_X(\Gamma)|_{X_p})$. If Y is a normal Gorenstein variety with at worst rational singularities and if X_p is not contained in the support of $\mathcal{C}(\Gamma, e)$, then there exists an open neighbourhood U of p in Y such that $\mathcal{C}(\Gamma|_{f^{-1}(U)}, e) = 0$.*

Proof. If Γ does not contain X_p, then (2.5) follows from (2.4). In general we have

Claim 2.6 There exist a desingularization $\delta : Y' \to Y$ and an effective normal crossing divisor Δ on Y' with: let

$$\begin{array}{ccc} X' & \xrightarrow{\delta'} & X \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\ Y' & \xrightarrow{\delta} & Y \end{array}$$

be the fibre product and $\Gamma' = \delta'^* \Gamma - f'^* \Delta$. Then Γ' is an effective divisor which does not contain any fibre $X'_{p'} = f'^{-1}(p')$.

Proof. Of course this follows from the "flattening" of Hironaka. However in this simple situation one can as well argue in the following way:

In order to prove (2.6) we can replace Γ by $\Gamma + H$ for an ample divisor H. Hence we may assume that $\mathcal{M} = \mathcal{O}_X(\Gamma)$ has no higher cohomology on the fibres and hence that $f_* \mathcal{M}$ is locally free and compatible with base change. If $s : \mathcal{O}_Y \to f_* \mathcal{M}$ is the direct image of the section of $\mathcal{M}$ whose zero divisor is Γ, then we just have to choose $\delta : Y' \to Y$ such that the zero locus of $\delta^*(s)$ becomes a normal crossing divisor Δ. In particular

$$\mathcal{O}_{Y'}(\Delta) \to \delta^* f_* \mathcal{M} = f'_* \delta'^* \mathcal{M}$$

splits locally and we get (2.6).

Since $\mathcal{O}_{X'}(\Gamma')|_{X'_{p'}} \simeq \mathcal{O}_X(\Gamma)|_{X_p}$, for all $p' \in \delta^{-1}(p)$, we have $e \geq e(\Gamma'|_{X'_{p'}})$. Let us choose a desingularization $\tau : X'' \to X'$ such that $\Gamma'' + \Delta''$ is a normal crossing divisor for $\Gamma'' = \tau^* \Gamma'$ and $\Delta'' = \tau^* f'^* \Delta$. By (2.3) there is a neighbourhood W' of $f'^{-1} \delta^{-1}(p)$ such that

$$\tau_* \omega_{X''}(-[\frac{\Gamma'' + \Delta''}{e}]) \to \omega_{X'}(-f'^*[\frac{\Delta}{e}])$$

is an isomorphism over W'. Since τ is proper one can take $W' = \delta'^*(W)$ for some neighbourhood W of X_p. Hence the cokernel of

$$\delta'_* \tau_* \omega_{X''}(-[\frac{\Gamma'' + \Delta''}{e}]) \to \omega_X$$

is over W isomorphic to

$$\mathcal{C} = coker(\delta'_* \omega_{X'}(-f'^*[\frac{\Delta}{e}]) \to \omega_X).$$

By flat base change

$$\delta'^* \omega_{X'}(-f'^*[\frac{\Delta}{e}]) = \omega_{X/Y} \otimes f^* \delta_* \omega_{Y'}(-[\frac{\Delta}{e}])$$

and $$\mathcal{C} = f^*(coker(\delta_* \omega_{Y'}(-[\tfrac{\Delta}{e}]) \to \omega_Y)).$$

Since we assumed that X_p does not lie in the support of $\mathcal{C}(\Gamma, e)$, for sufficiently small W around X_p one has

$$\mathcal{C}(\Gamma, e)|_W = \mathcal{C}|_W = 0.$$

Since f is proper, W contains $f^{-1}(U)$ for some open neighbourhood U of p.

Theorem 2.7 *Let $f : X \to Y$ be a proper morphism satisfying (2.2) and Γ be an effective divisor not containing any fibre of f. Then the function $e(\Gamma|_{X_y})$ is upper semicontinuous on Y.*

Proof. For $p \in Y$ given, let $e = e(\Gamma|_{X_p})$. Define

$$\Delta := \{y \in Y; e(\Gamma|_{X_y}) > e\}.$$

We have to show that p does not lie in the Zariski closure $\overline{\Delta}$ of Δ. Assume that $p \in \overline{\Delta}$ and let $\sigma : T \to \overline{\Delta}_0$ be the desingularization of some component $\overline{\Delta}_0$ of $\overline{\Delta}$ containing p. If $g : S \to T$ is the pullback of f and B the transform of Γ on S, then g and B satisfy the assumptions made in (2.3). Hence, if $\tau : S' \to S$ is a desingularization and $B' = \tau^* B$ a normal crossing divisor, then

$$\tau_* \omega_{S'}(-[\frac{B'}{e}]) \to \omega_S$$

will be an isomorphism over some open neighbourhood U of $g^{-1}(\sigma^{-1}(p))$. Since g is proper, U contains $g^{-1}(W)$ for some neighbourhood W of $\delta^{-1}(p)$, and for simplicity we may assume $W = T$. Let T_0 be the open subvariety of T over which $g' = g \circ \tau : S' \to T$ is smooth and B' is a relative normal crossing divisor. In contradiction to our assumption we have:

Claim 2.8 For $t \in T_0$ *one has* $e(B|_{g^{-1}(t)}) \leq e$.

Proof. If D is a smooth divisor passing through t and $H = g^{-1}(D)$, then $H' = \tau^{-1}(H)$ is irreducible and smooth and B' intersects H' transversally.

We have

$$\begin{array}{ccc} \tau_*\omega_{S'}(-[\frac{B'}{e}] + H') & \longrightarrow & \tau_*\omega_{H'}(-[\frac{B'|_{H'}}{e}]) \\ \| & & \downarrow \\ \tau_*\omega_{S'}(-[\frac{B'}{e}]) \otimes \mathcal{O}_S(H) & & \\ \| & & \downarrow \\ \omega_S(H) & \longrightarrow & \omega_H \end{array}$$

and therefore $\tau_*\omega_{H'}(-[\frac{B'|_{H'}}{e}]) = \omega_H$. Since $H' \to D$ is again smooth and $B'|_{H'}$ is a relative normal crossing divisor we can repeat this step until we obtain

$$\tau_*\omega_{g'^{-1}(t)}(-[\frac{B'|g'^{-1}(t)}{e}]) = \omega_{g'^{-1}(t)}.$$

Theorem 2.9 *Let Z be a projective normal Gorenstein variety with at most rational singularities and $X = Z \times \ldots \times Z$ the r-fold product. Let $\mathcal{L}$ be an invertible sheaf on Z and $\mathcal{M} = \otimes_{i=1}^r pr_{i^*}\mathcal{L}$. Then $e(\mathcal{M}) = e(\mathcal{L})$.*

Proof. Obviously $e(\mathcal{M}) \geq e(\mathcal{L}) = e$. Let Γ be any effective divisor with $\mathcal{M} = \mathcal{O}_X(\Gamma)$. By induction we may assume that (2.9) holds true for $(r-1)$-fold products. Hence (2.5) applied to $pr_i : X \to Z$ tells us that the support of $\mathcal{C}(\Gamma, e)$ is $pr_i^{-1}(T_i)$ for some subscheme T_i of Z. Since this holds true for all projections, $\mathcal{C}(\Gamma, e)$ must be zero.

In [3], 2.3, we obtained for smooth Z and very ample sheaves $\mathcal{L}$ that

$$e(\mathcal{M}^\nu) = e(\mathcal{L}^\nu) \leq \nu \cdot c_1(\mathcal{L})^{dimZ} + 1.$$

One has the slightly more general

Corollary 2.10 *If in (2.9) Z is non singular and $\mathcal{H}$ a very ample invertible sheaf on Z, then*

$$e(\mathcal{M}) = e(\mathcal{L}) \leq c_1(\mathcal{H})^{dimZ-1} \cdot c_1(\mathcal{L}) + 1.$$

Proof. By (2.9) it is enough to verify the inequality. If Γ is the zero set of a section of $\mathcal{L}$, H a smooth divisor with $\mathcal{O}_Z(H) = \mathcal{H}$, then we choose $\tau : Z' \to Z$ such that Γ' is a normal crossing divisor and such that the proper transform H' of H in Z' intersects Γ' transversally. As in (2.3) we have the diagram

$$\begin{array}{ccccc} \tau_*\omega_{Z'}(-[\frac{\Gamma'}{e}] + H') & \xrightarrow{\alpha} & \tau_*\omega_{H'}(-[\frac{\Gamma'|_{H'}}{e}]) & \xrightarrow{\beta_H} & \omega_H \\ \downarrow & & & & \| \\ \tau_*\omega_{Z'}(-[\frac{\Gamma'}{e}]) \otimes_{\mathcal{O}_Z} \mathcal{H} & \xrightarrow{\beta} & \omega_Z(H) & \longrightarrow & \omega_H \end{array}$$

where again α is surjective. By induction we can assume that for

$$e \geq c_1(\mathcal{H}|_H)^{dim(H)-1} \cdot c_1(\mathcal{L}) + 1$$

β_H is surjective. Moving H we obtain 2.10.

Corollary 2.11 *Assume in (2.9) that $\mathcal{L}$ is an ample invertible sheaf on Z and that there exist a desingularization $\tau : Z' \to Z$ and an effective exceptional divisor E such that $\tau^*\mathcal{L} \otimes \mathcal{O}_{Z'}(-E)$ is very ample. Then*

$$e(\mathcal{M}) = e(\mathcal{L}) \leq c_1(\mathcal{L})^{dimZ} + 1.$$

Proof. Obviously $e(\mathcal{L}) \leq e(\tau^*\mathcal{L})$ and $E \cdot (c_1(\tau^*\mathcal{L}) - E)^j \cdot c_1(\tau^*\mathcal{L})^{dimZ-1-j} \geq 0$ for all $0 \leq j \leq dimZ - 1$. Hence

$$e(\mathcal{L}) \leq (c_1(\tau^*\mathcal{L}) - E)^{dimZ-1} \cdot c_1(\tau^*\mathcal{L}) + 1 \leq c_1(\mathcal{L})^{dimZ} + 1.$$

Remark 2.12 In [14], III, 2.2 the second author claimed that (2.10) holds true for all Z with rational Gorenstein singularities. He overlooked that a hyperplane section through a rational singularity might have non rational singularities. In fact, we doubt that (2.10) can be generalized in that way. However, the results of [14], III, are not really affected. It was only used that $e(\mathcal{H}^\nu)$ is bounded for all pairs $(X, \mathcal{H}) \in \mathcal{M}_h(\mathbb{C})$ where $\mathcal{M}_h$ is a bounded moduli functor of polarized varieties with rational Gorenstein singularities. This follows from (2.7) anyway. (2.11) gives a second way to correct this ambiguity: In [14], III, just before (1.4), one has to add (see also §5):

1.3': *Since $\mathcal{M}''_h$ is bounded we may even choose ν large enough such that for all $(F, \mathcal{H}) \in \mathcal{M}''_h(\mathbb{C})$ one has:*

> *(*) There exist a desingularization $\tau : F' \to F$ and an effective exceptional divisor E on F' such that $\tau^*\mathcal{H} \otimes \mathcal{O}_{F'}(-E)$ is very ample.*

The same property (*) with $\mathcal{H}$ replaced by $\mathcal{M}_0|_F$ has to be added to (2.7, a) in [14], III, (See Remark 3.10, c)). æ

3 Ampleness of certain direct image sheaves

Notations 3.1 Throughout this section $f : X \to Y$ will denote a flat projective Gorenstein morphism between reduced quasi-projective schemes whose fibres $X_p = f^{-1}(p)$ are irreducible normal reduced varieties with at worst rational singularities.

Definition 3.2 For $f : X \to Y$ as above let $\mathcal{L}$ be an invertible sheaf on X. We will call $\mathcal{L}$ *relatively semi-ample over Y (or relatively semi-ample for f) if for some $N > 0$ the map*

$$f^*f_*\mathcal{L}^N \to \mathcal{L}^N$$

is surjective.

Theorem 3.3 *Using the notations (3.1) let $\mathcal{L}$ be an invertible sheaf on Y, Γ be an effective Cartier divisor on X and $e \in \mathbb{N} - \{0\}$. Assume that:*
a) For $p \in Y$ the divisor Γ does not contain X_p and $e(\Gamma|_{X_p}) \leq e$.
b) $\mathcal{L}^e(-\Gamma)$ is relatively semi-ample over Y.
Then one has:
i) For $i \geq 0$, $R^i f_*(\mathcal{L} \otimes \omega_{X/Y})$ is locally free and compatible with arbitary base change.
ii) If for some $M_0 > 0$ and all multiples M of M_0 the sheaf $f_*(\mathcal{L}^e(-\Gamma))^M$ is locally free and weakly positive over Y, then $f_*(\mathcal{L} \otimes \omega_{X/Y})$ is weakly positive over Y.

Remark 3.4 In fact, (3.3, ii) is quite similar to the main technical result of [14], III. There however we assumed in Theorem 2.6, that beside of (3.3, a) we know that $\mathcal{L}^e(-\Gamma)$ is semi-ample. Moreover we assumed $f_*(\mathcal{L} \otimes \omega_{X/Y})$ to be locally free and compatible with arbitary base change, which by part i) is automatically true.

The proof of (3.3, i) is mainly due to J. Kollár, as explained in [14], II, 2.8, 4 (see also [7]).

Proof of 3.3, i. By "Cohomology and base change" (see [10]) it is enough to show that for $i \geq 0$ the sheaves $R^i f_*(\mathcal{L} \otimes \omega_{X/Y})$ are locally free. Moreover (loc. cit.) there is a finite complex $\mathcal{E}^\cdot$ of locally free sheaves such that for a coherent sheaf $\mathcal{C}$ on Y the i-th cohomology of $\mathcal{E}^\cdot \otimes \mathcal{C}$ is $R^i f_*(\mathcal{L} \otimes \omega_{X/Y} \otimes f^*\mathcal{C})$. Hence, to verify the local freeness we may assume that Y is a curve and, taking for $\mathcal{C}$ the integral closure of $\mathcal{O}_Y$, that Y is non singular. If $\tau : X' \to X$ is a desingularization and $\Gamma' = \tau^*\Gamma$ a normal crossing divisor then $R^i\tau_*\omega_{X'}(-[\frac{\Gamma'}{e}]) = 0$ for $i > 0$ (see [13], 2.3) and $\tau_*\omega_{X'}(-[\frac{\Gamma'}{e}]) = \omega_X$ by assumption a) and (2.3). Hence for $\mathcal{L}' = \tau^*\mathcal{L}$ and $f' = f \circ \tau$ we have

$$R^i f'_*(\mathcal{L}'(-[\frac{\Gamma'}{e}]) \otimes \omega_{X'/Y}) = R^i f_*(\mathcal{L} \otimes \omega_{X/Y}).$$

Now the proof ends, as usual, by using Kollár-Tankeev's vanishing theorem: Assume that for some $i \geq 0$, $R^i f'_*(\mathcal{L}'(-[\frac{\Gamma'}{e}]) \otimes \omega_{X'})$ is not locally free. Then for some ideal $I = \mathcal{O}_Y(-q)^a$ on Y,

$$\alpha_i : R^i f'_*(\mathcal{L}'(-[\frac{\Gamma'}{e}]) \otimes \omega_{X'}) \otimes I \to R^i f'_*(\mathcal{L}'(-[\frac{\Gamma'}{e}]) \otimes \omega_{X'})$$

will have a kernel. In order to get a contradiction we can replace e and Γ by some common multiple such that $f'^* f'_* \mathcal{L}^e(-\Gamma') \to \mathcal{L}^e(-\Gamma')$ is surjective. Moreover we can compactify Y and X' and assume that Γ' extends to a normal crossing divisor on the compactification. Replacing $\mathcal{L}$ by $\mathcal{L} \otimes f^*\mathcal{H}$ for some very ample sheaf $\mathcal{H}$ on Y, we can assume moreover that the sheaves $R^i f'_*(\mathcal{L}'(-[\frac{\Gamma'}{e}] \otimes \omega_{X'})$ have no higher cohomology on Y and that for some N_0 and all multiples N of N_0, $\mathcal{L}'^e(-\Gamma')^N \otimes f^*I$ is generated by its global sections.

Hence, if D is the zero divisor of a general section of this sheaf then

$$B = D + N \cdot \Gamma' + f^*a \cdot q$$

is a normal crossing divisor and $[\frac{B}{N \cdot e}] = [\frac{\Gamma'}{e}]$ for N sufficiently big. The non injectivity of α_i implies that

$$H^i(X', \mathcal{L}'(-[\frac{B}{N \cdot e}]) \otimes \omega_{X'}) \to H^i(X', \mathcal{L}'(-[\frac{B}{N \cdot e}]) \otimes \omega_{X'}(f^*a \cdot q))$$

is not injective. However, this contradicts [2], 3.3,1, where we have shown, that the dual of this map

$$H^{n-i}(X', \mathcal{L}'(-[\frac{B}{N\cdot e}])^{-1} \otimes \mathcal{O}_{X'}(-f^*a\cdot q)) \to H^{n-i}(X', \mathcal{L}'(-[\frac{B}{N\cdot e}])^{-1})$$

is surjective for all i.

Proof of 3.3, ii. The assumptions a) and b) are compatible with arbitary base change and the assumption added in ii) is compatible with flat base change. Using (3.3, i) and the equivalence of a) and c) in (1.7), it is enough to show that for an ample invertible sheaf $\mathcal{A}$ on Y the sheaf $f_*(\mathcal{L}\otimes\omega_{X/Y})\otimes\mathcal{A}$ is weakly positive over Y. For some multiple M of M_0 the map

$$f^*f_*(\mathcal{L}^e(-\Gamma)^M)\otimes f^*\mathcal{A}^{M\cdot e} \to (\mathcal{L}^e(-\Gamma)\otimes f^*\mathcal{A}^e)^M$$

will be surjective by assumption b) and by the assumption made in ii)

$$S^b(f_*(\mathcal{L}^e(-\Gamma))^M)\otimes\mathcal{A}^{M\cdot b\cdot e}$$

is globally generated for $b >> 0$. Hence, $(\mathcal{L}\otimes f^*\mathcal{A})^e(-\Gamma)$ is semi-ample. (3.3, i) allows to apply [14], III, 2.6, to obtain the weak positivity over Y of

$$f_*((\mathcal{L}\otimes f^*\mathcal{A})\otimes\omega_{X/Y}) = \mathcal{A}\otimes f_*(\mathcal{L}\otimes\omega_{X/Y}).$$

In some way (3.3, ii) can be seen as a generalization of [3], 1.7. As in [3], 1.9, we obtain:

Proposition 3.5 *Let $f: X\to Y$ be as in (3.1) and let $\mathcal{L}$ be an invertible sheaf on X. Assume that:*
a) $\mathcal{L}$ is relatively semi-ample over Y.
b) For some $M>0$ and all multiples M of M_0 the sheaf $f_*(\mathcal{L}^M)$ is locally free and weakly positive over Y.
c) For some $N>0$ there is an ample invertible sheaf $\mathcal{A}$ on Y and an effective Cartier divisor Γ on X, not containing any fibre of f, with $\mathcal{L}^N = f^*\mathcal{A}\otimes\mathcal{O}_X(\Gamma)$.
Then $f_*(\mathcal{L}\otimes\omega_{X/Y})$ is ample.

Addendum 3.6 *Under the assumptions of (3.5) let $e \geq sup\{N, e(\Gamma|_{X_p})$ for $p\in Y\}$ (which exists by (2.7)). Then the ampleness of $f_*(\mathcal{L}\otimes\omega_{X/Y})$ is measured by (see (1.4)) :*

$$f_*(\mathcal{L}\otimes\omega_{X/Y}) \succeq \frac{1}{e}\cdot\mathcal{A}.$$

Proof. We have to show that $S^e(f_*(\mathcal{L}\otimes\omega_{X/Y}))\otimes\mathcal{A}^{-1}$ is weakly positive over Y. Using the equivalence of a) and b) in (1.5) we are allowed to replace Y by a finite flat cover $\tau: Y'\to Y$ such that $\mathcal{O}_Y\to\tau_*\mathcal{O}_{Y'}$ splits. Hence, using (1.6, b) we may assume that $\mathcal{A}$ is the e-th power of some invertible sheaf $\mathcal{A}'$ on Y and we have to show that $f_*(\mathcal{L}' \otimes \omega_{X/Y})$ is weakly positive over Y for $\mathcal{L}' = \mathcal{L} \otimes f^*\mathcal{A}'^{-1}$. We have $\mathcal{L}'^e(-\Gamma) = \mathcal{L}^{e-N}$ and the assumptions a) and b) of (3.3) hold true for $\mathcal{L}'$ and Γ. The additional assumption made in (3.3, ii) is nothing but b) in (3.5) if $N < e$ and, since $f_*\mathcal{O}_X = \mathcal{O}_Y$, it is trivial for $N = e$. Therefore we can apply (3.3, ii) to end the proof of (3.3) and (3.4).

Just copying the arguments used in [3], 2.4, we can now prove theorem (0.1) as well as

Addendum 3.7 *Using the notations and assumptions from (0.1) we have:*
Let $e \geq sup\{\mu, e(\omega_{X_p}^{\mu\cdot(\eta-1)})$ *for* $p \in Y\}$. *Then*

$$f_*\omega_{X/Y}^{\eta} \succeq \frac{\eta - 1}{e \cdot rank(f_*\omega_{X/Y}^{\mu})} \cdot det(f_*\omega_{X/Y}^{\mu}).$$

Remark 3.8 In particular if f is smooth, if $\omega_{X/Y}$ is relatively ample over Y and if Y is connected, let $N > 0$ be an integer such that $\omega_{X_p}^N$ is very ample for all p. Then we can choose

$$e = \mu \cdot (\eta - 1) \cdot N^{dim X_p - 1} \cdot c_1(\omega_{X_p}^{dim X_p}) + 1$$

for any $p \in Y$. If f is not smooth but $\omega_{X/Y}$ relatively ample, then the same choice of e is possible, if we take N to be big enough, such that the condition (*) in (2.12) holds true for $\mathcal{H} = \omega_{X/Y}^N$.

Proof of 3.7 and 0.1 Set $r = rank(f_*\omega_{X/Y}^{\mu})$ and consider $f^r : X^r \to Y$, where X^r is the r-fold product of X over Y. Of course, f^r is again a flat projective Gorenstein morphism and the fibres of f^r still have at worst rational singularities (see [14], III, 2.9, for example). By flat base change one finds that

$$\omega_{X^r/Y} = \otimes_{i=1}^r pr_i^*\omega_{X/Y}$$

is again relatively semi-ample over Y. By [14], II, Theorem 2.7, $f_*^r\omega_{X^r/Y}^{\gamma}$ is weakly positive over Y for all $\gamma > 0$. In fact, there we added some assumptions on base change properties, which by (3.3,i) are no longer necessary. In order to apply (3.5) and (3.6) to $\mathcal{L} = \omega_{X/Y}^{\eta-1}$ we consider the natural inclusion

$$det(f_*\omega_{X/Y}^{\mu}) \to \otimes^r f_*\omega_{X/Y}^{\mu} = f_*^r\omega_{X^r/Y}^{\mu}.$$

Since this inclusion splits locally, the zero divisor Γ of

$$f^{r*}det(f_*\omega_{X/Y}^{\mu})^{\eta-1} \to \omega_{X^r/Y}^{\mu\cdot(\eta-1)}$$

does not contain any fibre and by (2.9)

$$e(\Gamma|_{X_p\times\ldots\times X_p}) \leq e(\omega_{X_p\times\ldots\times X_p)}^{\mu\cdot(\eta-1)}) = e(\omega_{X_p}^{\mu\cdot(\eta-1)}).$$

By (3.5) and (3.6) we obtain

$$f_*^r(\omega^\eta_{X^r/Y}) = \otimes^r f_*\omega^\eta_{X/Y} \ \succeq \ \frac{1}{e}\cdot det(f_*\omega^\mu_{X/Y})^{\eta-1}$$

and hence

$$S^r(f_*\omega^\eta_{X/Y}) \ \succeq \ \frac{1}{e}\cdot det(f_*\omega^\mu_{X/Y})^{\eta-1}.$$

By definition of $\succeq$ in (1.4) one obtains:

$$f_*\omega^\eta_{X/Y} \ \succeq \ \frac{1}{e\cdot r}\cdot det(f_*\omega^\mu_{X/Y})^{\eta-1}.$$

For application to moduli of polarized varieties we need a second application of (3.3) and (3.5) generalizing (0.1) (by taking $\mathcal{M} = \mathcal{O}_X$).

Theorem 3.9 *For $f : X \to Y$ as in (3.1) let $\mathcal{M}$ be an invertible sheaf on X. Assume that for some $e \in \mathbb{N} - \{0\}$ one has:*
a) $\mathcal{M}$ is relatively semi-ample over Y.
b) $f_*\mathcal{M}$ is locally free of rank r'.
c) $e \geq e(\mathcal{M}|_{X_p})$ for all $p \in Y$.
d) $\mathcal{M}\otimes\omega^e_{X/Y}$ is relatively semi-ample over Y.
e) $f_*(\mathcal{M}\otimes\omega^e_{X/Y})^N$ is locally free for all $N > 0$.
Then one has:
i) $(\otimes^{r'} f_*(\mathcal{M}\otimes\omega^e_{X/Y}))\otimes det(f_*\mathcal{M})^{-1}$ is weakly positive over Y.
ii) If for some $\mu > 0$ the sheaf $det(f_*(\mathcal{M}\otimes\omega^{e+1}_{X/Y})^\mu)^{r'}\otimes det(f_*\mathcal{M})^{-\mu\cdot r(\mu)}$ is ample for $r(\mu) = rank(f_*(\mathcal{M}\otimes\omega^{e+1}_{X/Y})^\mu)$, then $(\otimes^{r'} f_*(\mathcal{M}\otimes\omega^{e+1}_{X/Y}))\otimes det(f_*\mathcal{M})^{-1}$ is ample.

Remarks 3.10 a) As in (3.5) and (3.7) one can give effective bounds on the degree of ampleness in part ii) of (3.9).
b) Part i) of (3.9) is a straightforward generalization of [14], III, 2.7. However, since we weakend the assumptions we sketch the proof.
c) If as, in [14], III, 2.7, we assume that $\mathcal{M}$ is ample we can give bounds for e:
Assume that for all $p \in Y$ there exists a desingularization $\tau : X'_p \to X_p$ and an effective exceptional divisor E_p on X'_p such that $\tau^*\mathcal{M}|_{X_p}\otimes\mathcal{O}_{X'_p}(-E_p)$ is very ample, then (3.9, c) can be replaced by the assumption

c') $$e \geq c_1(\mathcal{M}|_{X_p})^{dim X_p} + 1.$$

Proof of 3.9 Obviously the assumptions are compatible with flat base change. Using (1.6, b) and (1.5) we can assume that $det(f_*\mathcal{M}) = \lambda'^{r'}$ for some invertible sheaf λ' on Y. Replacing $\mathcal{M}$ by $\mathcal{M}\otimes f^*\lambda'^{-1}$ we may as well assume that $det(f_*\mathcal{M}) = \mathcal{O}_Y$. We have to show in i) that $f_*(\mathcal{M}\otimes\omega^e_{X/Y})$ is weakly positive over Y and in ii) that the ampleness of $\lambda_\mu = det(f_*(\mathcal{M}\otimes\omega^{e+1}_{X/Y})^\mu)$ implies the ampleness of $f_*(\mathcal{M}\otimes\omega^{e+1}_{X/Y})$.

For $r \in \mathbb{N}$ let $f^r : X^r \to Y$ be the r-fold product of X over Y. We write

$$\mathcal{N} = \otimes_{i=1}^r pr_i^* \mathcal{M}.$$

In order to prove i) we choose $r = r'$. Hence $\mathcal{N}$ has a section induced by

$$det(f_* \mathcal{M}) = \mathcal{O}_Y \to f_*^r \mathcal{N} = \otimes^r f_* \mathcal{M}.$$

Let Γ be the zero divisor. Γ does not contain any fibre of f^r and for $N > 0$ and $\Gamma' = N \cdot \Gamma$ we have by (2.9) and by definition

$$e(\Gamma'|_{X_p \times \ldots \times X_p}) \leq N \cdot e(\Gamma|_{X_p \times \ldots \times X_p}) \leq N \cdot e(\mathcal{M}|_{X_p}) \leq N \cdot e.$$

Let $\mathcal{H}$ be an ample invertible sheaf and $m \geq 0$ be an integer. For $e' = N \cdot e$ let us consider the sheaf

$$\mathcal{L} = \mathcal{N}^N \otimes \omega_{X^r/Y}^{e'-1} \otimes f^{r*} \mathcal{H}^{m \cdot (e'-1) \cdot r}.$$

Then

$$\mathcal{L}^{e'}(-\Gamma') = (\mathcal{N}^N \otimes \omega_{X^r/Y}^{e'} \otimes f^{r*} \mathcal{H}^{m \cdot r \cdot e'})^{(e'-1)}$$

and the assumptions a) and b) of (3.3) hold true. By (1.8) weak positivity is compatible with tensor products and (3.3) implies:

Claim 3.11 *If for some $M_0 > 0$ and all multiples M of M_0 the sheaf*

$$f_*(\mathcal{M} \otimes \omega_{X/Y}^e)^{N \cdot M} \otimes \mathcal{H}^{m \cdot e \cdot N \cdot M}$$

is weakly positive over Y, then

$$f_*(\mathcal{M} \otimes \omega_{X/Y}^e)^N \otimes \mathcal{H}^{m \cdot (e \cdot N - 1)}$$

is weakly positive over Y.

Since $\mathcal{M} \otimes \omega_{X/Y}^e$ is relatively semi-ample over Y we can find some N_0 such that for all multiples N of N_0 and $M >> 0$ the multiplication maps

$$\alpha(N, M) : S^M(f_*(\mathcal{M} \otimes \omega_{X/Y}^e)^N) \to f_*(\mathcal{M} \otimes \omega_{X/Y}^e)^{N \cdot M}$$

are surjective. For those N and

$$m = Min\{\mu > 0; f_*(\mathcal{M} \otimes \omega_{X/Y}^e)^N \otimes \mathcal{H}^{\mu \cdot e \cdot N} \text{ weakly positive over } Y\}$$

the surjectivity of $\alpha(N, M)$ implies the weak positivity over Y of

$$f_*(\mathcal{M} \otimes \omega_{X/Y}^e)^{N \cdot M} \otimes \mathcal{H}^{m \cdot e \cdot N \cdot M}$$

for all $M >> 0$. Hence (3.11) gives that

$$f_*(\mathcal{M} \otimes \omega_{X/Y}^e)^N \otimes \mathcal{H}^{m \cdot e \cdot N - m}$$

is weakly positive over Y. By the choice of m this implies that

$$(m-1) \cdot e \cdot N < m \cdot e \cdot N - m$$

or that $m < e \cdot N$. Hence $f_*(\mathcal{M} \otimes \omega^e_{X/Y})^N \otimes \mathcal{H}^{e^2 \cdot N^2}$ is weakly positive. Since this holds for all finite flat coverings of Y as well we obtain by (1.7) the weak positivity of $f_*(\mathcal{M} \otimes \omega^e_{X/Y})^N$. Applying (3.11) for $m = 0, N = 1$ and $M_0 = N_0$ defined above, we obtain part i) of (3.9).

To prove ii) we take $r = r' \cdot r(\mu)$ and $\mathcal{L} = \mathcal{N} \otimes \omega^e_{X^r/Y}$. We have natural inclusions, splitting locally

$$\mathcal{O}_Y = det(f_*\mathcal{M})^{r(\mu)} \to f^r_* \mathcal{N} = \otimes^{r' \cdot r(\mu)} f_* \mathcal{M}$$

and

$$\lambda^{r'}_\mu \to f^r_*(\mathcal{N} \otimes \omega^{e+1}_{X^r/Y})^\mu = \otimes^{r' \cdot r(\mu)} f_*(\mathcal{M} \otimes \omega^{e+1}_{X/Y})^\mu.$$

If Δ_1 and Δ_2 denote the corresponding zero-divisors on X^r, then $\Delta_1 + \Delta_2$ does not contain any fibre of f^r and

$$\mathcal{L}^{(e+1)\cdot\mu} = (\mathcal{N} \otimes \omega^{e+1}_{X^r/Y})^{e\cdot\mu} \otimes \mathcal{N}^\mu = f^{r*}\lambda^{r'\cdot e}_\mu \otimes \mathcal{O}_X(e \cdot \Delta_2 + \mu \cdot \Delta_1).$$

Hence for $\mathcal{A} = \lambda^{r'\cdot e}_\mu$ and $N = (e+1)\cdot\mu$ the assumption (3.5, c) holds true. The assumption (3.5, b) is just part i) and e) of (3.9) and (3.5, a) is implied by (3.9, d). Hence

$$f^r_*(\mathcal{N} \otimes \omega^{e+1}_{X^r/Y}) = \otimes^r f_*(\mathcal{M} \otimes \omega^{e+1}_{X/Y})$$

is ample. æ

4 Ample sheaves on moduli schemes of canonically polarized manifolds

Let us consider the moduli functor $\mathcal{C}'_h$ of canonically polarized normal Gorenstein varieties with Hilbert polynomial h and with at worst rational singularities. Hence for a scheme S defined over $\mathbb{C}$,

> $\mathcal{C}'_h(S) = \{f : X \to S; f$ flat, projective, Gorenstein; $\omega_{X/Y}$ relatively ample for f; all fibres F of f are irreducible normal varieties with at most rational singularities and $h(\nu) = \chi(\omega^\nu_F)\}/\simeq$.

Let $\mathcal{C}_h$ be a submoduli functor of $\mathcal{C}'_h$ such that $\mathcal{C}_h$ is bounded, separated and such that for $f : X \to S \in \mathcal{C}'_h(S)$ the subset $S_0 = \{s \in S; f^{-1}(s) \in \mathcal{C}_h(Spec\mathbb{C})\}$ is constructible in S.

By [6] we can choose $\mathcal{C}_h = \mathcal{C}'_h$, if $n = deg(h) \leq 2$, and in all dimensions

$$\mathcal{C}_h(S) = \{f \in \mathcal{C}'_h(S); f \text{ smooth } \}$$

will work by "Makusaka's big theorem". In any case we have (see [14], I, §1 and II, §6)

Assumption 4.1
i) There exists $\nu > 1$ such that for all $F \in \mathcal{C}_h(Spec(\mathbb{C}))$ the sheaf ω_F^ν is very ample.
ii) There exists a Hilbert scheme H and a universal family $g : \mathcal{X} \to H \in \mathcal{C}_h(H)$ together with an H-isomorphism

$$\mathbb{P}(g_* \omega_{\mathcal{X}/H}^\nu) \cong \mathbb{P}^{r-1} \times H.$$

iii) The action of $G = \mathbb{P}Gl(r, \mathbb{C})$ on H obtained by "change of coordinates in $\mathbb{P}^{r-1}$" is proper.
iv) Let us write $\lambda_\eta = det(g_* \omega_{\mathcal{X}/H}^\eta)$ and $r(\eta) = rank(g_* \omega_{\mathcal{X}/H}^\eta)$. Then for $\mu >> 0$ the sheaf $\mathcal{L}_0 = \lambda_{\nu \cdot \mu}^{r(\nu)} \otimes \lambda_\nu^{-\mu \cdot r(\nu \cdot \mu)}$ is ample on H.

Corollary 4.2 *λ_ν is ample on H.*

Proof. Of course we may assume H to be reduced. By [14], II, 2.7 the sheaf $g_* \omega_{\mathcal{X}/H}^\nu$ is weakly positive over H and hence λ_ν has the same property. By (1.2) $\lambda_{\nu \cdot \mu}$ is ample and as we have seen in (0.1), $g_* \omega_{\mathcal{X}/H}^\eta$ as well as λ_η will be ample for all $\eta > 1$ with $r(\eta) > 0$.

Let $\mathcal{L}$ be a G-linearized sheaf on H ([11], Def. 1.6). In [14], I, 5.2, we denoted by $H(\mathcal{L})^s$ the set of stable points of H with respect to $\mathcal{L}$ and under the G-action (see [11], Def. 1.7). Obviously the sheaves λ_η are G-linearized for all $\eta > 0$.

Corollary 4.3 *One has $H = H(\lambda_\nu)^s$.*

Proof. In [14], II, we proved that $H = H(\mathcal{L}_0 \otimes \lambda_\nu^\eta)^s$ for some $\eta >> 0$. However we only used "$\mathcal{L}_0$ ample" and not at all the specific shape of $\mathcal{L}_0$. Hence, if we start with λ_ν instead of $\mathcal{L}_0$ we obtain 4.3.

The stability criterion used in [14], II, was formulated in a more general set up in [15], 3.2.

By [11], 4.3 implies the existence of a geometric quotient H/G and H/G is embedded into some projective space by G-invariant sections of λ_ν^p for $p >> 0$. Therefore λ_ν^p descends to some ample sheaf on H/G, which we denote by $\lambda_\nu^{(p)}$. As it is shown in [11] $C_h = H/G$ is a coarse moduli scheme for $\mathcal{C}_h$. Hence we obtained:

Theorem 4.4 *Let $\mathcal{C}_h$ be the moduli functor of canonically polarized manifolds with Hilbert polynomial h (or any moduli functor satisfying (4.1)). Then there exists a coarse moduli scheme C_h and an ample invertible sheaf $\lambda_\nu^{(p)}$ for ν as in (4.1, i) and $p >> 0$, such that: For $f : X \to S \in \mathcal{C}_h(S)$ let $\varphi : S \to C_h$ be the induced morphism. Then*

$$\varphi^* \lambda_\nu^{(p)} = det(f_* \omega_{X/S}^\nu)^p.$$

Corollary 4.5 *For $f : X \to S \in \mathcal{C}_h(S)$ assume that $\varphi : S \to C_h$ is affine over its image. Then for all $\eta > 1$ with $h(\eta) > 0$ the sheaf $f_* \omega_{X/S}^\eta$ is ample.*

Proof. By (4.4) the sheaf $det(f_*\omega^{\nu}_{X/S})$ is ample and (4.5) follows from (0.1).

It is easy to show, that for all $\eta > 0$ with $h(\eta) > 0$ the sheaves λ_η on H descend to invertible sheaves $\lambda_\eta^{(p(\eta))}$ on C_h for some $p(\eta) > 0$ (see for example [8]).

Corollary 4.6 *Let $\tau : C'_h \to C_h$ be the normalization. Then for all $\eta > 0$ with $h(\eta) > 0$ the sheaf $\tau^*(\lambda_\eta^{p(\eta))})$ is ample on C'_h.*

Proof. By [8], §2, there exist a finite cover $\varphi : Y \to C_h$ and $f : X \to Y \in \mathcal{C}_h(Y)$ such that φ is induced by f. We may assume that φ factors through $\varphi' : Y \to C'_h$. Hence

$$\varphi'^*\tau^*(\lambda_\eta^{(p(\eta))}) = (det f_*\omega^{\eta}_{X/Y})^{p(\eta)}$$

is ample by (4.5) and by (1.5) ampleness descends to C'_h.

(4.6) suggests the following question, which, in fact, would have an affirmative answer if we could choose in the proof of (4.6) $\varphi : Y \to C_h$ such that $\mathcal{O}_{C_h} \to \varphi_*\mathcal{O}_Y$ splits.

Question 4.7 Are the sheaves $\lambda_\eta^{(p(\eta))}$ ample on C_h for all $\eta > 1$ with $h(\eta) > 0$?

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5 Ample sheaves on moduli schemes of polarized manifolds

As in [14], III, §1 let us consider the moduli functor $\mathcal{M}'_h$ with

> $\mathcal{M}'_h(S) = \{(f : X \to S, \mathcal{H}); f$ flat, projective and Gorenstein and $\mathcal{H}$ invertible, relatively ample over S, such that: for all $p \in S$ $X_p = f^{-1}(p)$ is a normal variety with at worst rational singularities, $\chi(\mathcal{H}|_{X_p}) = h(p)$ and X_p is not uniruled $\}/\simeq$.

Differently from the definition of moduli of polarized varieties in [11] or [16], we define

$$(f : X \to S, \mathcal{H}) \simeq (f' : X' \to S, \mathcal{H}')$$

if there is an S-isomorphism $\tau : X \to X'$ and an invertible sheaf $\mathcal{B}$ on S such that

$$\tau^*\mathcal{H}' \simeq \mathcal{H} \otimes f^*\mathcal{B}.$$

We take $\mathcal{M}''_h$ to be a bounded and separated submoduli functor of $\mathcal{M}'_h$ such that for all S and all $(f : X \to S, \mathcal{H}) \in \mathcal{M}'_h(S)$ the subset

$$S_0 = \{s \in S; (f^{-1}(s), \mathcal{H}|_{f^{-1}(s)}) \in \mathcal{M}'_h(Spec(\mathbb{C}))\}$$

is constructible in S.

By [6] again, we can choose $\mathcal{M}''_h = \mathcal{M}'_h$, if $deg(h) = 2$ and by "Makusaka's big theorem"

$$\mathcal{M}''_h(S) = \{(f : X \to S, \mathcal{H}) \in \mathcal{M}'_h(S), f \text{ smooth } \}$$

will always work. By boundedness we can find some $\nu > 0$ such that for all $(F, \mathcal{H}) \in \mathcal{M}''_h(Spec(\mathbb{C}))$ the sheaf $\mathcal{H}^\nu$ is very ample. We can even choose ν big enough to have:

Assumption 5.1 There is some $\nu > 0$ such that for all $(F, \mathcal{H}) \in \mathcal{M}''_h(Spec(\mathbb{C}))$ one has:
a) There is a desingularization $\tau : F' \to F$ and an effective exceptional divisor E on F' such that $\tau^*\mathcal{H}^\nu \otimes \mathcal{O}_{F'}(-E)$ is very ample.
b) $H^i(F, \mathcal{H}^\nu) = 0$ for $i > 0$.
c) For all numerically effective sheaves $\mathcal{L}$ on F, the sheaf $\mathcal{H}^\nu \otimes \omega_F \otimes \mathcal{L}$ is very ample and without higher cohomology.

Proof. By boundedness, for some smooth Y, there is a family $(g : X \to Y, \mathcal{H}') \in \mathcal{M}''_h(Y)$ such that all $(F, \mathcal{H}) \in \mathcal{M}''_h(Spec(\mathbb{C}))$ occur as fibres. For a) we consider a desingularization of X. We find a) to be true for all $(F, \mathcal{H})$ over some dense open subscheme of Y. Repeating this for the complement we obtain a).

In a) we are allowed to replace ν by any multiple. Then b) is obvious and c) follows from [14], III, 1.3, if we replace ν by $\nu(n+1)$ for $n = deg(h)$.

Notations 5.2 For ν as in 5.1 let c be the highest coefficient of h and

$$e \geq (n!) \cdot c \cdot \nu^n + 2$$

for $n = deg(h)$. Especially $e \geq e(\mathcal{H}^\nu) + 1$ for all $(F, \mathcal{H}) \in \mathcal{M}''_h(Spec(\mathbb{C}))$ by (2.11). Let us choose

$$\mathcal{M}_h(S) = \{(f : X \to S, \mathcal{H}) \in \mathcal{M}''_h(S); \mathcal{H}^\nu \otimes \omega^e_{X/S} \text{ relatively very ample over } S\}.$$

As in [14], III, 1.4, we may assume, that $\chi(F, (\mathcal{H}^\nu \otimes \omega_F^e)^\eta)$ and $\chi(F, \mathcal{H}^{\nu\cdot\eta+1} \otimes \omega_F^{e\cdot\eta})$ are the same for all $(F, \mathcal{H}) \in \mathcal{M}_h(Spec(\mathbb{C}))$ regarded as polynomials in η.

By (5.1, c) $\mathcal{M}_h(Spec(\mathbb{C}))$ will contain all $(F, \mathcal{H}) \in \mathcal{M}''_h(Spec(\mathbb{C}))$ with ω_F numerically effective. But, since we do not know wether "ω_F nef" is a constructible condition we are not able to consider just

$$\mathcal{M}_h^{nef}(S) = \{(f : X \to S, \mathcal{H}); \omega_{X/Y} \text{ numerically effective on each fibre } \}.$$

However, the functor $\mathcal{M}_h^{sa}$ can replace $\mathcal{M}_h$ in the results following where

$$\mathcal{M}_h^{sa} = \{(f : X \to X, \mathcal{H}); \omega_{X/Y} \text{ relatively semi-ample over } S\}.$$

Lemma 5.3 *Using the notations and assumption introduced above one has for all*

$$(f : X \to S, \mathcal{H}) \in \mathcal{M}_h(S) :$$

a) $\mathcal{H}^\nu$ *is relatively ample over* S *and* $f_*\mathcal{H}^\nu$ *is locally free of rank* r'.
b) $\mathcal{H}^\nu \otimes \omega^e_{X/S}$ *is relatively ample over* S *and* $f_*(\mathcal{H}^\nu \otimes \omega^e_{X/S})^\eta$ *is locally free of rank* $r(\eta)$ *for* $\eta > 0$.
c) $f_*(\mathcal{H}^\nu \otimes \omega^e_{X/S})$ *is weakly positive over* Y.
d) If for some $\mu > 0$ *the sheaf* $\lambda_\mu = det(f_*(\mathcal{H}^\nu \otimes \omega^e_{X/S})^\mu)^{r'} \otimes det(f_*\mathcal{H}^\nu)^{-\mu\cdot r(\mu)}$ *is ample, then* $(\otimes^{r'} f_*(\mathcal{H}^\nu \otimes \omega^e_{X/S})) \otimes det(f_*\mathcal{H}^\nu)^{-1}$ *is ample.*

Proof. a) holds true since we have no higher cohomology along the fibres and everything is compatible with base change. Moreover $\mathcal{H}^\nu \otimes \omega^e_{X/S}$ is relatively ample over S by definition of $\mathcal{M}_h$. Hence $\mathcal{H}^{\nu\cdot\eta} \otimes \omega^{e'}_{X/S}$ is relatively ample over S for $o \le e' \le \eta \cdot e$. By (3.3, i) all the sheaves

$$f_*(\mathcal{H}^\nu \otimes \omega^{e'}_{X/S})^\eta$$

are locally free for $e' = e$ or $e' = e - 1$. Since $e \ge e(\mathcal{H}|_F) + 1$ for all fibres F of f, c) and d) are implied by (3.9, i) and (3.9, ii).

Let H be the Hilbert scheme considered in [14], III, 1.5, d. Especially we have a "universal family"

$$(g : \mathcal{X} \to H, \mathcal{H}) \in \mathcal{M}_h(H)$$

and an isomorphism

$$\varphi : \mathbb{P}(g_*(\mathcal{H}^\nu \otimes \omega^e_{\mathcal{X}/H})) \to \mathbb{P}^{r-1} \times H.$$

Again, $G = \mathbb{P}Gl(r, \mathbb{C})$ acts on H properly and G/H will be a candidate for M_h (see [11]).

Lemma 5.4 *For* ν *as in (5.1) the sheaf* λ_1 *is ample on* H *(see (5.3) for the notations).*

Proof. For $\mu >> 0$ the Plücker coordinates give an embedding of H into some projective space and the induced ample sheaf is

$$\mathcal{L}_0 = det(g_*(\mathcal{H}^\nu \otimes \omega^e_{\mathcal{X}/H}))^{-\mu\cdot r(\mu)} \otimes det(g_*(\mathcal{H}^{\mu\cdot\nu} \otimes \omega^{e\cdot\mu}_{\mathcal{X}/H}))^{r(1)}.$$

Hence $\mathcal{L}_0^{r'} = \lambda_1^{-\mu\cdot r(\mu)} \otimes \lambda_\mu^{r(1)}$ is ample. By (3.9, i) λ_1 as the determinant of a weakly positive sheaf is weakly positive over Y and hence λ_μ is ample. Since we have choosen $e \ge e(\mathcal{H}) + 1$ for $(\mathcal{H}, F) \in \mathcal{M}_h(Spec(\mathbb{C}))$ we can apply (3.9, ii) for $(e-1)$ and we obtain the ampleness of

$$(\otimes^{r'} g_*(\mathcal{H}^\nu \otimes \omega^e_{\mathcal{X}/H})) \otimes det(f_*\mathcal{H}^\nu)^{-1}$$

and hence of λ_1.

As in (4.3) and (4.4) we obtain

Corollary 5.5 *One has* $H = H(\lambda_1)^s$.

Theorem 5.6 *Let* $\mathcal{M}_h$ *be the moduli functor of (5.2),* ν *as in (5.1) and* e *as in (5.2). Then there exists a coarse moduli scheme* M_h *and an ample invertible sheaf* $\lambda^{(p)}$ *for some* $p > 0$*, such that:*
For $(f : X \to S, \mathcal{H}) \in \mathcal{M}_h(S)$ *let* $\varphi : S \to M_h$ *be the induced morphism. Then*

$$\varphi^*\lambda^{(p)} = (det(f_*(\mathcal{H}^\nu \otimes \omega^e_{X/S}))^{r'} \otimes det(f_*\mathcal{H}^\nu)^{-r})^p$$

for $r' = rank(f_*\mathcal{H}^\nu)$ *and* $r = rank\ f_*(\mathcal{H}^\nu \otimes \omega^e_{X/S})$.

Remarks 5.7 a) In fact, if one compares (5.4) with [14], III, 1.11, then the choice of e is slightly different. However in [14], III, we only used that $e \geq (n!) \cdot c \cdot \nu^n + 1$ and not, as stated there, that one has equality. This is obvious if one takes the stability criterion [15], 3.2.
b) Kollár [8] and Fujiki-Schumacher [4] developed independently methods to study sheaves on analytic moduli spaces, the first one by estimating the degree on complete curves of certain natural sheaves on moduli spaces, the two others by curvature estimates. Both methods give ampleness criteria for sheaves on compact subspaces of moduli spaces. The comparison of the results of [4] with those of this note should give some candidates beside of $\lambda^{(p)}$ for ample sheaves on M_h and some hope that question (4.7) has an affirmative answer.

Corollary 5.8 *For* $(f : X \to S, \mathcal{H}) \in \mathcal{M}_h(S)$ *assume that* $\varphi : S \to M_h$ *is affine over its image. Then for* ν, e, r' *as above the sheaf* $\otimes^{r'} f_*(\mathcal{H}^\nu \otimes \omega_{X/S}) \otimes det(f_*\mathcal{H}^\nu)^{-1}$ *is ample.*

Proof. Use (5.6) and (3.9, ii)

Notations 5.9 It is quite easy to see, that invertible G-linear sheaves on H have some power which descends to M_h (see [8] for example). Since $det(g_*\omega^\delta_{\mathcal{X}/H})^q$ is G-invariant, for $q >> 0$ we can descend this sheaf to some sheaf $\gamma^{(q)}_\delta$ on M_h. Especially, if we have choosen $\mathcal{M}_h$ such that for some $\delta > 0$ and all $(F, \mathcal{H}) \in \mathcal{M}_h(Spec(\mathbb{C}))$ one has $\omega^\delta_F = \mathcal{O}_F$, then for $(f : X \to S, \mathcal{H}) \in \mathcal{M}_h(S)$ and the induced morphism $\varphi : S \to M_h$ one has

$$f^*\varphi^*\gamma^{(q)}_\delta = \omega^{q\cdot\delta}_{X/S}.$$

Corollary 5.10 *Assume that for some* $\delta > 0$ *and all* $(F, \mathcal{H}) \in \mathcal{M}_h(Spec(\mathbb{C}))$ *one has* $\omega^\delta_F = \mathcal{O}_F$*. Assume moreover that the integer* e *used in the definition of* $\mathcal{M}_h$ *is divisible by* δ*. Then* $\gamma^{(q)}_\delta$ *is ample on* M_h*.*

Proof. By (5.6) $\lambda^{(p)}$ is ample on M_h. For $e = \delta \cdot e'$ we have for

$$(f : X \to S, \mathcal{H}) \in \mathcal{M}_h(S)$$

that

$$f_*(\mathcal{H}^\nu \otimes \omega^e_{X/S}) = f_*(\mathcal{H}^\nu) \otimes (f_*\omega^\delta_{X/S})^{e'}$$

and $\varphi^*\lambda^{(p)} = (f_*\omega^\delta_{X/S})^{e'\cdot r'\cdot p}$. Hence $\lambda^{(p)} = \gamma_\delta^{(e'\cdot r'\cdot p)}$.

(5.10) generalizes the result of Pjatetskij-Šapiro and Šafarevich on moduli of $K3$ surfaces [12]. In fact, the quite simple ample sheaf obtained in [12] was one of the motivations to reconsider the proof of the existence of quasi-projective moduli schemes and to try to improve the ample sheaves obtained in [14], III.

Remark 5.11 a) Replacing "isomorphisms of polarizations" by "numerical equivalence of polarizations" (see for example [16], §4) one obtains the "right" functor $\mathcal{P}_h$ of polarized varieties. In [16] we constructed a coarse quasi-projective moduli space P_h for $\mathcal{P}_h$. The natural morphism

$$\Sigma : M_h \to P_h$$

is proper, and, replacing $\mathcal{H}$ by some high power and $h(t)$ by $h(\eta \cdot t)$, we may assume moreover that the fibres of Σ are connected. If for all $(F, \mathcal{H})$ one has

$$q(F) = h^0(F, \Omega^1_F) = 0,$$

then Σ will be an isomorphism and (5.6) gives an ample sheaf on P_h as well.
b) On the other hand, if $\mathcal{M}_h$ satisfies the assumptions of (5.10), then the ampleness of $\gamma_\delta^{(q)}$ just implies, that M_h can not contain projective curves C such that the induced family $(f : X \to C', \mathcal{H})$ over some finite cover C' of C satisfies $X \simeq F \times C'$. Hence Σ is finite and (5.10) gives an ample sheaf on P_h again.
c) Regarding the construction of P_h in §3 of [16], the fact that Σ is finite just means that one has:

> *If F is a manifold with $\omega_F^\delta = \mathcal{O}_F$, $\mathcal{H}$ ample invertible on F and ν sufficiently large (as in (5.1) for example), then for each $\mathcal{L} \in Pic^0(F)$ one can find an automorphism $\tau : F \to F$ such that $\tau^*(\mathcal{H}^\nu) = \mathcal{H}^\nu \otimes \mathcal{L}$ or $\mathcal{L} = \tau^*(\mathcal{H}^\nu) \otimes \mathcal{H}^{-\nu}$.*

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6 Degenerate fibres

The ampleness criteria (0.1) and (3.9) are strong enough to be applied to moduli schemes. However, from the point of view of fibrespaces it seems natural to keep track of the behaviour of sections with respect to compactifications, as we also did in [3].

For simplicity we stay in the category of quasi-projective complex schemes, even if the result remain true for analytic spaces, if one modifies the definition of weak positivity, as it was done in [14], II, §5.

Since it is not at all understood how to get the natural compactification of the total space of a smooth morphism to a singular scheme, we have to return to the notations used in [14], II and III:

Assumptions 6.1 Let

$$\begin{array}{ccccc} X_0 & \xrightarrow{\ i\ } & X & \xleftarrow[\sigma]{} & X' \\ {\scriptstyle f_0}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y_0 & \xrightarrow{\ j\ } & Y & \xleftarrow[\delta]{} & Y' \end{array}$$

be a commutative diagram of morphisms of reduced quasi-projective schemes such that
a) i and j are open dense embeddings and σ and δ are desingularizations
b) f and f' are surjective and $X_0 = f^{-1}(Y_0)$.
c) f_0 is flat and Gorenstein. All fibres $X_p = f_0^{-1}(p)$ are reduced irreducible normal varieties of dimension n with at worst rational singularities.
d) If (...) is a sheaf or a divisor on X (or Y) then $(...)_0$ will always denote the restriction to X_0 (or Y_0) and $(...)'$ will be the pullback to X' (or Y').
e) ω_{X_0/Y_0} is the dualizing sheaf of f_0 and $\omega_{X'/Y'} = \omega_{X'} \otimes f'^{*}\omega_{Y'}^{-1}$.

The main purpose of this chapter is to sketch the changes of the arguments employed in §3 and §4 to obtain:

Claim 6.2 *Under the assumption (6.1) assume that $f' : X' \to Y'$ is semi stable in codimension one (see [9], 4.6), that ω_{X_0/Y_0} is relatively semi-ample over Y_0 and that for some $\mu > 0$ the sheaf $det(f_{0*}\omega^{\mu}_{X_0/Y_0})$ is ample with respect to $(Y', det(f'_*\omega^{\mu}_{X'/Y'}))$. Then for all $\eta \geq 2$ with $f_{0*}\omega^{\eta}_{X_0/Y_o} \neq 0$, the sheaf $f_{0*}\omega^{\eta}_{X_0/Y_0}$ is ample with respect to $(Y', f'_*\omega^{\eta}_{X'/Y'})$.*

Claim 6.3 *Under the assumptions (6.1) assume that $f_0 : X_0 \to Y_0 \in \mathcal{C}_h(Y_0)$ and that the induced morphism $\varphi : Y_0 \to C_h$ is quasi-finite over its image. Then for all $\eta > 1$ with $k(\eta) > 0$ the sheaf $f_{0*}\omega^{\eta}_{X_0/Y_0}$ is ample with respect to $(Y', f'_*\omega^{\eta}_{X'/Y'})$.*

Contrary to [14], II and III, we do not assume here in (6.1) that Y, Y', X and X' are compact. The main reason is that (6.2) will be needed for partial compactifications in order to prove (6.3). Hence to give complete proofs of (6.2) and (6.3), one has to verify first, that neither in the definition of "weak positivity with respect to" nor in [14], II, 2.7, or [14], III, 2.6, the compactness of Y (or X) was really necessary.

Since this can not be done in all details here, and since we just give a coarse outline of the proofs, the reader should regard (6.2) and (6.3) with certain doubts.

Remark 6.4 In (6.2) and (6.3) the conditions (1.9, i and ii) hold true. In fact, using (3.3, i) one finds that $f_{0*}\omega^{\eta}_{X_0/Y_0}$ is locally free and compatible with base change and hence the inclusion of sheaves asked for is just given by

$$S^{\nu}(f_{0*}\omega^{\eta}_{X_o/Y_o}) \to S^{\nu}(f_{0*}\omega^{\eta}_{X_0/Y_0}) \otimes j^*\delta_*\mathcal{O}_{Y'} \simeq j^*\delta_*S^{\nu}(f_*\omega^{\eta}_{X'/Y'}).$$

Sketch of the proof of 6.2 First of all, the equivalence of a) and c) in (1.7) extends to "weakly positive with respect to" by [14], II, 2.4, b, and one can even assume there that

$\tau : Z \to Y$ is flat. In [14], III, 2.6 the assumption that:

$(\mathcal{L}_0^e(-\Gamma_0))^N$ *is globally generated by* $H^0(X_0, (\mathcal{L}_0^e(-\Gamma_0))^N) \cap H^0(X', (\mathcal{L}'^e(-\Gamma'))^N)$ *over* X_0

can be replaced by:

a) $\mathcal{L}_0^e(-\Gamma_0)$ *is relatively semi-ample over* Y_0
b) *For some* $M_0 > 0$ *and all multiples* M *of* M_0 *the sheaf* $f_{0*}(\mathcal{L}_0^e(-\Gamma_0))^M$ *is locally free and weakly positive over* Y_0 *with respect to* $(Y', f'_*(\mathcal{L}'^e(-\Gamma'))^M)$.

The proof remains the same. In order to be able to apply [14], II, 2.4, b one has to use [14], II, 2.5. In the same way (3.5) can be modified to

Claim 6.5 *For* $f : X \to Y$ *as in (6.1) let* $\mathcal{L}$ *be an invertible sheaf on* X. *Assume that:*
a) $\mathcal{L}_0$ *is relatively semi-ample over* Y_0 .
b) For some $M > 0$ *and all multiples* M *of* M_0 *the sheaf* $f_{0*}(\mathcal{L}_0^M)$ *is locally free and weakly positive over* Y_0 *with respect to* $(Y', f'_*(\mathcal{L}'^M))$.
c) For some $N > 0$ *there is an ample invertible sheaf* $\mathcal{A}$ *on* Y *and an effective Cartier divisor* Γ *on* X, *not containing* X_p *for* $p \in Y_0$, *with* $\mathcal{L}^N = f^*\mathcal{A} \otimes \mathcal{O}_X(\Gamma)$.
Then $f_{0*}(\mathcal{L}_0 \otimes \omega_{X_o/Y_0})$ *is ample with respect to* $(Y', f'_*(\mathcal{L}' \otimes \omega_{X'/Y'}))$.

Now, using (2.10, c) of [14], III, the proof of (0.1) carries over to prove (6.2).

Sketch of the proof of 6.3 Using [9], 4.6 and [14], II, 1.10 and 2.4, a, we can assume that $f' : X' \to Y'$ is semi-stable in codimension one. Hence, by (6.2) it is enough to show that $det(f_{0*}\omega^{\mu}_{X_0/Y_0})$ is ample with respect to $(Y', det(f'_*\omega^{\mu}_{X'/Y'}))$.

If one forgets about the compactification, i.e. if $Y = Y_0$, (6.3) is obtained in (4.5) using the ampleness of λ_ν on C_h. However [14], I, §2 and §4 contain a direct proof of the ampleness of $det(f_{0*}\omega^{\nu}_{X_0/Y_0})$ in that case, parallel to methods from "Geometric Invariant Theory". The only necessary modification is that in §4 of [14], I, one uses the ampleness of $\mathcal{L}_0$ and (6.2) to get the ampleness of $\pi^* det(\mathcal{E})$.

However, since in (6.3) we want to allow degenerate fibres, one has to modify the arguments used to prove [14], II, 5.2. The necessary changes are more difficult to explain:

The "Ampleness Criterion" [14], II, 5.7, should be applied to the multiplication maps

$$S^{\mu}(f_{0*}\omega^{\nu}_{X_0/Y_0}) \to f_{0*}\omega^{\nu\cdot\mu}_{X_0/Y_0}$$

and
$$S^{\mu}(f'_*\omega^{\nu}_{X'/Y'}) \to f'_*\omega^{\nu\cdot\mu}_{X'/Y'},$$

hence for $s = 1$, $T_1 = S^{\mu}$ and $\mathcal{F}_0^{(1)} = f_*\omega^{\nu}_{X_0/Y_0}$ in the notation of [14], II, 5.6. Using the notations from the proof of [14], II, 5.7 we have to consider

$$\mathcal{F} = \delta_*(f'_*\omega^{\nu}_{X'/Y'}) \cap j_*(f_{0*}\omega^{\nu}_{X_0/Y_0})$$

and $$Q = \delta_*(f'_*\omega^{\nu\cdot\mu}_{X'/Y'}) \cap j_*(f_{0*}\omega^{\nu\cdot\mu}_{X'_0/Y'_0})$$

and we may assume that both sheaves are locally free. On some blowing up $\mathbb{P}'$ of $\mathbb{P} = \mathbb{P}(\mathcal{F})$ we found an effective divisor E, not meeting $\mathbb{P}_0$, such that

$$\pi^{'*}(det(Q)^a \otimes det(\mathcal{F})^{-a}) \otimes \mathcal{O}_{\mathbf{P}'}(E)|_{\mathbf{P}'-\tau^*(D)}$$

is ample on $\mathbb{P}' - \tau^*(D)$.

Since $det(\mathcal{F})$ is weakly positive over Y_0, $\pi^{'*}det(Q)^a$ will be ample over $\mathbb{P}_0 - D$. From (6.2) applied to be the pullback families over $\mathbb{P}' - \tau^* D$ we find that

$$\pi^{'*}(det(\mathcal{F}))^{a'} \otimes \mathcal{O}_{\mathbf{P}}(E')|_{\mathbf{P}'-\tau^*(D)}$$

will again be ample for some E' and $a' > 0$. As in [14], II, p 220, we will get for $\alpha >> 0$ and for an effective divisor D' supported in $\tau^* D$ that

$$\pi^{'*}(det(\mathcal{F}))^{\alpha} \otimes \mathcal{O}_{\mathbf{P}'}(E' + D')$$

is ample. As in [14], I, 4.7 one can descend this to obtain the ampleness of $det(\mathcal{F})$.
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