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Characteristic foliation on a hypersurface of general type in a projective symplectic manifold

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Abstract

A foliation on a non-singular projective variety is algebraically integrable if all leaves are algebraic subvarieties. A non-singular hypersurface X in a non-singular projective variety M equipped with a symplectic form has a naturally defined foliation, called the characteristic foliation on X. We show that if X is of general type and $\dim M \geqslant 4$, then the characteristic foliation on X cannot be algebraically integrable. This is a consequence of a more general result on Iitaka dimensions of certain invertible sheaves associated with algebraically integrable foliations by curves. The latter is proved using the positivity of direct image sheaves associated to families of curves.

1. Introduction

An important question in classical mechanics is whether the orbit of the motion of a celestial body is periodic. In Hamiltonian formalism, this question is formulated in terms of symplectic geometry as follows. Let (M, ω) be a symplectic manifold. Given a non-singular hypersurface $X \subset M$, the restriction of ω to the tangent space of X at each point $x \in X$ has one-dimensional kernel, defining a foliation of rank one which we will call the *characteristic foliation on* X induced by ω. The question about periodicity of orbits corresponds to the following geometric problem.

Question 1.1. Given a symplectic manifold (M, ω) and a hypersurface $X \subset M$, when are the leaves of the characteristic foliation on X compact?

In Hamiltonian mechanics, M is a *real* symplectic manifold corresponding to the phase space of the mechanical system, and X corresponds to the level set of the energy, which is a *real* hypersurface in M. It is interesting that Question 1.1 makes perfect sense in the setting of complex geometry, where M is a *holomorphic* symplectic manifold and X is a *complex* hypersurface. Recall that a holomorphic symplectic manifold is a complex manifold M equipped with a closed holomorphic 2-form $\omega \in H^0(M, \Omega^2_M)$ such that $\omega^n \in H^0(M, K_M)$ is nowhere vanishing. The holomorphic version of Question 1.1 was studied in [HO09], and an example for which it has an affirmative answer was examined in detail in that paper.

In the introduction of [HO09], the authors wrote that they expect the answer to Question 1.1 to be negative for a 'general' hypersurface X. The aim of this paper is to verify this expected

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result in the case where M is a non-singular projective variety over $\mathbb{C}$. For a holomorphic foliation of rank one on a projective variety, compactness of the leaf is equivalent to algebraicity of the leaf. Let us say that a holomorphic foliation on an algebraic variety is an *algebraically integrable foliation* if all of its leaves are algebraic subvarieties. Our main result is the following.

Theorem 1.2. *Let M be a non-singular projective variety of dimension at least 4 with a symplectic form ω. Let $X \subset M$ be a non-singular hypersurface of general type. Then the characteristic foliation on X induced by ω cannot be algebraically integrable.*

This result applies to non-singular ample (or nef and big) hypersurfaces $X \subset M$, because $K_M \cong \mathcal{O}_M$ via the symplectic form. Theorem 1.2 is a direct consequence of the following more general result.

Theorem 1.3. *Let X be a non-singular projective variety of dimension at least 2, and let*

$$0 \longrightarrow \mathcal{Q} \longrightarrow \Omega^1_X \longrightarrow \mathcal{F} \longrightarrow 0$$

be a foliation on X all of whose leaves are algebraic curves. If $\mathcal{F}$ is big, i.e. if its Iitaka dimension $\kappa(\mathcal{F})$ is equal to $\dim(X)$, then $\kappa(\det(\mathcal{Q})) = \dim(X) - 1$.

In the setting of Theorem 1.2, assume that the foliation is algebraically integrable and apply Theorem 1.3. By the definition of the characteristic foliation, ω induces a symplectic form on $\mathcal{Q}$, hence $\det(\mathcal{Q}) \cong \mathcal{O}_X$. Since X is of general type, this implies that $\mathcal{F}$ is big. This yields a contradiction:

$$0 = \kappa(\mathcal{O}_X) = \kappa(\det(\mathcal{Q})) = \dim(X) - 1.$$

Thus Theorem 1.2 is a consequence of Theorem 1.3. Conversely, $X \subset M$ as in Theorem 1.2 shows that the condition in Theorem 1.3 that all leaves be algebraic is necessary.

To prove Theorem 1.3, we first develop some general structure theory for algebraically integrable foliations. In particular, we will prove an étale version of the classic Reeb stability theorem from foliation theory. Furthermore, for algebraically integrable foliations by curves, we will prove a global version of this result. Using this general structure theorem, the relation between Iitaka dimensions will be obtained by borrowing a result from the theory of positivity of direct image sheaves associated to families of curves [Vie01]. This latter theory originated from the study of the Shafarevich conjectures over function fields and properties of sheaves on fine moduli spaces of curves. It is interesting to observe that these questions of modern algebraic geometry are related to the question of periodicity of motions of celestial bodies.

By the decomposition theorem of [Bea83], non-singular projective varieties with symplectic forms are, up to finite étale cover, products of abelian varieties and projective hyperkähler manifolds. As far as we know, Theorem 1.2 is new even for abelian varieties. In the simplest case, we can formulate the result explicitly as follows.

Corollary 1.4. *Let $A = \mathbb{C}^{2n}/\Lambda$ be an even-dimensional principally polarized abelian variety with smooth theta divisor. Fix any linear coordinate $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ on $\mathbb{C}^{2n}$, and let $\theta(p_1, \ldots, p_n, q_1, \ldots, q_n)$ be the Riemann theta function on $\mathbb{C}^{2n}$ associated to the period Λ. For a very general (i.e. outside a countable union of proper subvarieties) point $(a_1, \ldots, a_n, b_1, \ldots, b_n)$ on the theta divisor, the solution $(p_i(t), q_i(t))$ of the Hamiltonian flow on $\mathbb{C}^{2n}$,*

$$\frac{dp_i}{dt} = -\frac{\partial\theta}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial\theta}{\partial p_i} \quad \text{for } i = 1, \ldots, n,$$

with initial value $(p_i(0), q_i(0)) = (a_i, b_i)$, $i = 1, \ldots, n$, cannot descend to an algebraic curve on A.

In fact, the symplectic form $dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$ on $\mathbb{C}^{2n}$ descends to a symplectic form on the abelian variety A. The zero locus of the Riemann theta function descends to the theta divisor on A, which is ample. The solution of the Hamiltonian flow with initial value at a point on the theta divisor is exactly the leaf of the characteristic foliation on the theta divisor through that point. Thus the above is a direct consequence of Theorem 1.2.

It is natural to ask whether at least *some* leaf of the characteristic foliation in Theorem 1.2 can be an algebraic curve. This question is completely beyond the reach of the methods employed in the present paper. We cannot even make a guess as to whether the answer would be affirmative or not.

2. Étale Reeb stability for algebraically integrable foliations

Let X be a non-singular projective variety over $\mathbb{C}$. A foliation on X is given by an exact sequence of locally free sheaves

$$0 \longrightarrow \mathcal{Q} \longrightarrow \Omega^1_X \longrightarrow \mathcal{F} \longrightarrow 0$$

where $\mathcal{Q}$ is integrable. The integrability of $\mathcal{Q}$ is equivalent to saying that through each point $x \in X$, there exists a complex submanifold C such that $\mathcal{Q}$ corresponds to the conormal bundle of C at every point of C. This submanifold C is called the leaf of the foliation through x. We say that the foliation is algebraically integrable if each leaf is an algebraic subvariety of X. Our aim in this section is to describe the behavior of the leaves of an algebraically integrable foliation as a family of algebraic subvarieties. For this purpose, we need to recall some standard results on the structure of differentiable foliations.

Let X be a differentiable manifold with a differentiable foliation. A *transversal section* at a point $x \in X$ means a (not necessarily closed) submanifold S through x whose dimension is equal to the codimension of the leaves such that the intersection of each leaf of the foliation with S is transversal (or empty). Let C be the leaf through x. A choice of a transversal section S at x determines a group homomorphism

$$\pi_1(C, x) \longrightarrow \mathrm{Diff}_x(S),$$

called the *holonomy homomorphism*, from the fundamental group of C to the group $\mathrm{Diff}_x(S)$ of germs of the diffeomorphisms of S at x. For a precise definition of this homomorphism, we refer the reader to [MM03, § 2.1]. Roughly speaking, a loop γ on C representing an element of $\pi_1(C, x)$ acts on S by moving a point $y \in S$ close to x along the leaf through y following γ. The image of the holonomy homomorphism will be called the *holonomy group* of the leaf C. The isomorphism class of this group depends only on C and is independent of the choice of x and S. The following is a well-known criterion for finiteness of the holonomy group.

PROPOSITION 2.1. *Let X be a differentiable manifold with a foliation all of whose leaves are compact. The holonomy group of a leaf C is finite if there exist a transversal section S at a point $x \in C$ and a fixed positive integer N such that the cardinality of the intersection of S with any leaf of the foliation is bounded by N.*

Proof. This assertion is contained in [Eps76, Theorem 4.2], and the proof can be found in [Eps76, § 7]. In fact, all we really need is the simple fact that if a finitely generated group G acts effectively on a set S such that the cardinality of each orbit is bounded by a positive integer N, then the group is finite. We recall the proof for the reader's convenience. Denote by S_r the permutation group of r points. Since G is finitely generated, there are only finitely many homomorphisms $G \to \mathrm{S}_r$. Let $H \subset G$ be the intersection of the kernels of all such homomorphisms for $r \leqslant N$.

Then G/H is finite. Since each orbit of G in S determines a group homomorphism $G \to \mathrm{S}_r$ for some $r \leqslant N$, H must act trivially on S. Thus H is the trivial subgroup. It follows that G is finite. □

We recall the construction of the *flat bundle foliation* in [MM03, p. 17]. Let G be a finite group which acts freely on a manifold $\tilde{C}$ on the right. Suppose G acts effectively on another manifold S on the left, with a fixed point $x \in S$. Let $\tilde{C} \times_G S$ be the quotient of $\tilde{C} \times S$ by the equivalence relation $(yg, s) \sim (y, gs)$ for $g \in G$ and $(y, s) \in \tilde{C} \times S$. Let $C \subset \tilde{C} \times_G S$ be the image of $\tilde{C} \times \{x\}$. We have the following commutative diagram.

$$\begin{array}{ccc} \tilde{C} \times S & \longrightarrow & \tilde{C} \times_G S \\ \big\downarrow & & \big\downarrow \\ S & \longrightarrow & G \backslash S \end{array} \tag{2.1}$$

The foliation on the manifold $\tilde{C} \times_G S$ given by the vertical fibers is called the *flat bundle foliation* arising from the actions of G on $\tilde{C}$ and S. Note that C is a leaf of this foliation and G is the holonomy group of the leaf C. For any $y \in \tilde{C}$, the image of $\{y\} \times S$ gives a transversal section of this foliation at the image $\bar{y} \in C$. Then $\tilde{C}$ is the G-Galois cover of C associated to the holonomy homomorphism $\pi_1(C, \bar{y}) \to G$. The following is easy to check.

Lemma 2.2. *Let $S' \subset \tilde{C} \times_G S$ be a closed submanifold with $S' \cap C =: \{y\}$ that is a transversal section of the flat bundle foliation at $y \in C$. Then (2.1) factors through the (set-theoretic) fiber product of $S' \to G\backslash S$ and $\tilde{C} \times_G S \to G\backslash S$ via a finite map*

$$\tilde{C} \times S \longrightarrow S' \times_{G\backslash S} (\tilde{C} \times_G S)$$

which is one-to-one over a dense open subset.

We say that a subset of a manifold with a foliation is *saturated* if it is the union of leaves intersecting it. The next result is the classic Reeb stability theorem, whose proof can be found in [MM03, Theorem 2.9].

Theorem 2.3. *For a differentiable manifold X with a foliation, suppose that C is a compact leaf with finite holonomy group G. Then there exist a saturated open neighborhood U of C in X and a transversal section S in U such that, with G denoting the finite holonomy group acting on S and $\tilde{C} \to C$ denoting the G-Galois covering, there exists a diffeomorphism $\tilde{C} \times_G S \cong U$ such that the foliation on U corresponds to the flat bundle foliation on $\tilde{C} \times_G S$.*

Now assume that X is a complex manifold with a holomorphic foliation. Then we have the following holomorphic version of Reeb stability.

Theorem 2.4. *For a complex manifold X with a holomorphic foliation, suppose that C is a compact leaf with finite holonomy group G. Then there exist a saturated open neighborhood U of C in X, a holomorphic transversal section S in U with a G-action, an unramified G-Galois cover $\tilde{U} \to U$, a smooth proper morphism $h : \tilde{U} \to S$ and a proper morphism $g : U \to G\backslash S$ that satisfy the following commutative diagram.*

$$\begin{array}{ccc} \tilde{U} & \longrightarrow & U \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ S & \longrightarrow & G \backslash S \end{array} \tag{2.2}$$

Moreover, for each closed submanifold $\Sigma \subset U$ which intersects all leaves transversally, the normalization of the fiber product $\Sigma \times_{G\backslash S} U$ is an unramified cover of U.

Proof. We can apply Theorem 2.3. The diagram (2.2) is just (2.1), where $U = \tilde{C} \times_G S$, $\tilde{U} = \tilde{C} \times S$ and $\tilde{U}$ is given the complex structure as an unramified covering of U. The differentiable maps h and g are holomorphic maps because the foliation is holomorphic. The last statement is a consequence of Lemma 2.2. □

We can apply this theorem to algebraically integrable foliations as follows.

PROPOSITION 2.5. *Let X be a non-singular projective variety with an algebraically integrable foliation. Then each leaf has finite holonomy group. Denoting by Chow_X the Chow variety of X, there exists a natural morphism $\mu : X \to \mathrm{Chow}_X$ sending all points on a leaf C with holonomy group G_C to the cycle $|G_C| \cdot C$.*

Proof. Given a leaf C and a point $x \in C$, we can find a complete intersection S of very ample hypersurfaces which intersects C transversally with $x \in C \cap S$. In an analytic neighborhood of C, a component of $S \cap U$ is a transversal section. By Noetherian induction, the intersection number of each leaf of the foliation with S is bounded by a positive number N. Thus we can apply Proposition 2.1 to conclude that the holonomy group is finite. The fact that the cycles $|G_C| \cdot C$ form a nice family follows from the local description of the family of leaves in Theorem 2.4. □

The above morphism μ is a special case of a more general construction in [Gom89, Theorem 3].

For the next theorem, we need the following lemma.

LEMMA 2.6. *In the setting of Proposition 2.5, let $\Sigma \subset X$ be a subvariety such that*

$$\nu := \mu|_\Sigma : \Sigma \longrightarrow \mu(X)$$

is a finite morphism. Let $M_0 \subset \mu(X)$ be a connected analytic open subset such that for each point $y \in M_0$, the reduction of the fiber $\mu^{-1}(y)_{\mathrm{red}}$ intersects Σ transversally. Set $X_0 := \mu^{-1}(M_0)$ and $\Sigma_0 := \nu^{-1}(M_0)$. Then the normalization of the fiber product of $\mu_0 : X_0 \to M_0$ and $\nu_0 : \Sigma_0 \to M_0$ is an unramified covering of X_0.

Proof. The statement is local on $\mu(X)$, so we can verify it for any neighborhood of a given point $y \in \mu(X)$. In other words, we may assume that X_0 is contained in the neighborhood U of Theorem 2.4. The result is then immediate from the last statement in Theorem 2.4. □

The following is the étale version of the local Reeb stability theorem for an algebraically integrable foliation.

THEOREM 2.7. *Let X be a non-singular projective variety with an algebraically integrable foliation*

$$0 \longrightarrow \mathcal{Q} \longrightarrow \Omega^1_X \longrightarrow \mathcal{F} \longrightarrow 0. \tag{2.3}$$

Then, for each leaf $C \subset X$, there exists an étale neighborhood $\tau : U \to X$ of C, a smooth projective morphism $h : U \to M$ and isomorphisms

$$\tau^*\mathcal{Q} \cong h^*\Omega^1_M \quad \text{and} \quad \tau^*\mathcal{F} \cong \Omega^1_{U/M}$$

such that the pullback of (2.3) is isomorphic to the tautological exact sequence

$$0 \longrightarrow h^*\Omega^1_M \longrightarrow \Omega^1_U \longrightarrow \Omega^1_{U/M} \longrightarrow 0.$$

Proof. Just take a general complete intersection $\Sigma \subset X$ of very ample divisors intersecting C transversally. Then apply Lemma 2.6, taking M_0 to be the Zariski open subset where the reduction of fibers of μ intersect Σ transversally. The étale neighborhood U is given by the normalization of the fiber product $X_0 \times_{M_0} \Sigma_0$. The existence of the smooth morphism $U \to M$ follows from Theorem 2.4. □

3. Global étale Reeb stability for algebraically integrable foliations by curves

For algebraically integrable foliations by curves, we can globalize Theorem 2.7. The essential point is the existence of the moduli scheme M_g of curves of genus g, or of the moduli schemes $M_g^{[N]}$ of curves of genus g with a level-N structure, which are fine for $N \geqslant 3$.

Lemma 3.1. *Let $f : V \to W$ be a smooth morphism of curves and let $N \in \mathbb{N}$. Then there exists an étale finite morphism $\tilde{W} \to W$ such that*

$$\tilde{V} = V \times_W \tilde{W} \longrightarrow \tilde{W}$$

carries a level-N structure. In particular, for $N \geqslant 3$ the family $\tilde{V} \to \tilde{W}$ is the pullback of the universal family over the fine moduli scheme $M_g^{[N]}$ of curves with a level-N structure.

Proof. One can choose a level-N structure if the N-division points of the relative Jacobian are generated by sections. Since the N-division points of a family of abelian varieties are étale and finite over the base, this can be achieved over a finite étale cover. □

The global version of Theorem 2.7 is the following.

Theorem 3.2. *Let X be a non-singular projective variety with an algebraically integrable foliation*

$$0 \longrightarrow \mathcal{Q} \longrightarrow \Omega^1_X \longrightarrow \mathcal{F} \longrightarrow 0 \tag{3.1}$$

whose leaves are curves of genus 2 or greater. Then there exist a generically finite projective morphism $\sigma : V \to X$, a non-singular projective variety W, a smooth projective morphism $f : V \to W$ and injections

$$\sigma^*\mathcal{Q} \xrightarrow{\alpha} f^*\Omega^1_W \quad \text{and} \quad \sigma^*\mathcal{F} \xrightarrow{\cong} \Omega^1_{V/W}$$

such that the pullback of (3.1) is a subcomplex of the tautological exact sequence

$$0 \longrightarrow f^*\Omega^1_W \longrightarrow \Omega^1_V \longrightarrow \Omega^1_{V/W} \longrightarrow 0.$$

For each point $w \in W$ one can choose a neighborhood W_0 and an étale open neighborhood U of the image of $f^{-1}(w)$ in X, satisfying the condition in Theorem 2.7, such that the morphism $\sigma|_{f^{-1}(W_o)} : f^{-1}(W_o) \to X$ factors through U. In particular, $f : V \to W$ is a smooth family of curves. Moreover, we can assume that the associated classifying morphism $W \to M_g$ factors like

$$W \xrightarrow{\varphi'} M_g^{[N]}$$

for a given positive integer N.

Proof. Let us choose a finite set of étale neighborhoods as in Theorem 2.7, say $\tau_i : U_i \to X$ for $i \in \{1, \ldots, \ell\}$, such that

$$\bigcup_{i=1}^{\ell} \tau_i(U_i) = X. \tag{3.2}$$

The families $h_i : U_i \to M_i$ induce morphisms $\phi_i : M_i \to \mathrm{Chow}_X$ such that

$$\begin{array}{ccc} U_i & \xrightarrow{\tau_i} & X \\ {\scriptstyle h_i}\downarrow & & \downarrow{\scriptstyle \mu} \\ M_i & \xrightarrow{\phi_i} & \mathrm{Chow}_X \end{array}$$

where μ is the morphism in Proposition 2.5.

Next, we fix some projective compactification $\bar{M}_i$ such that ϕ_i extends to a morphism $\bar{\phi}_i : \bar{M}_i \to \mathrm{Chow}_X$. Replacing $\bar{M}_i$ by the Stein factorization, we may as well assume that $\bar{\phi}_i$ is finite. Let $\bar{M}'$ be an irreducible component of

$$\bar{M}_1 \times_{\mathrm{Chow}_X} \cdots \times_{\mathrm{Chow}_X} \bar{M}_\ell,$$

with induced morphism $\bar{\phi}' : \bar{M}' \to \mathrm{Chow}_X$. Let W be the normalization of $\bar{M}'$ in the Galois hull of the function field $\mathbb{C}(\bar{M}')$ over $\mathbb{C}(\bar{\phi}'(\bar{M}'))$. Hence, writing $\bar{\phi} : W \to \mathrm{Chow}_X$ for the induced morphism, there is a finite group G acting on W with quotient $\bar{\phi}(W)$. The condition (3.2) implies that

$$\bigcup_{i=1}^{\ell} \phi_i(M_i) = \bar{\phi}(W) = \mu(X).$$

Let $\tilde{M}_i$ denote the preimage of M_i under

$$\bar{M}' \subset \bar{M}_1 \times_{\mathrm{Chow}_X} \cdots \times_{\mathrm{Chow}_X} \bar{M}_\ell \xrightarrow{\mathrm{pr}_i} \bar{M}_i.$$

By pullback, there is a smooth projective morphism $\tilde{U}_i \to \tilde{M}_i$. For $\gamma \in G$, one obtains the pullback family

$$\tilde{U}_i^\gamma \longrightarrow \tilde{M}_i^\gamma := \gamma^{-1}(\tilde{M}_i).$$

For different i and i' and for γ and γ', the closed fibers of those families coincide on

$$\tilde{M}_i^\gamma \cap \tilde{M}_{i'}^{\gamma'}.$$

In fact, the isomorphism class of a fiber is determined by the image in X, hence it is invariant under G and independent of the étale neighborhood.

In particular, the morphisms $\tilde{U}_i^\gamma \to M_g$ mapping a point w to the moduli point of the isomorphism class of the fiber over w glue to a morphism $W \to M_g$. Replacing W by a finite covering, we may assume by Lemma 3.1 that this morphism factors through the fine moduli scheme $M_g^{[N]}$. Then the different families over the open subsets $\tilde{M}_i^\gamma$ are pullbacks of the universal family over $M_g^{[N]}$. Hence they coincide over the pairwise intersections and glue to a smooth family $V \to W$.

With an abuse of notation, we replace W by a desingularization and $f : V \to W$ by the pullback family. It satisfies all the required properties. □

4. Positivity property of algebraically integrable foliations by curves

Let us recall some notions of positivity for locally free sheaves.

Definition 4.1. Let $\mathcal{G}$ be a locally free sheaf on a quasi-projective variety Z, and let $Z_0 \subset Z$ be an open dense subvariety. Let $\mathcal{H}$ be an ample invertible sheaf on Z.

(i) $\mathcal{G}$ is *globally generated over* Z_0 if the natural morphism

$$H^0(Z,\mathcal{G})\otimes\mathcal{O}_Z\longrightarrow\mathcal{G}$$

is surjective over Z_0.

(ii) $\mathcal{G}$ is *ample with respect to* Z_0 if for some $k>0$, $S^k(\mathcal{G})\otimes\mathcal{H}^{-1}$ is globally generated over Z_0. In particular, $\mathcal{G}$ is *ample* if it is ample with respect to $Z_0=Z$.

(iii) $\mathcal{G}$ is *big* if it is ample with respect to some open dense subvariety Z_0. If $\mathrm{rk}(\mathcal{G})=1$, this is equivalent to saying that the Iitaka dimension $\kappa(\mathcal{G})$ is equal to $\dim Z$.

In the literature one finds a second definition of bigness of a locally free sheaf, which requires $\mathcal{O}_{\mathbb{P}(\mathcal{G})}(1)$ to be big on the projective bundle $\pi:\mathbb{P}(\mathcal{G})\to Z$ induced by $\mathcal{G}$. Our notion of bigness is stronger. It is equivalent to the ampleness of $\mathcal{O}_{\mathbb{P}(\mathcal{G})}(1)$ with respect to an open set of the form $\pi^{-1}(Z_0)$.

Lemma 4.2. *Let $\mathcal{G}$ be a locally free sheaf on a quasi-projective non-singular variety Z.*

(i) *If $\mathcal{G}$ is big, then for a locally free sheaf $\mathcal{G}'$ and a non-zero homomorphism $\eta:\mathcal{G}\to\mathcal{G}'$, the sheaf $\det(\eta(\mathcal{G}))$ is big.*

(ii) *If $\mathcal{G}$ is ample, then for any generically finite morphism $\rho:Y\to Z$, the pullback $\rho^*\mathcal{G}$ is big.*

The proof of the lemma is straightforward. Let us just point out that we define $\det(\eta(\mathcal{G}))$ to be $\iota_*\det(\eta(\mathcal{G}|_{Z'}))$, where $\iota:Z'\to Z$ is the largest open subscheme with $\eta(\mathcal{G}|_{Z'})$ locally free. The 'non-singular' assumption is needed for obtaining an invertible sheaf; without it, one would have to allow torsion-free coherent sheaves $\mathcal{G}$ in Definition 4.1, making the notation more complicated.

We will need the following result from [Vie01, Proposition 2.4].

Proposition 4.3. *Let $\mathfrak{f}:\mathcal{C}_g\to M_g^{[N]}$, $N\geqslant 3$, be the universal family over the fine moduli scheme of curves with level-N structures. Then $\mathfrak{f}_*\omega^{\nu}_{\mathcal{C}_g/M_g^{[N]}}$ is ample for all $\nu\geqslant 2$.*

Theorem 1.3 is a direct consequence of the following proposition.

Proposition 4.4. *In the setting of Theorem 3.2, let $v=\kappa(\mathcal{F})-1$. Then the following hold.*

(a) *We have $v=\mathrm{Var}(f)$, the dimension of the image of W in M_g.*

(b) *There is a subsheaf $\mathcal{V}\subset\sigma^*\mathcal{Q}$ of rank v with $\kappa(\det(\mathcal{V}))=v$.*

Proof. Fix $N\geqslant 3$. Since $M_g^{[N]}$ is a fine moduli scheme, $V\to W$ is the pullback of the universal family $\mathcal{C}_g\to M_g^{[N]}$ under the morphism $\varphi':W\to M_g^{[N]}$. Consider a factorization

$$W\xrightarrow{\varphi}Z\xrightarrow{\rho}M_g^{[N]},$$

with φ surjective and with connected fibres, such that ρ is generically finite. Upon blowing up and replacing the families with the pullbacks, we may assume that Z is non-singular. Let us write $g:T\to Z$ for the pullback of the universal family to Z. Then $V\cong T\times_Z W$ and the second projection defines a morphism $p:V\to T$.

Since $\omega_{V/W}=p^*\omega_{T/Z}$, one finds that

$$\kappa(\mathcal{F})=\kappa(\sigma^*\mathcal{F})=\kappa(\omega_{V/W})=\kappa(\omega_{T/Z}).$$

Now we assert the following.

Claim 4.5. *The invertible sheaf $\omega_{T/Z}$ is big.*

Since $\dim(Z) = \mathrm{Var}(f)$, Claim 4.5 shows that

$$v + 1 = \kappa(\omega_{T/Z}) = \dim(Z) + 1 = \mathrm{Var}(f) + 1,$$

which proves assertion (a) of Proposition 4.4.

Proof of Claim 4.5. From Proposition 4.3, $\mathfrak{f}_*\omega^2_{\mathcal{C}_g/M_g^{[N]}}$ is ample. Let $\mathcal{N}$ be an ample invertible sheaf on $M_g^{[N]}$. For some $k \gg 1$, the sheaf $\mathcal{N}^{-1} \otimes S^k(\mathfrak{f}_*\omega^2_{\mathcal{C}_g/M_g^{[N]}})$ is globally generated. Writing $\mathcal{H}$ for the pullback of $\mathcal{N}$ to Z, one finds by base change that $\mathcal{H}^{-1} \otimes S^k(g_*\omega^2_{T/Z})$ is globally generated. Using the multiplication map, one gets an inclusion

$$\bigoplus \mathcal{H}^\ell \hookrightarrow g_*\omega^{2\cdot\ell\cdot k}_{T/Z},$$

where $\bigoplus \mathcal{H}^\ell$ denotes the sum of $\mathrm{rk}(g_*\omega^{2\cdot\ell\cdot k}_{T/Z})$ copies of $\mathcal{H}^\ell$. So $h^0(T, \omega^{2\cdot\ell\cdot k}_{T/Z})$ is larger than a polynomial in ℓ of degree $\dim(Z) + 1$ with positive leading coefficient. □

To prove Proposition 4.4(b), recall that we have the Kodaira–Spencer homomorphism

$$\mathfrak{f}_*\omega^2_{\mathcal{C}_g/M_g^{[N]}} \longrightarrow \Omega^1_{M_g^{[N]}},$$

whose pullback

$$\eta : g_*\omega^2_{T/Z} \longrightarrow \Omega^1_Z$$

must be surjective over a Zariski open subset of Z because Z is generically finite over $M_g^{[N]}$. Let $\Omega \subset \Omega^1_Z$ be the image of the homomorphism η. Since $\mathfrak{f}_*\omega^2_{\mathcal{C}_g/M_g^{[N]}}$ is ample, $g_*\omega^2_{T/Z} = \rho^*\mathfrak{f}_*\omega^2_{\mathcal{C}_g/M_g^{[N]}}$ is big by Lemma 4.2(ii); hence $\det(\Omega)$ is big by Lemma 4.2(i). Thus

$$\kappa(\varphi^* \det(\Omega)) = \kappa(\det(\varphi^*\Omega)) = \kappa(\det(\Omega)) = v.$$

Let $\mathcal{V} := f^*\varphi^*\Omega$. Since $\det(\mathcal{V}) = f^*\varphi^*\det(\Omega)$, its Iitaka dimension is equal to $\dim(Z)$. Therefore, to prove (b), it suffices to verify the following.

Claim 4.6. *$\mathcal{V} = f^*\varphi^*\Omega$ is a subsheaf of $\sigma^*\mathcal{Q}$.*

Proof of Claim 4.6. One has a natural inclusion $\varphi^*\Omega^1_Z \to \Omega^1_W$, and hence $f^*\varphi^*\Omega \to f^*\Omega^1_W$. Showing that its image lies in the smaller sheaf $\sigma^*\mathcal{Q}$ is a local problem, so it will be sufficient to verify this in the neighborhood W_0 considered in Theorem 3.2. Over W_0, the morphism is the pullback of the morphism $h : U \to M$ in Theorem 2.7. So the morphism $W_0 \to M_g^{[N]}$ factors like $W_0 \to M \to M_g^{[N]}$ and the pullback of $\mathfrak{f}_*\omega^2_{\mathcal{C}_g/M_g^{[N]}}$ is $h_*\omega^2_{U/M}$, which is sent to Ω^1_M by the Kodaira–Spencer map. It follows that the pullback of Ω lies in $h^*\Omega^1_M = \tau^*\mathcal{Q}$. □

This completes the proof of Proposition 4.4. □

Proof of Theorem 1.3. Since $\mathcal{F}$ is big, general leaves have genus 2 or greater. By Theorem 2.3, every leaf has genus at least 2. Moreover, the number v in Proposition 4.4 is equal to $\dim(X) - 1$ and hence to $\mathrm{rank}(\mathcal{Q})$. So the subsheaf $\mathcal{V}$ in Proposition 4.4 has the same rank as $\sigma^*\mathcal{Q}$ and

$$\kappa(\det(\mathcal{Q})) \geqslant \kappa(\det(\mathcal{V})) = v.$$

Finally, to see that $\kappa(\det(\mathcal{Q})) < \dim(X)$, let C be a general leaf of the foliation. The restriction of $\mathcal{Q}$ to C corresponds to the conormal sheaf of C. Since C is general, the normal sheaf $\mathcal{Q}^\vee$

of C has global sections which generate the fiber of $\mathcal{Q}^\vee$ at a general point of C. It follows that $\det(\mathcal{Q})|_C$ cannot be ample. This implies that $\det(\mathcal{Q})$ is not big. □

References

Bea83 A. Beauville, *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Differential Geom. **18** (1983), 755–782.

Eps76 D. B. A. Epstein, *Foliations with all leaves compact*, Ann. Inst. Fourier **26** (1976), 265–282.

Gom89 X. Gomez-Mont, *Integrals for holomorphic foliations with singularities having all leaves compact*, Ann. Inst. Fourier **39** (1989), 451–458.

HO09 J.-M. Hwang and K. Oguiso, *Characteristic foliation on the discriminant hypersurface of a holomorphic Lagrangian fibration*, Amer. J. Math. **131** (2009), 981–1007.

MM03 I. Moerdijk and J. Mrčun, *Introduction to foliations and Lie groupoids*, Cambridge Studies in Advanced Mathematics, vol. 91 (Cambridge University Press, Cambridge, 2003).

Vie01 E. Viehweg, *Positivity of direct image sheaves and applications to families of higher dimensional manifolds,* in *Vanishing theorems and effective results in algebraic geometry*, ICTP Lecture Notes, vol. 6 (The Abdus Salam International Centre for Theoretical Physics, Trieste, 2001), 249–284.

Jun-Muk Hwang jmhwang@kias.re.kr
Korea Institute for Advanced Study, Hoegiro 87, Seoul 130-722, Korea

Eckart Viehweg viehweg@uni-due.de
Fakultät für Mathematik, Universität Duisburg-Essen, 45117 Essen, Germany