

On the Shafarevich conjecture for surfaces of general type over function fields

Egor Bedulev¹, Eckart Viehweg²

¹ Independent Moscow University, Moscow, Russia
(e-mail: bedulev@mccme.ru)

² Universität GH Essen, FB6 Mathematik, D-45117 Essen, Germany
(e-mail: viehweg@uni-essen.de)

Oblatum 10-V-1999 & 9-IX-1999 / Published online: 29 November 1999

Let B be a projective algebraic curve over an algebraically closed field k of characteristic zero, and let $S \subseteq B$ be a finite set of points. Shafarevich's conjecture for families of curves, proved by Parshin and Arakelov (see [2]), states that

- (I) There are only finitely many isomorphism classes of smooth non-isotrivial families of curves over $B - S$.
- (II) If $2q - 2 + \#S \leq 0$, then there are no such families.

Similar questions can be asked for smooth families of higher dimensional manifolds. (II) was recently verified for families of minimal surfaces of general type and for canonically polarized manifolds, by Migliorini [14], Kovács [11], [12], [13] and Zhang [18]. As a byproduct we reprove their result, but the reader familiar with their articles will recognize strong similarities between their and our approach.

Unfortunately our method does not work for families whose relative dualizing sheaf is big and numerically effective on the smooth fibers. In this case, for non-isotrivial families over \mathbb{P}^1 or over an elliptic curve, Kovács (Thesis, Utah, 1995, see [11]) showed the existence of at least one degenerate fibre.

In the higher dimensional case (I) might be too much to hope for. For fixed B and S there exist non-trivial deformations of families of abelian varieties over $B - S$ (see [6]). So (I) splits up in two questions: boundedness and rigidity.

To be more precise, let us fix some polynomial $h \in \mathbb{Q}[t]$. If $\deg(h) = 2$, we define M_h to be the moduli scheme of minimal surfaces S of general

type with $h(\mu) = \chi(\omega_S^\mu)$, or allowing singularities, the moduli scheme of canonically polarized normal surfaces with at most rational double points and with Hilbert polynomial h (see [17], for example).

If $\deg(h) > 2$, we denote by M_h the moduli scheme of canonically polarized manifolds, with Hilbert polynomial h . In both cases (I) should be replaced by two sub-problems:

(B) The non-trivial morphisms $B - S \rightarrow M_h$, which are induced by smooth projective maps $g_0 : Y_0 \rightarrow B - S$, are parameterized by some scheme of finite type.

(R) Under which additional conditions are the morphisms $B - S \rightarrow M_h$ in (B) rigid.

The corresponding questions, for abelian varieties and their moduli schemes, were solved by Faltings [6]. In this note we prove (B) for surfaces of general type, and for canonically polarized manifolds, in case $S = \emptyset$. The only obstruction to extend (B) to arbitrary families of canonically polarized manifolds, is the lack of a proof for the existence of relative minimal models for semi-stable families of such varieties over curves. We have nothing to contribute to problem (R).

For smooth families $g : Y \rightarrow B$ of surfaces, i.e. for $S = \emptyset$, (B) has been obtained by the first author in [3], using different methods.

This note benefited from discussions between the first author and A. Parshin and between the second author and Qi Zhang. We thank both of them, and K. Uguiso for remarks and comments.

1. Families of canonically polarized manifolds and of surfaces of general type

Let B be a curve and Y a variety of dimension $n + 1$, both non-singular, projective and defined over an algebraically closed field k of characteristic zero. Let $g : Y \rightarrow B$ be a flat morphism with connected fibers $Y_b = g^{-1}(b)$. Let $S \subset B$ be a finite set of points and $D = g^{-1}(S)$. Frequently we will denote the divisors

$$\sum_{s \in S} s \quad \text{and} \quad \sum_{s \in S} g^{-1}(s)$$

by S and D , as well.

We want to study both, families of surfaces of general type and of canonically polarized manifolds, and D will be supposed to be the set of bad fibers. To cover both cases we formulate the following assumption:

Assumption 1.1.

a) $g|_{Y-D} : Y - D \rightarrow B - S$ is smooth and $\omega_{Y/B}|_{Y-D}$ is relatively semi-ample, i.e. for some $\nu \gg 0$ the natural map

$$\iota_\nu : g^* g_* \omega_{Y/B}^\nu \rightarrow \omega_{Y/B}$$

is surjective over $Y - D$.

b) The fibers Y_b should be manifolds of general type and the v -canonical map should contract at most curves. Hence if

$$\pi_v : Y - D \rightarrow V \subseteq \mathbb{P}(g_*\omega_{Y/B}^v|_{B-S})$$

is the morphism, induced by ι_v , then π_v should be birational and the maximal fibre dimension of π_v should be one, for some $v \gg 0$, satisfying the condition a).

Recall from [4], § 2, or [17], § 5, that for an invertible sheaf \mathcal{L} on a normal projective variety F the integer $e(\mathcal{L})$ is defined to be the smallest positive integer e for which the multiplier sheaf $\omega_F \left\{ -\frac{D}{e} \right\}$ is isomorphic to ω_F , for all divisors D of global sections of \mathcal{L} . For $e(\mathcal{L})$ to exist, one has to require F to have at most rational singularities. Then, by [4], 2.3, or [17], 5.11, 5.12 and 5.21, $e(\mathcal{L})$ exists and in some cases there are explicit bounds:

Lemma 1.2. Assume F is projective with at most rational singularities and let \mathcal{L} be an ample invertible sheaf on F .

- a) If F is non-singular and \mathcal{L} very ample, then $e(\mathcal{L}) \leq c_1(\mathcal{L})^{\dim F} + 1$.
- b) In general, let \mathcal{L} be very ample and assume that there exists a desingularization $\sigma : F' \rightarrow F$ and an effective divisor E such that $\sigma^* \mathcal{L} \otimes \mathcal{O}_{F'}(-E)$ is very ample. Then $e(\mathcal{L}) \leq c_1(\mathcal{L})^{\dim F} + 1$.
- c) If F has rational Gorenstein singularities and if $Z = F \times \dots \times F$ and $\mathcal{M} = \otimes pr_i^* \mathcal{L}$, then $e(\mathcal{M}) = e(\mathcal{L})$.

Recall that $g : Y \rightarrow B$ is called isotrivial, if for some variety F , defined over k , there is a birational map $Y \times_B \overline{k(B)} \rightarrow F \times_k \overline{k(B)}$.

In [17], 2.7 and 2.9, we defined a locally free sheaf \mathcal{G} on B to be numerically effective (or nef), if for all $\mu > 0$ and for a point $p \in B$ the sheaf $S^\mu(\mathcal{G}) \otimes \mathcal{O}_B(p)$ is ample. We will use:

Proposition 1.3.

- a) If $g : Y \rightarrow B$ is a projective morphism between non-singular varieties, then $g_*\omega_{Y/B}^v$ is nef, for all $v > 0$.
- b) If g satisfies the assumptions made in 1.1, a), if the general fibre of g is of general type and if g is non-isotrivial, then $\kappa(\omega_{Y/B}) = n + 1$ and $\det(g_*\omega_{Y/B}^\eta)$ is ample, provided $g_*\omega_{Y/B}^\eta \neq 0$ and $\eta > 1$.

a) is a special case of [15], Theorem III, and b) can be found in [16], Theorem II. In fact, there the ampleness of $\det(g_*\omega_{Y/B}^\eta)$ is shown for some η , but as in [4], 3.1, or [17] this implies that $g_*\omega_{Y/B}^\eta$ is ample for all $\eta > 1$.

Theorem 1.4. Let $g : Y \rightarrow B$ be a non-isotrivial morphism, satisfying the assumptions made in 1.1. Let us fix some $v > 1$, such that 1.1 a) and b) hold true. We write

- $q = g(B)$ for the genus of B .
- $s = \#S$ for the number of degenerate fibers.
- $e(v) = e(\omega_F^v)$ for some general fibre F of g .
- $r(v) = \text{rank}(g_*\omega_{Y/B}^v)$.

Then one has:

- a) $2q - 2 + s > 0$.
- b) If $g : Y \rightarrow B$ is semi-stable, then

$$n \cdot (2q - 2 + s) \cdot v \cdot e(v) \cdot r(v) \geq \text{deg}(g_*\omega_{Y/B}^v).$$

c) In general

$$(n \cdot (2q - 2 + s) + s) \cdot v \cdot e(v) \cdot r(v) \geq \text{deg}(g_*\omega_{Y/B}^v).$$

a) will follow from 1.3 and b). This part of theorem 1.4 is due to Migliorini [14] and Kovács [11], [12], [13]. Qi Zhang [18] gave an elegant proof in case B is an elliptic curve. His proof easily extends to $B = \mathbb{P}^1$, as he and the second author found out discussing his result. In this note we take up their approach, together with positivity properties of direct image sheaves, as stated in [4].

Remark 1.5. The constants v and $r(v)$ are determined, using Matsusaka's big theorem, by the Hilbert polynomial of the fibers Y_b for $b \in B - S$. For canonically polarized manifolds, $e(v) \leq v^n \cdot c_1(\omega_F)^n + 1$, as we have seen in 1.2, a).

For surfaces of general type, the canonical model \tilde{F} of F has $A - D - E$ singularities, and the number of (-2) -curves on F is bounded by $\dim H^1(F, \Omega_F^1)$, a number which is constant in families. One can use 1.2, b) to bound $e(\omega_F^v) = e(\omega_{\tilde{F}}^v)$.

Such a bound exists for different reasons. By [17], the moduli space of normal canonically polarized surfaces with at most rational double points and with given Hilbert polynomial is quasi-projective, in particular it has only finitely many irreducible components. Moreover, Kollár [9] (see also [17], 9.25) constructed a finite covering Z of the moduli scheme, together with a "universal family". By [17], 5.17, $e(\omega_{\tilde{F}}^v)$ is bounded on each irreducible component of Z . Hence there exists some constant e depending on h with $e(\omega_{\tilde{F}}^v) \leq e$, for all surfaces F with Hilbert polynomial h .

2. Ampleness and vanishing theorems

Let X be a projective manifold, $U \subset X$ an open dense submanifold, and let \mathcal{L} be an invertible sheaf on X .

Definition 2.1.

a) \mathcal{L} is semi-ample with respect to U if for some $\eta > 0$

$$\iota_\eta : H^0(X, \mathcal{L}^\eta) \otimes_k \mathcal{O}_X \longrightarrow \mathcal{L}^\eta$$

is surjective over U .

b) \mathcal{L} is ℓ -ample and big with respect to U , if one can choose η in a) such that the morphism

$$\Psi_\eta : U \rightarrow V \subset \mathbb{P}(H^0(X, \mathcal{L}^\eta))$$

induced by ι_η is proper, birational and $\text{Max}\{\dim \Psi_\eta^{-1}(v); v \in V\} \leq \ell$.

A slight modification of the argument used to prove 6.4 in [5], yields a generalization of the Akizuki-Kodaira-Nakano vanishing theorem, similar to the one used in the proof of [12], 1.1 and [13], 1.1.

Proposition 2.2. *Let \mathcal{L} be ℓ -ample and big with respect to U . Assume that for η as in 2.1, b), the image of V of Ψ_η allows a projective flat morphism $\gamma : V \rightarrow W$ to a non-singular affine variety W .*

Then there exists a blowing up $\tau : X' \rightarrow X$ with centers in $\Delta = X - U$, such that $\Delta' = \tau^{-1}(X - U)$ is a normal crossing divisor, and an effective divisor Γ' with $\Gamma'_{\text{red}} \leq \Delta'$, such that for all numerically effective invertible sheaves \mathcal{N} and for $p + q < \dim X - \text{Max}\{0, \ell - 1\}$,

$$H^p(X', \Omega_{X'}^q(\log \Delta') \otimes \tau^*(\mathcal{L}^{-1} \otimes \mathcal{N}^{-1}) \otimes \mathcal{O}_{X'}(\Gamma')) = 0.$$

Proof. If $\Delta = \emptyset$, this is [5], 6.6. Hence we will assume that $\Delta \neq \emptyset$, and fix some η for which the assumption 2.1 b) on Ψ_η holds true. Let $\tau : X' \rightarrow X$ be a blowing up, such that X' is non-singular, $\Delta' = \tau^{-1}(\Delta)$ a normal crossing divisor and such that, for $\mathcal{L}' = \tau^*\mathcal{L}$, the image of

$$\iota'_\eta : H^0(X', \mathcal{L}'^\eta) \otimes_k \mathcal{O}_{X'} \longrightarrow \mathcal{L}'^\eta$$

is an invertible sheaf, equal to \mathcal{L}'^η over $U' = \tau^{-1}(U)$.

So $\text{Im}(\iota'_\eta) = \mathcal{L}'^\eta(-\Gamma_1)$, for some divisor Γ_1 , supported in Δ' . Let

$$\Psi' : X' \rightarrow Z \subset \mathbb{P}(H^0(X', \mathcal{L}'^\eta(-\Gamma_1)))$$

be the induced morphism. $\Psi'|_{U'}$ is a proper morphism with image V . Let I be the ideal sheaf of $Z - V$ and $J = \Psi'^*I / \text{torsion}$. Blowing up again, we may assume that $J = \mathcal{O}(-\Gamma_2)$ for some divisor Γ_2 with $(\Gamma_2)_{\text{red}} = \Delta'_{\text{red}}$. For all $\mu \gg 0$ the sheaf $\mathcal{L}'^{\mu \cdot \eta}(-\mu \cdot \Gamma_1 - \Gamma_2)$ will be generated by global sections. For some $\mu \gg 0$, the number $\mu \cdot \eta$ will not divide the multiplicity of any of the components of $\mu \cdot \Gamma_1 + \Gamma_2$. Allowing Δ' to have multiplicities, we thereby got to the following situation:

Assumption 2.3. For some η and some normal crossing divisor Δ' with

$$\Delta'_{\text{red}} = X' - U' = \left(\Delta' - \eta \cdot \left[\frac{\Delta'}{\eta} \right] \right)_{\text{red}},$$

the sheaf $\mathcal{L}'^\eta(-\Delta') = \mathcal{L}'^\eta \otimes \mathcal{O}_{X'}(-\Delta')$ is generated by global sections, the induced morphism $\Psi' : X' \rightarrow Z$ is birational and the fiber-dimension of $U' = \Psi'^{-1}(V) \rightarrow V$ is at most ℓ . Moreover there exists a projective flat morphism $\gamma : V \rightarrow W$ to a non-singular affine variety W . Let \mathcal{N}' be any numerically effective invertible sheaf on X' and choose $[\frac{\Delta'}{\eta}] = \Gamma'$.

We will prove, by induction on $\dim X' - \dim W$, that the assumptions 2.3 imply

$$(2.3.1) \quad H^p(X', \Omega_{X'}^q(\log \Delta') \otimes \mathcal{L}'^{-1} \otimes \mathcal{N}'^{-1} \otimes \mathcal{O}_{X'}(\Gamma')) = 0$$

for $p + q < \dim X - \text{Max}\{0, \ell - 1\}$.

If $\dim X' = \dim V > \dim W$ we may replace η by $\mu \cdot \eta$ and Δ' by $\mu \cdot \Delta'$ for $\mu \gg 0$. Thereby we are allowed to assume that $(\mathcal{L}' \otimes \mathcal{N}')^\eta \otimes \mathcal{O}_{X'}(-\Delta')$ is generated by global sections, and that for the zero divisor H of a general section of this sheaf $\Psi'(H) \cap V \rightarrow W$ is again projective and flat. Moreover $\Delta' + H$ is a normal crossing divisor and $\Gamma'|_H = [\frac{\Delta'|_H}{\eta}]$. Therefore $H, \mathcal{L}'|_H$ and $\Delta'|_H$ satisfy again the assumptions 2.3. If $\dim V = \dim W$, we choose $H = 0$. In both cases, the morphism

$$V - \Psi'(H) \longrightarrow W$$

is affine and $V - \Psi'(H)$ is an affine variety. $(\mathcal{L}' \otimes \mathcal{N}')^\eta$ has a section with zero divisor $\Delta' + H$. By [5], § 3,

$$(\mathcal{L}' \otimes \mathcal{N}')^{-1} \otimes \mathcal{O}_{X'}(\Gamma') = (\mathcal{L}' \otimes \mathcal{N}')^{-1} \otimes \mathcal{O}_{X'} \left(\left[\frac{\Delta' + H}{\eta} \right] \right)$$

has an integrable connection which satisfies the E_1 -degeneration. The assumption $\Delta'_{\text{red}} = (\Delta' - \eta \cdot [\frac{\Delta'}{\eta}])_{\text{red}}$ allows to apply [5], 4.12, and to obtain

$$H^p(X', \Omega_{X'}^q(\log (\Delta' + H')) \otimes (\mathcal{L}' \otimes \mathcal{N}')^{-1} \otimes \mathcal{O}_{X'}(\Gamma')) = 0,$$

for $p + q < n - \text{Max}\{0, \ell - 1\}$. If $\dim V = \dim W$, we are done. If $\dim V > \dim W$ one uses the exact sequence

$$0 \rightarrow \Omega_{X'}^p(\log \Delta') \rightarrow \Omega_{X'}^p(\log (\Delta' + H')) \rightarrow \Omega_{H'}^{p-1}(\log \Delta'|_{H'}) \rightarrow 0$$

to obtain 2.3.1 by induction. □

3. The proof of Theorem 1.4

We will assume that D is a normal crossing divisor. Enlarging S and D we may assume that $2q - 2 + s \geq 0$, hence that $\omega_B(S)$ is nef. If $2q - 2 + s = 0$, then either B is elliptic and g smooth, or $B = \mathbb{P}^1$ and $S = \{b_1, b_2\}$. In the second case, there exists a finite covering $\mathbb{P}^1 \rightarrow \mathbb{P}^1$, totally ramified in S , such that the pullback family has stable reduction. Altogether, part a) of 1.4 follows from 1.3 and part b) of 1.4.

If the fibers of $Y - D \rightarrow B - S$ are canonically polarized manifolds, the next proposition follows from [4], 2.4. Since we will use it for families of minimal models of surfaces of general type, as well, we will recall the proof.

Proposition 3.1. *Under the assumptions made in 1.4 let \mathcal{N} be an invertible sheaf on B with $\deg \mathcal{N} < \deg g_*\omega_{Y/B}^v$. Then the sheaf*

$$\omega_{Y/B}^{v \cdot e(v) \cdot r(v)} \otimes g^* \mathcal{N}^{-1}$$

is 1-ample and big with respect to $Y - D$.

Proof. Let $m : S^{r(v) \cdot e(v)}(g_*\omega_{Y/B}^v) \otimes \mathcal{N}^{-1} \rightarrow g_*\omega_{Y/B}^{v \cdot r(v) \cdot e(v)} \otimes \mathcal{N}^{-1}$ be the map given by the multiplication on the first factor and by the identity on \mathcal{N}^{-1} . The assumptions a) and b) in 1.1 imply that the composite

$$\begin{aligned} g^* S^{r(v) \cdot e(v)}(g_*\omega_{Y/B}^v) \otimes g^* \mathcal{N}^{-1} &\xrightarrow{g^* m} g^* g_*\omega_{Y/B}^{v \cdot r(v) \cdot e(v)} \otimes g^* \mathcal{N}^{-1} \\ &\longrightarrow \omega_{Y/B}^{v \cdot r(v) \cdot e(v)} \otimes g^* \mathcal{N}^{-1} \end{aligned}$$

is surjective over $Y - D$ and that the induced map

$$\pi : Y \longrightarrow \mathbb{P} = \mathbb{P}(S^{r(v) \cdot e(v)}(g_*\omega_{Y/B}^v) \otimes \mathcal{N}^{-1})$$

is birational, and its restriction to $Y - D$ regular with maximal fibre dimension one. Let $(\pi^* \mathcal{O}_{\mathbb{P}}(\mu))^\sim$ denote the reflexive hull of the pullback of $\mathcal{O}_{\mathbb{P}}(\mu)$. By construction the induced embedding

$$(\pi^* \mathcal{O}_{\mathbb{P}}(\mu))^\sim \longrightarrow \omega_{Y/B}^{\mu \cdot v \cdot r(v) \cdot e(v)} \otimes g^* \mathcal{N}^{-\mu}$$

is an isomorphism over $Y - D$. If $\mathcal{O}_{\mathbb{P}}(\mu)$ is very ample and $\mu' = \mu \cdot v \cdot r(v) \cdot e(v)$ the composite

$$\pi^* H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(\mu)) \otimes_k \mathcal{O}_Y \subset H^0(Y, \omega_{Y/B}^{\mu'} \otimes g^* \mathcal{N}^{-\mu}) \otimes_k \mathcal{O}_Y \longrightarrow \omega_{Y/B}^{\mu'} \otimes g^* \mathcal{N}^{-\mu}$$

is surjective over $Y - D$ and the induced maps factor like

$$(3.1.1) \quad \begin{array}{ccc} Y & \xrightarrow{\Psi} & \mathbb{P}(H^0(Y, \omega_{Y/B}^{\mu'} \otimes g^* \mathcal{N}^{-\mu})) \\ \pi \downarrow & & \downarrow \\ \mathbb{P} & \xrightarrow{\subset} & \mathbb{P}(H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(\mu))). \end{array}$$

In particular Ψ is birational. Since all the rational morphisms in (3.1.1) are regular on $Y - D$, the fibre dimension of $\Psi|_{Y-D}$ is smaller than the one of $\pi|_{Y-D}$, hence at most one. The proper morphism $Y - D \rightarrow B - S$ factors through $\Psi|_{Y-D}$ and the latter is proper as well.

To prove 3.1 it is thereby sufficient to show the ampleness of $\mathcal{O}_{\mathbb{P}}(1)$, or that

$$S^{r(v) \cdot e(v)}(g_*\omega_{Y/B}^v) \otimes \mathcal{N}^{-1}$$

is ample. By definition of nef, this holds true, if the sheaf

$$S^{r(v) \cdot e(v)}(g_*\omega_{Y/B}^v) \otimes \det(g_*\omega_{Y/B}^v)^{-1}$$

is nef. To this aim (see [17], 2.8) we can replace B by a covering, unramified in S , and Y by the pullback family. This allows to assume that $\det(g_*\omega_{Y/B}^v) = \mathcal{A}^{e(v)}$ for an invertible sheaf \mathcal{A} on B . For $r = r(v)$ let

$$f : X = Y \times_B \dots \times_B Y \longrightarrow B$$

be the r -fold fibre product, let $\sigma : X' \rightarrow X$ be a desingularization, and

$$f' = f \circ \sigma : X' \rightarrow B.$$

We write

$$\mathcal{M} = \sigma^*(\otimes_{i=1}^r pr_i^* \omega_{Y/B}).$$

The morphism f is Gorenstein and the general fibre is non-singular. Hence there are natural injective maps

$$(3.1.2) \quad \otimes^r g_*\omega_{Y/B}^v = f_* \otimes_{i=1}^r pr_i^* \omega_{Y/B}^v \longrightarrow f'_* \mathcal{M}^v$$

$$(3.1.3) \quad \text{and } f'_*(\mathcal{M}^{v-1} \otimes \omega_{X'/B}) \longrightarrow f'_*\omega_{X'/B}^v = \otimes^r g_*\omega_{Y/B}^v,$$

both isomorphisms on some open dense subset of B . (3.1.2) induces

$$\mathcal{A}^{e(v)} = \det(g_*\omega_{Y/B}^v) \longrightarrow \otimes^r g_*\omega_{Y/B}^v \longrightarrow f'_* \mathcal{M}^v,$$

hence a section of $\mathcal{M}^v \otimes f'^* \mathcal{A}^{-e(v)}$ with zero-divisor Γ . Blowing up X' , with centers in a finite number of fibers, we may assume that the image $\mathcal{M}^v \otimes J$ of

$$f'^* \otimes^r g_*\omega_{Y/B}^v \longrightarrow f'^* f'_* \mathcal{M}^v \longrightarrow \mathcal{M}^v$$

is invertible. The ideal sheaf J is trivial in a neighborhood of the general fibre. By 1.3 $g_*\omega_{Y/B}^v$ is nef, hence $\mathcal{M}^v \otimes J$ is nef, as well.

Let us write $\mathcal{L} = \mathcal{M}^{v-1} \otimes f'^* \mathcal{A}^{-1}$. Then

$$\mathcal{L}^{e(v) \cdot v}(-v \cdot \Gamma) = \mathcal{M}^{e(v) \cdot v \cdot (v-1)} \otimes f'^* \mathcal{A}^{-v \cdot e(v)} \otimes \mathcal{O}_{X'}(-v \cdot \Gamma) = \mathcal{M}^{e(v) \cdot v \cdot (v-1) - v^2}$$

contains a nef subsheaf, equal to $\mathcal{L}^{e(v) \cdot v}(-v \cdot \Gamma)$ in a neighborhood of the general fibre. Moreover, by 1.3, b),

$$\kappa(\mathcal{L}^{e(v) \cdot v}(-v \cdot \Gamma)) = \dim X'.$$

[4], 1.7 implies that the sheaf $f'_*(\mathcal{L} \otimes \omega_{X'/B} \{-\frac{\Gamma}{e(v)}\})$ is nef. By the definition of $e(v)$ and by 1.2, c), the natural inclusion

$$f'_*\left(\mathcal{L} \otimes \omega_{X'/B} \left\{-\frac{\Gamma}{e(v)}\right\}\right) \longrightarrow f'_*(\mathcal{L} \otimes \omega_{X'/B}) = f'_*(\mathcal{M}^{v-1} \otimes \omega_{X'/B}) \otimes \mathcal{A}^{-1}$$

is an isomorphism on some open dense subset of B . Using (3.1.3) one obtains a nef subsheaf of $(\otimes^r g_* \omega_{Y/B}^v) \otimes \mathcal{A}^{-1}$ of full rank, hence the latter is nef, as well. Then

$$S^{e(v)}(\otimes^r g_* \omega_{Y/B}^v) \otimes \mathcal{A}^{-e(v)} = S^{e(v)}(\otimes^r g_* \omega_{Y/B}^v) \otimes \det(g_* \omega_{Y/B}^v)^{-1}$$

as well as its quotient $S^{e(v) \cdot r}(g_* \omega_{Y/B}^v) \otimes \det(g_* \omega_{Y/B}^v)^{-1}$ are nef. □

Let us return to the proof of 1.4 b) and c). If g is semi-stable, i.e. if D is reduced, we choose $\delta = 0$, otherwise $\delta = 1$. Let us assume that 1.4 b) or c) are false. Hence

$$(3.1.4) \quad (n \cdot (2q - 2 + s) + \delta \cdot s) \cdot v \cdot e(v) \cdot r(v) < \deg(g_* \omega_{X/B}^v)$$

and 3.1 implies that for

$$\mathcal{A} = \omega_B(S)^n \otimes \mathcal{O}_B(\delta \cdot S)$$

the sheaf $\omega_{Y/B}^{v \cdot e(v) \cdot r(v)} \otimes \mathcal{A}^{-v \cdot e(v) \cdot r(v)}$, hence $\mathcal{L} = \omega_{Y/B} \otimes g^* \mathcal{A}^{-1}$, is 1-ample and big with respect to $Y - D$. Since we assumed $\omega_B(S)$ to be nef, 2.2 implies that:

Claim 3.2. There exists a blowing up $\tau : X \rightarrow Y$ with centers in D , such that $\Delta = \tau^* D$ is a normal crossing divisor, and there exists an effective divisor Γ , supported in Δ , with

$$H^p(X, \Omega_X^q(\log \Delta) \otimes \tau^*(\omega_{Y/B}^{-1} \otimes g^*(\omega_B(S)^{n-m} \otimes \mathcal{O}_B(\delta \cdot S))) \otimes \mathcal{O}_X(\Gamma)) = 0$$

for $p + q < \dim X = n + 1$ and for all $m \geq 0$.

The morphism $f = g \circ \tau$ is smooth outside of Δ , and one has an exact sequence of locally free sheaves

$$(3.2.1) \quad 0 \rightarrow f^* \omega_B(S) \rightarrow \Omega_X^1(\log \Delta) \rightarrow \tilde{\Omega}_{X/B}^1 \rightarrow 0.$$

Comparing the determinants one finds

$$\tilde{\Omega}_{X/B}^n = \det(\tilde{\Omega}_{X/B}^1) = \omega_X(\Delta_{\text{red}}) \otimes f^* \omega_B(S)^{-1} = \omega_{X/B}(\Delta_{\text{red}} - \Delta).$$

Claim 3.3. $H^0(X, \tilde{\Omega}_{X/B}^n \otimes \tau^* \omega_{Y/B}^{-1} \otimes f^* \mathcal{O}_B(\delta \cdot S)) \neq 0.$

Proof. If $\delta = 1$, this holds true with $\tau^*\omega_{Y/B}^{-1}$ replaced by the smaller sheaf $\omega_{X/B}^{-1}$, since

$$\tilde{\Omega}_{X/B}^n \otimes \omega_{X/B}^{-1} \otimes f^*\mathcal{O}_B(S) = \mathcal{O}(\Delta_{\text{red}}).$$

If $\delta = 0$, i.e. if g is semi-stable, then $\Omega_{Y/B}^n = \omega_{Y/B}$. By [5], 3.21, one has an inclusion $\tau^*\Omega_Y^1(\log D) \rightarrow \Omega_X^1(\log \Delta)$, hence $\tau^*\omega_{Y/B} = \tau^*\tilde{\Omega}_{Y/B}^n \subset \tilde{\Omega}_{X/B}^n$. \square

The m -th wedge product, applied to the sequence (3.2.1) induces an exact sequence

$$(\Sigma_m^\bullet) \quad 0 \rightarrow f^*\omega_B(S) \otimes \tilde{\Omega}_{X/B}^{m-1} \rightarrow \Omega_X^m(\log \Delta) \rightarrow \tilde{\Omega}_{X/B}^m \rightarrow 0.$$

3.2 implies, that the $(n - m)$ -th cohomology of the sheaf in the middle of the sequence

$$\Sigma_m^\bullet \otimes \tau^*\omega_{Y/B}^{-1} \otimes f^*\omega_B^{n-m}((n - m + \delta) \cdot S) \otimes \mathcal{O}_X(\Gamma)$$

is zero, hence

$$H^{n-m}(X, \tilde{\Omega}_{X/B}^m \otimes \tau^*\omega_{Y/B}^{-1} \otimes f^*\omega_B^{n-m}((n - m + \delta) \cdot S) \otimes \mathcal{O}_X(\Gamma))$$

injects into

$$H^{n-m+1}(X, \tilde{\Omega}_{X/B}^{m-1} \otimes \tau^*\omega_{Y/B}^{-1} \otimes f^*\omega_B^{n-m+1}((n - m + 1 + \delta) \cdot S) \otimes \mathcal{O}_X(\Gamma)).$$

Altogether

$$H^0 = H^0(X, \tilde{\Omega}_{X/B}^n \otimes \tau^*\omega_{Y/B}^{-1} \otimes f^*\mathcal{O}_B(\delta \cdot S) \otimes \mathcal{O}_X(\Gamma))$$

is a subspace of

$$H^n = H^n(X, \tau^*\omega_{Y/B}^{-1} \otimes f^*\omega_B^n((n + \delta) \cdot S) \otimes \mathcal{O}_X(\Gamma)).$$

Using 3.2 again, one finds H^n and thereby H^0 to be zero. Since Γ is effective, this contradicts 3.3. Hence the inequality stated in (3.1.4) does not hold true, and we obtain 1.4 b) and c).

4. Moduli schemes and boundedness

Let \mathcal{M}_h denote the moduli functor of minimal surfaces of general type, if $\deg h = 2$, and of canonically polarized manifolds, if $\deg h > 2$, both times with fixed Hilbert polynomial $h \in \mathbb{Q}[t]$. The corresponding moduli scheme is denoted by M_h .

If $\deg h = 2$, i.e. in the surface case, Kollár and Shepherd-Barron [10] defined stable surfaces, Alexeev [1] proved that the index of the singularities is bounded in terms of the coefficients of the Hilbert polynomial, which

implies by [9] that M_h has a compactification \bar{M}_h , parameterizing families of stable surfaces (see also [17], section 9.6).

For $\text{deg } h \geq 2$, those results were generalized by Karu [7], assuming the minimal model conjecture for semi-stable families of n -folds over curves ($MMP(n + 1)$). Let again \bar{M}_h be the compactification and $\bar{\mathcal{M}}_h$ the corresponding moduli functor. The existence of \bar{M}_h implies that for some N_0 , depending on h , the reflexive hull $\omega_X^{[N_0]}$ of $\omega_X^{N_0}$ is invertible for all $x \in \bar{\mathcal{M}}_h(k)$.

In both cases, for some $\eta \gg 0$ the sheaf $\omega_X^{[\eta]}$ is very ample ($\text{deg } h > 2$) or semi-ample and the induced morphism the contraction of (-2) curves ($\text{deg } h = 2$).

Kollár [9] (see also [17]) has shown, that for $\eta \gg 0$ and for some $p > 0$ there exists a very ample invertible sheaf λ on \bar{M}_h such that:

For $f : X \rightarrow Z \in \bar{\mathcal{M}}_h(Z)$ and for the induced morphism $\varphi : Z \rightarrow \bar{M}_h$,

$$\varphi^* \lambda = \det \left(f_* \omega_{X/Z}^{[\eta]} \right)^p .$$

Corollary 4.1. *Assume $\text{deg } h = n = 2$. Let B be a projective non-singular curve, $S \subset B$ a finite subset, and let $g_0 : Y_0 \rightarrow B - S$ be a smooth projective morphism, whose fibers are surfaces of general type with Hilbert polynomial h . Let $\Phi : B \rightarrow \bar{M}_h$ be the induced morphism. Then $\text{deg } \Phi^* \lambda$ is bounded above by a constant, depending on h , $g(B)$ and $\#S$.*

Addendum 4.2. *Assuming $MMP(\text{deg } h + 1)$, 4.1 remains true for $\text{deg } h > 2$ and for $g_0 : Y_0 \rightarrow B - S$ a family of canonically polarized manifolds with Hilbert polynomial h .*

Proof. Choose a non-singular projective compactification Y of Y_0 such that g_0 extends to $g : Y \rightarrow B$. The assumptions 1.1 hold true for $v = \eta$. By 1.4 the degree of $\det(g_* \omega_{Y/B}^\eta)$ is smaller than a constant depending on h , $g(B)$ and $\#S$. There exists a finite covering $\gamma : C \rightarrow B$ and $f : X \rightarrow C \in \bar{\mathcal{M}}_h(C)$ which induces

$$C \xrightarrow{\gamma} B \xrightarrow{\Phi} \bar{M}_h .$$

We may assume, in addition, that f has a semi-stable model $f' : X' \rightarrow C$, with X' non-singular. By the definition of stable surfaces in [10] (or of stable canonically polarized varieties in [7]), $f_* \omega_{X/C}^{[\eta]} = f'_* \omega_{X'/C}^\eta$. [15], 3.2, gives an injective map $f'_* \omega_{X'/C}^\eta \rightarrow \gamma^* g_* \omega_{Y/B}^\eta$ which is an isomorphism over $\gamma^{-1}(B - S)$. Hence

$$\text{deg } \Phi^* \lambda = \frac{\text{deg} \left(f_* \omega_{X/C}^{[\eta]} \right)^p}{\text{deg } \gamma} \leq \text{deg} \left(g_* \omega_{Y/B}^\eta \right)^p$$

is bounded, as well. □

Let us denote by $\mathbf{H} = \mathbf{Hom}((B, B - S), (\bar{M}_h, M_h))$ the scheme parameterizing morphism $\Phi : B \rightarrow \bar{M}_h$ with $\Phi(B - S) \subset M_h$. Since λ is ample, 4.1 and 4.2 imply:

Corollary 4.3. *Under the assumptions made in 4.1 (or 4.2) there exists a subscheme $T \subset \mathbf{H}$, of finite type over k , which contains all points $[\Phi] \in \mathbf{H}$, induced by smooth morphisms*

$$g_0 : Y_0 \rightarrow B - S \in \mathcal{M}_h(B - S).$$

Without assuming the minimal model conjecture $MMP(\deg h + 1)$ 4.2 and 4.3 remain true, if $S = \emptyset$. In fact, the existence of the very ample sheaf λ on M_h has been shown in [17]. For the corresponding embedding $M_h \rightarrow \mathbb{P}^m$ choose \bar{M}_h to be the closure of M_h in \mathbb{P}^m . Then for complete curves B and for morphisms $B \rightarrow \bar{M}_h$ with image in M_h , the arguments used to prove 4.2 remain valid.

In [17], section 8.5, one finds the definition of a moduli functor $\mathcal{D}_h^{[N_0]}$ of canonically polarized normal varieties with canonical singularities of index N_0 . Kawamata [8] has shown that this moduli functor is locally closed. Unfortunately it is not known to be bounded, i.e. whether Matsusaka’s big theorem holds true.

As in [17], 1.20, one can enforce boundedness by considering the submoduli functor $\mathcal{D}_h^{[N_0](M)}$ with

$$\mathcal{D}_h^{[N_0](M)}(k) = \{X \in \mathcal{D}_h^{[N_0]}; \omega_X^{[N_0 \cdot M]} \text{ very ample}\}.$$

Then there exists a moduli scheme $\mathcal{D}_h^{[N_0](M)}$ for $\mathcal{D}_h^{[N_0](M)}$. For $\nu = N_0 \cdot M$ and some $p > 0$ there exists again a very ample invertible sheaf λ on $\mathcal{D}_h^{[N_0](M)}$ which induces

$$\det \left(f_* \omega_{X/Z}^{[\nu]} \right)^p \quad \text{for } f : X \rightarrow Z \in \mathcal{D}_h^{[N_0](M)}(Z).$$

Choosing \bar{M}_h again as the closure of $D_h^{[N_0](M)}$ for an embedding $\phi : D_h^{[N_0](M)} \rightarrow \mathbb{P}^m$ with $\phi^* \mathcal{O}_{\mathbb{P}^m}(1) = \lambda$, one obtains:

Corollary 4.4. *For $\deg h > 2$, there exists a subscheme $T \subset \mathbf{H}$, of finite type over k , which contains all points $[\Phi] \in \mathbf{H}$, induced by morphisms*

$$g : Y \rightarrow B \in \mathcal{D}_h^{[N_0](M)}(B),$$

smooth over $B - S$.

References

1. Alexeev, V.: Boundedness and K^2 for log surfaces. *Int. Journal Math.* **5** (1994) 779–810
2. Arakelov, A.: Families of algebraic curves with fixed degeneracies. *Izv. Ak. Nauk. S.S.S.R., ser. Math.* **35** (1971) [Math. U.S.S.R. *Izv.* **5** (1971) 1277–1302]
3. Bedulev, E.: On the Shafarevich conjecture for surfaces of general type over function fields. preprint, 1998, ICTP, Trieste
4. Esnault, H.; Viehweg, E.: Effective bounds for semipositive sheaves and for the height of points on curves over complex function fields. *Compositio Math.* **76** (1990) 69–85
5. Esnault, H.; Viehweg, E.: Lectures on Vanishing Theorems. *DMV Seminar* **20** (1992), Birkhäuser
6. Faltings, G.: Arakelov’s Theorem for abelian varieties. *Invent. math.* **73** (1983) 337–348
7. Karu, K.: Minimal models and boundedness of stable varieties. *J. Alg. Geom.* to appear
8. Kawamata, Y.: Deformations of canonical singularities. *J. Amer. Math. Soc.* **12** (1999) 85–92
9. Kollár, J.: Projectivity of complete moduli. *J. Diff. Geom.* **32** (1990) 235–268
10. Kollár, J.; Shepherd-Barron, N. I.: Threefolds and deformations of surface singularities. *Invent. math.* **91** (1988) 299–338
11. Kovács, S.: Smooth families over rational and elliptic curves. *J. Alg. Geom.* **5** (1996) 369–385
12. Kovács, S.: On the minimal number of singular fibres in a family of surfaces of general type. *Journ. Reine Angew. Math.* **487** (1997) 171–177
13. Kovács, S.: Algebraic hyperbolicity of fine moduli spaces. to appear in *J. Alg. Geom.*
14. Migliorini, L.: A smooth family of minimal surfaces of general type over a curve of genus at most one is trivial. *J. Alg. Geom.* **4** (1995) 353–361
15. Viehweg, E.: Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces. *Adv. Stud. Pure Math.* **1** (1983) 329–353 North-Holland
16. Viehweg, E.: Weak positivity and the additivity of the Kodaira dimension II: The local Torelli map. *Progr. Math.* **39** (1983) 567–589 Birkhäuser
17. Viehweg, E.: Quasi-projective Moduli for Polarized Manifolds. *Ergebnisse 3. Folge*, **30** (1995) Springer
18. Zhang, Qi: Holomorphic one-forms on projective manifolds. *J. Alg. Geom.* **6** (1997) 777–787