

The universal regular quotient of the Chow group of points on projective varieties

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Let X be a projective variety of dimension n defined over an algebraically closed field k . For X irreducible and non-singular, Matsusaka [Ma] constructed an abelian variety $\text{Alb}(X)$ and a morphism $\alpha : X \rightarrow \text{Alb}(X)$ (called the Albanese variety and mapping respectively), depending on the choice of a base-point on X , which is universal among the morphisms to abelian varieties (see Lang [La], Serre [Se] for other constructions). Over the field of complex numbers the existence of $\text{Alb}(X)$ and α was known before, and has a purely Hodge-theoretic description (see Igusa [I] for the Hodge theoretic construction). Incidentally, the terminology “Albanese variety” was introduced by A. Weil, for reasons explained in his commentary on the article [1950a] of Volume I of his collected works (see [W]), one of which is that the paper [Alb] of Albanese defines it (for a surface) as a quotient of the group of 0-cycles of degree 0 modulo an equivalence relation.

Let $CH^n(X)_{\text{deg } 0}$ denote the Chow group of 0-cycles of degree 0 on X modulo rational equivalence. When X is irreducible and non-singular, a remarkable feature of the Albanese morphism α is that it factors through a regular homomorphism $\varphi : CH^n(X)_{\text{deg } 0} \rightarrow \text{Alb}(X)$, that is a homomorphism, which when composed with the cycle map $\gamma : X \rightarrow CH^n(X)_{\text{deg } 0}$, gives an algebraic morphism. This follows immediately from the fact that an abelian variety does not contain any rational curve. Thus one can reformulate Matsusaka’s theorem as the

statement that there is a universal regular quotient of $CH^n(X)_{\text{deg}0}$ as an abelian variety.

For $n = 1$, φ is an isomorphism, and moreover $CH^1(X)_{\text{deg}0} \cong \text{Pic}^0(X)$, the Picard variety of X . Since Pic is a well understood functor, one defines the Chow group $CH^1(X)_{\text{deg}0}$ and the generalized Albanese variety $A^1(X)$ by

$$CH^1(X)_{\text{deg}0} \cong A^1(X) \cong \text{Pic}^0(X)$$

even in the singular case. This has several consequences for the expected structure of the generalized Albanese variety $A^n(X)$ of a projective reduced variety X of dimension n . First, it forces the correct definition of the Chow group $CH^n(X)_{\text{deg}0}$, as proposed by Levine and Weibel in [LW]. Second, it shows that $A^n(X)$ should be a smooth commutative algebraic group, that is an extension of an abelian variety by a linear group, where the latter is a product of additive and multiplicative factors. Third, the cycle map to $CH^n(X)_{\text{deg}0}$ is only defined on the regular locus X_{reg} of X . Consequently the expected generalized Albanese mapping should be only defined on X_{reg} . But already for curves, a morphism from X_{reg} to a smooth commutative algebraic group G need not factor through $CH^1(X)_{\text{deg}0}$, as G contains rational subvarieties. Therefore, the expected $A^n(X)$ should be constructed as a regular quotient of $CH^n(X)_{\text{deg}0}$ in the category of smooth commutative algebraic groups. Note here that the difficulty comes from the non-normality of X . In fact for normal surfaces and for irreducible normal varieties in characteristic zero (see [S]), it is known that $A^n(X) = \text{Alb}(\tilde{X})$, where \tilde{X} is a resolution of singularities, and that the cycle map does factor through $CH^n(X)_{\text{deg}0}$.

Roitman [R] proved that, when k is a universal domain, φ is an isomorphism precisely when the Chow group of 0-cycles is *finite dimensional* in the sense of Mumford [M]. This result was generalized for irreducible normal varieties in characteristic zero for $A^n(X) = \text{Alb}(\tilde{X})$ (see [S]).

In this article we prove the existence of a universal regular quotient of the Chow group of 0-cycles for singular projective varieties. We note that the term “variety” is used to mean a reduced quasi-projective scheme of finite type over a field; in particular it need not be irreducible, or equidimensional. A regular homomorphism is defined in 1.14 and the finite dimensionality of the Chow group in 7.1.

Theorem 1. *Let X be a projective variety of dimension n , defined over an algebraically closed field k .*

(i) *There exists a smooth connected commutative algebraic group $A^n(X)$, together with a regular homomorphism $\varphi : CH^n(X)_{\text{deg } 0} \rightarrow A^n(X)$, such that φ is universal among regular homomorphisms from $CH^n(X)_{\text{deg } 0}$ to smooth commutative algebraic groups.*

(ii) *Over a universal domain k the Chow group is finite dimensional precisely when φ is an isomorphism.*

(iii) *$A^n(X \times_k K) = A^n(X) \times_k K$, for all algebraically closed fields K containing k .*

We also give a second construction of $A^n(X)$ and φ using transcendental arguments when $k = \mathbb{C}$. Over $k = \mathbb{C}$, there is a natural semi-abelian variety

$$J^n(X) = \frac{H^{2n-1}(X, \mathbb{C}(n))}{F^0 H^{2n-1}(X, \mathbb{C}(n)) + \text{image } H^{2n-1}(X, \mathbb{Z}(n))} ,$$

that is a commutative algebraic group without additive factors, whose construction is implicit in Deligne’s article [D]. (For $k \not\subseteq \mathbb{C}$, one can in fact define $J^n(X)$ over k ; see [BS] and also [FW]; a related algebraic construction of 1-motives is given in [Ra]). From the discussion above, one sees that $A^n(X)$ cannot be isomorphic to $J^n(X)$. However, there is an Abel-Jacobi mapping

$$AJ^n : CH^n(X)_{\text{deg } 0} \longrightarrow J^n(X)$$

with very good properties. For example, if X is irreducible and non-singular, then Roitman [R2] proved that the Albanese mapping φ is an isomorphism on torsion subgroups. For any reduced, projective X of dimension n , the Abel-Jacobi map AJ^n induces an isomorphism on torsion subgroups as well (see [BiS] for the general result, and [L], [C], [BPW] for earlier partial results). This indicates that $J^n(X)$ should differ from $A^n(X)$ only by additive factors. This, together with the classical theory for curves, was the main motivation for our construction. Note that M. Levine also studied Kähler differentials in the singular case, see [L].

Theorem 2. *Let X be a projective variety over \mathbb{C} . For any $m \geq 0$, define the Deligne complex*

$$\mathcal{D}(m)_X = (0 \longrightarrow \mathbb{Z}_X(m) \longrightarrow \mathcal{O}_X \longrightarrow \Omega_{X/\mathbb{C}}^1 \longrightarrow \cdots \longrightarrow \Omega_{X/\mathbb{C}}^{n-1} \longrightarrow 0) ,$$

and associated cohomology group $D^m(X) = \mathbb{H}^{2m}(X, \mathcal{D}(m)_X)$. For $n = \dim X$, let

$$A^n(X) = \ker(D^n(X) \longrightarrow H^{2n}(X, \mathbb{Z}(n)))$$

be the kernel of the map induced by the natural surjection $\mathcal{D}(n)_X \rightarrow \mathbb{Z}(n)_X$ of complexes. Then

(i) the analytic group $A^n(X)$ has an underlying algebraic structure and for some $s \geq 0$ a presentation via an exact sequence of commutative algebraic groups

$$0 \longrightarrow (\mathbb{G}_a)^s \longrightarrow A^n(X) \longrightarrow J^n(X) \longrightarrow 0$$

(ii) there is a cycle class homomorphism $CH^n(X) \rightarrow D^n(X)$, such that the composite $CH^n(X) \rightarrow H^{2n}(X, \mathbb{Z}(n))$ is the degree homomorphism, induced by the topological cycle class map, and giving rise to a commutative diagram

$$\begin{array}{ccc} CH^n(X)_{\text{deg } 0} & \xrightarrow{\varphi} & A^n(X) \\ \downarrow & & \downarrow \\ CH^n(X) & \longrightarrow & D^n(X) \end{array}$$

(iii) the homomorphism φ is the universal regular homomorphism from the Chow group $CH^n(X)_{\text{deg } 0}$ to commutative algebraic groups over \mathbb{C}

(iv) φ is an isomorphism precisely when the Chow group is finite dimensional.

We give examples in 3.10 illustrating two pathological properties of $A^n(X)$. First we give examples of irreducible projective varieties X and Y of dimensions n and m , respectively, for which $\dim(A^{n+m}(X \times Y)) > \dim(A^n(X) \times A^m(Y))$. Next, we exhibit a flat family $\mathcal{X} \rightarrow S$, with geometrically integral fibres, for which the dimension of $A^n(X_s)$ is not locally constant on S .

We do not study higher dimensional cycles in this article. For this reason we do not analyze $D^m(X)$, $A^m(X) = \ker(D^m(X) \rightarrow H^{2m}(X, \mathbb{Z}(m)))$ and their relation to the Chow ring $CH^\bullet(X, X_{\text{sing}})$ considered by Levine in [L2].

After recalling the definition ([LW]) of the Chow group $CH^n(X)_{\text{deg } 0}$ and its relation to the Picard group of curves, and a moving lemma from [BiS], we construct the analytic cycle class map in section 2 and prove theorem 2, (ii). In section 3 we prove theorem 2, (i) and give a cohomological description for the Lie algebra of $A^n(X)$. The regularity of $\varphi : CH^n(X)_{\text{deg } 0} \rightarrow A^n(X)$ and the universal property (iii) for φ are shown by analytic methods in Section 4.

The next two sections, independent of the transcendental arguments used before, contain the algebraic part of the article. We recall

in Section 5 some of the properties of Picard groups of curves and apply them to general curves in X . The main technical tool in Section 6 is the boundedness of the dimension of a regular quotient. This being guaranteed $A^n(X)$ is constructed essentially using Lang’s arguments [La].

In Section 7 we give two slightly different proofs for part (iv) of theorem 2 and (ii) of theorem 1, the first one building up on the transcendental methods, the second one using the algebraic arguments developed in Sections 1, 5 and 6.

1. Chow groups and regular homomorphisms

We begin by recalling the definition of the Chow group of 0-cycles $CH^n(X)$, as given in [LW] (see also [BiS]). As in [BiS], we adopt the convention that a point lying on a lower dimensional component of X is deemed to be singular. Let X_{sing} denote the (closed) subset of singular points, and $X_{\text{reg}} = X - X_{\text{sing}}$ the complementary open set. The closure of X_{reg} is the union of the n -dimensional components of X .

The group $Z^n(X)$ of 0-cycles is defined to be the free abelian group on the closed points of X_{reg} . The subgroup $R^n(X)$ of cycles rationally equivalent to 0 is defined using the notion of a Cartier curve.

Definition 1.1. *A Cartier curve is a subscheme $C \subset X$, defined over k , such that*

- (i) *C is pure of dimension 1*
- (ii) *no component of C is contained in X_{sing}*
- (iii) *if $x \in C \cap X_{\text{sing}}$, then the ideal of C in $\mathcal{O}_{x,X}$ is generated by a regular sequence (consisting of $n - 1$ elements).*

If C is a Cartier curve on X , with generic points η_1, \dots, η_s , and $\mathcal{O}_{S,C}$ is the semilocal ring on C of the points of $S = (C \cap X_{\text{sing}}) \cup \{\eta_1, \dots, \eta_s\}$, there is a natural map on unit groups

$$\theta_{C,X} : \mathcal{O}_{S,C}^* \longrightarrow \bigoplus_{i=0}^s \mathcal{O}_{\eta_i,C}^* .$$

Define $R(C, X) = \text{image } \theta_{C,X}$. For $f \in R(C, X)$, define the divisor of $(f)_C$ as follows: let C_i denote the maximal Cohen-Macaulay subscheme of C supported on the component with generic point η_i . Then for any $x \in C_i$ the map

$$\mathcal{O}_{x,C_i} \longrightarrow \mathcal{O}_{\eta_i,C_i} = \mathcal{O}_{\eta_i,C}$$

is the injection of a Cohen-Macaulay local ring of dimension 1 into its total quotient ring. If f_i is the component of f in $\mathcal{O}_{\eta_i, C}$, then $f_i = a_x/b_x$ for some non zero-divisors $a_x, b_x \in \mathcal{O}_{x, C_i}$. Define

$$(f)_C = \sum_{i=1}^s (f_i)_{C_i} = \sum_{i=1}^s \sum_{x \in C_i} (\ell(\mathcal{O}_{x, C_i}/a_x \mathcal{O}_{x, C_i}) - \ell(\mathcal{O}_{x, C_i}/b_x \mathcal{O}_{x, C_i})) \cdot [x] .$$

Standard arguments imply that this is well-defined (i.e., the coefficient of $[x]$ is independent of the choice of the representation $f_i = a_x/b_x$, and vanishes for all but a finite number of x).

Suppose C is reduced. Then in the above considerations, \mathcal{O}_{x, C_i} is an integral domain with quotient field $\mathcal{O}_{\eta_i, C}$. If v_1, \dots, v_m are the discrete valuations of $\mathcal{O}_{\eta_i, C}$ centered at x , then the multiplicity of x in $(f)_{C_i}$ is

$$(1.1) \quad \ell(\mathcal{O}_{x, C_i}/a_x \mathcal{O}_{x, C_i}) - \ell(\mathcal{O}_{x, C_i}/b_x \mathcal{O}_{x, C_i}) = \sum_{j=1}^m v_j(f_i)$$

(compare [Ful], Example A.3.1.). In fact, let R be the integral closure of $O = \mathcal{O}_{x, C_i}$ in $\mathcal{O}_{\eta_i, C}$. The Chinese remainder theorem implies that

$$\ell(R/a_x R) = \sum_{j=1}^m v_j(a_x) ,$$

and similarly for b_x . Multiplying a_x and b_x by the same element of O we may assume that both $a_x R$ and $b_x R$ are contained in O , and

$$\ell(O/a_x O) + \ell(R/O) = \ell(R/a_x R) + \ell(a_x R/a_x O) .$$

Since $a_x \neq 0$ the second terms on both sides are equal.

Definition 1.2. Let $U \subset X_{\text{reg}}$ be an open dense subscheme. $R^n(X, U)$ is defined to be the subgroup of $Z^n(U)$ generated by elements $(f)_C$ as C ranges over all Cartier curves with $C \cap U$ dense in C , and $f \in R(C, X)$ with $(f)_C \in Z^n(U)$. For $U = X_{\text{reg}}$ we write $R^n(X)$ instead of $R^n(X, X_{\text{reg}})$ and define

$$CH^n(X) = Z^n(X)/R^n(X) .$$

Mapping a point $x \in X_{\text{reg}}$ to its rational equivalence class defines a map

$$\gamma : X_{\text{reg}} \longrightarrow CH^n(X) .$$

If U_1, \dots, U_r denote the irreducible components of X_{reg} , then $Z^n(X)_{\text{deg } 0}$ and $CH^n(X)_{\text{deg } 0}$ denote the subgroups of $Z^n(X)$ and $CH^n(X)$, respectively, of cycles δ with $\text{deg}(\delta|_{U_i}) = 0$ for $i = 1, \dots, r$.

As noted in [BiS], lemma 1.3 of [LW] allows one to restrict to considering only curves C such that $C \cap X_{\text{reg}}$ has no embedded points, and any irreducible component C' of C which lies entirely in X_{reg} occurs in C with multiplicity 1. The moving lemmas 2.2.2 and 2.2.3 of [BiS] allow stronger restrictions on C :

Lemma 1.3. *Let $A \subset X_{\text{sing}}$ be a closed subset of dimension $\leq n - 2$, and let $D \subset X$ be a closed subset of dimension $\leq n - 1$. Then any element $\delta \in R^n(X)$ can be written in the form $\delta = (f)_C$ for a single (possibly reducible) Cartier curve C , such that*

- (a) C is reduced
- (b) $C \cap A = \emptyset$
- (c) $C \cap D$ is empty or consists of finitely many points.

Corollary 1.4. *If $U \subset X_{\text{reg}}$ is an open and dense subscheme, then*

$$CH^n(X) = Z^n(U)/R^n(X, U)$$

and $CH^n(X)_{\text{deg } 0} = Z^n(U)_{\text{deg } 0}/R^n(X, U)$.

Proof. First note that the zero cycles supported on U generate $CH^n(X)$ since the corresponding assertion holds true for curves. The moving lemma 1.3 for $D = X - U$ implies that $R^n(X) \cap Z^n(U) = R^n(X, U)$. □

Remark 1.5. Let $X^{(n)}$ denote the union of the n -dimensional irreducible components of X , and let $X^{<n}$ be the union of the lower dimensional components. Applying the corollary to $X^{(n)}$ and the open subset

$$U = X^{(n)} - X_{\text{sing}} = X - X_{\text{sing}} \subset X^{(n)} - X_{\text{sing}}^{(n)} ,$$

we see that the natural map from $CH^n(X)$ to $CH^n(X^{(n)})$ is surjective. It seems plausible that a stronger form of lemma 1.3 holds, where A is allowed to be any closed subset of X of codimension ≥ 2 which is disjoint from $\text{supp}(\delta)$. If this is true, then applying it to $X^{(n)}$ with $A = X^{(n)} \cap X^{<n}$, one sees that for any $\delta \in R^n(X^{(n)}) \cap Z^n(X)$ there exists a reduced Cartier curve C in $X^{(n)}$, disjoint from A , and $f \in R(C, X^{(n)})$ with $\delta = (f)_C$. Then C is also a Cartier curve on X , and $\delta \in R^n(X)$. We deduce that $CH^n(X) \rightarrow CH^n(X^{(n)})$ is an isomorphism. We have as yet been unable to prove this.

Remark 1.6. Keeping the notation from the previous remark, we note further that for $k = \mathbb{C}$, the natural maps

$$\begin{aligned}
 H^{2n}(X, \mathbb{Z}(n)) &\longrightarrow H^{2n}(X^{(n)}, \mathbb{Z}(n)), & D^n(X) &\longrightarrow D^n(X^{(n)}), \\
 A^n(X) &\longrightarrow A^n(X^{(n)})
 \end{aligned}$$

are isomorphisms, since $X^{<n}$ has constructible cohomological dimension $\leq 2(n - 1)$ and coherent cohomological dimension $\leq n - 1$.

As reflected by the notation, $R(C, X)$ depends on the pair (C, X) , and is not necessarily intrinsic to C . Since we have not imposed any unit condition at singular points of C which lie in X_{reg} , the functions $f \in R(C, X)$ are defined on some curve C' , birational to C .

Definition 1.7. *Let C' be a reduced projective curve and $\iota : C' \rightarrow X$ be a morphism. Then (C', ι) will be called admissible if $\iota : C' \rightarrow C = \iota(C')$ is birational, if C is a reduced Cartier curve and if for some open neighbourhood W of X_{sing} the restriction of ι to $\iota^{-1}(W)$ is a closed embedding.*

If (C', ι) is admissible one has an inclusion $R(C', C') \subset R(C, X)$ which is an equality if $\iota^{-1}(X_{\text{reg}})$ is non-singular.

Lemma 1.8. *Let (C', ι) be admissible. Then there exists a homomorphism (of abstract groups)*

$$\eta : \text{Pic}^0(C') \cong CH^1(C')_{\text{deg } 0} \longrightarrow CH^n(X)_{\text{deg } 0}$$

which maps the isomorphism class of $\mathcal{O}_{C'}(p - p')$ to $\gamma(\iota(p)) - \gamma(\iota(p'))$.

Proof. By definition $\text{Pic}(C') = Z^1(C'_{\text{reg}})/R(C', C')$ and one has a map

$$\gamma \circ \iota : Z^1(C'_{\text{reg}}) \longrightarrow CH^n(X) .$$

The equality (1.1) shows that for $f \in R(C', C')$ the image of $(f)_{C'}$ in $CH^n(X)$ is zero. □

Notations 1.9. Let Y be a non-singular scheme with irreducible components Y_1, \dots, Y_s , let G be an abstract or an algebraic group, and let $\pi : Y \rightarrow G$ a map or morphism.

(i) After choosing base points $p_i \in Y_i$ a map

$$\pi_m : S^m(Y) := S^m\left(\bigcup_{i=1}^s Y_i\right) \longrightarrow G ,$$

is defined by $\pi_m(y_1, \dots, y_m) = \sum_{j=1}^m (\pi(y_j) - \pi(p_{\rho(j)}))$, where $\rho(j) = i$ if $y_j \in Y_i$.

(ii) To avoid the reference to base points, we will frequently use different maps:

$$\pi^{(-)} : \Pi_Y = \bigcup_{i=1}^s Y_i \times Y_i \longrightarrow G$$

is defined by $\pi^{(-)}(y, y') = \pi(y) - \pi(y')$, and $\pi_m^{(-)} : S^m(\Pi_Y) \rightarrow G$ is the composite $S^m(\Pi_Y) \rightarrow S^m(G) \xrightarrow{\text{sum}} G$.

If G is an algebraic group, then the images of $\pi_m^{(-)}$ lie in the connected component of 0. In particular for U open and dense in X_{reg} we will frequently consider

$$\begin{aligned} \gamma^{(-)} &= \gamma_U^{(-)} : \Pi_U \longrightarrow CH^n(X)_{\text{deg } 0} \\ \text{and } \gamma_m &= \gamma_{U,m} : S^m(U) \longrightarrow CH^n(X)_{\text{deg } 0} . \end{aligned}$$

Lemma 1.10. *Let G be a d -dimensional smooth connected commutative algebraic group and let $\Gamma \subset G$ be a constructible subset which generates G as an abstract group. Then*

- (i) *the image of the composite map $S^d(\Gamma) \longrightarrow S^d(G) \xrightarrow{\text{sum}} G$ is dense*
- (ii) *$S^{2d}(\Gamma) \longrightarrow S^{2d}(G) \xrightarrow{\text{sum}} G$ is surjective*
- (iii) *if B is a non-singular scheme with connected components B_1, \dots, B_s and if $\vartheta : B \rightarrow G$ is a morphism with image Γ then the morphism*

$$\vartheta_d^{(-)} : S^d(\Pi_B) = S^d\left(\bigcup_{i=1}^s B_i \times B_i\right) \longrightarrow G$$

with $\vartheta_d^{(-)}((b_1, b'_1), \dots, (b_d, b'_d)) = \sum_{i=1}^d (\vartheta(b_i) - \vartheta(b'_i))$ is surjective.

Proof. Let $\bar{\Gamma}_1, \dots, \bar{\Gamma}_s$ be the irreducible components of the closure $\bar{\Gamma}$ of Γ , and let $\Gamma_i = \bar{\Gamma}_i \cap \Gamma$. It is sufficient to find non-negative integers d_1, \dots, d_s with $\sum_{i=1}^s d_i \leq d$ such that the image of $S^{d_1}(\Gamma_1) \times \dots \times S^{d_s}(\Gamma_s)$ is dense in G . To this end, we may assume that the identity of G lies on each Γ_i .

Let $\bar{\Gamma}_1^\vee$ be the closure of the image of $S^\vee(\Gamma_1)$ in G . Since $\bar{\Gamma}_1^\vee \subset \bar{\Gamma}_1^{\vee+1}$ there exists some $d_1 \leq d$ with $\bar{\Gamma}_1^{d_1} = \bar{\Gamma}_1^{d_1+1}$, and d_1 is minimal with this property. Hence $\bar{\Gamma}_1^{d_1} = \bar{\Gamma}_1^{2d_1}$ and $\bar{\Gamma}_1^{d_1}$ is a subgroup of G of dimension larger than or equal to d_1 . If $s = 1$, i.e. if $\bar{\Gamma}$ is irreducible, then $\bar{\Gamma}_1^{d_1} = G$.

In general, replacing G by $G/\bar{\Gamma}_1^{d_1}$ one obtains 1.10 (i) by induction on s .

The second part is an easy consequence of (i). Let U be an open dense subset of G , contained in $S^{d_1}(\Gamma_1) \times \cdots \times S^{d_s}(\Gamma_s)$. Given $p \in G$ the intersection of the two open sets U and $p - U$ is non-empty and hence there are points $a, b \in U$ with $p - b = a$.

Replacing B by some open subscheme we may assume for (iii) that the image of B_i is dense in Γ_i . By (i) one finds d_1, \dots, d_s with $\sum_{i=1}^s d_i = d$ such that the image of the composite

$$S^{d_1}(B_1) \times \cdots \times S^{d_s}(B_s) \longrightarrow S^d(\Gamma) \longrightarrow S^d(G) \longrightarrow G$$

contains a subset U which is open in G . Given $p \in G$ the intersection of U and of $p + U$ is non empty and hence $p = a - b$ for two points a and b in U . Obviously $a - b$ lies in the image of $\vartheta_d^{(-)}$. □

Corollary 1.11. *Let C' be a reduced curve, let B_1, \dots, B_s be the connected components of $B = C'_{\text{reg}}$, let $b'_j \in B_j$ be base points and let $\vartheta : B \rightarrow \text{Pic}^0(C')$ be the morphism with $\vartheta|_{B_j}(b) = \mathcal{O}_{C'}(b - b'_j)$. Then there exists some open connected subscheme W of $S^g(B)$, for $g = \dim_k(\text{Pic}^0(C'))$, such that $\vartheta_W := \vartheta_g|_W$ is an open embedding.*

Proof. By 1.10 we find some W with $\vartheta_W(W)$ open and ϑ_W finite over its image. On the other hand, any fibre of ϑ_g is an open subset of $\mathbb{P}(H^0(C', \mathcal{O}_{C'}(D)))$ for some divisor D on C' ; hence the projective spaces corresponding to points of $\vartheta_W(W)$ must be 0-dimensional. □

Lemma 1.12. *Let G be a smooth commutative algebraic group, $U \subset X_{\text{reg}}$ an open and dense subset, and $\pi : U \rightarrow G$ a morphism. Then the following two conditions are equivalent.*

(a) *There exists a homomorphism (of abstract groups) $\phi : CH^n(X)_{\text{deg}0} \rightarrow G$ such that $\pi^{(-)} = \phi \circ \gamma^{(-)}$ (as maps on the closed points).*

(b) *For all admissible pairs (C', ι) with $B = (\iota^{-1}(U))_{\text{reg}}$ dense in C' there exists a homomorphism of algebraic groups $\psi : \text{Pic}^0(C') \rightarrow G$ such that the diagram*

$$\begin{array}{ccc} \Pi_B & \xrightarrow{\vartheta^{(-)}} & \text{Pic}^0(C') \\ \iota \downarrow & & \downarrow \psi \\ \Pi_U & \xrightarrow{\pi^{(-)}} & G \end{array}$$

commutes. Here $\vartheta : B \rightarrow \text{Pic}(C')$ denotes the natural morphism, mapping a point p to the isomorphism class of the invertible sheaf $\mathcal{O}_{C'}(p)$.

Moreover, if the equivalent conditions (a) and (b) are true, the morphism ψ in (b) factors as

$$\begin{array}{ccc} \text{Pic}^0(C') & \xrightarrow{\eta} & CH^n(X)_{\text{deg } 0} \\ \psi \searrow & & \swarrow \phi \\ & G & \end{array}$$

and the image of $\phi : CH^n(X)_{\text{deg } 0} \rightarrow G$ is contained in the connected component of the identity of G .

Proof. Assume (a) and let (C', ι) be admissible and $g = \dim(\text{Pic}^0(C'))$. Choosing base points $b'_j \in B_j$, one finds by 1.11 an open subscheme W of $S^g(B)$ such that the morphism $\vartheta_W : W \rightarrow \text{Pic}^0(C')$ is an open embedding. By 1.8 one obtains a homomorphism

$$\psi : \text{Pic}^0(C') \xrightarrow{\eta} CH^n(X)_{\text{deg } 0} \xrightarrow{\phi} G$$

of abstract groups. By assumption $\pi^{(-)}$ is a morphism of schemes and the same holds true for $\pi^{(-)} \circ \iota : \Pi_B \rightarrow G$. Thereby the restriction of ψ to the open subscheme $W \subset \text{Pic}^0(C')$ is a morphism of schemes, and being a homomorphism on closed points ψ is a morphism of algebraic groups.

Since each point of X_{reg} lies on some Cartier curve, the images of the connected algebraic groups $\text{Pic}^0(C')$ generate $CH^n(X)_{\text{deg } 0}$ and the image $\phi(CH^n(X)_{\text{deg } 0})$ lies in the connected component of G , which contains the identity.

The morphism $\pi^{(-)}$ induces a map $\tilde{\phi} : Z^n(U)_{\text{deg } 0} \rightarrow G$ and it remains to verify that (b) implies that $\tilde{\phi}(R^n(X, U)) = 0$. By 1.3 each $\delta \in R^n(X, U)$ is of the form $(f)_C$ for a reduced Cartier curve C . There exists an admissible pair (C', ι) with $\iota(C') = C$ and with $\iota^{-1}(X_{\text{reg}})$ non-singular. $(f)_C$ is the image of $(f)_{C'}$ in $Z^n(U)$ and by assumption $\iota \circ \pi^{(-)}$ factors through $\text{Pic}^0(C')$. \square

Corollary 1.13. *Let $\phi : CH^n(X)_{\text{deg } 0} \rightarrow G$ be a homomorphism to a smooth commutative algebraic group G . Then the following conditions are equivalent.*

- (i) $\phi \circ \gamma^{(-)} : \Pi_{X_{\text{reg}}} \rightarrow G$ is a morphism of schemes.
- (ii) There exists an open dense subscheme U of X_{reg} such that $\phi \circ \gamma^{(-)}|_{\Pi_U}$ is a morphism of schemes.
- (iii) Given a base point p_i on each irreducible component U'_i of some open dense subscheme U of X_{reg} , the map $\pi : U \rightarrow G$ with $\pi|_{U'_i}(x) = \phi(x - p_i)$ is a morphism of schemes.

(iv) Given any $m > 0$ and base points p_i on each irreducible component U'_i of some open dense subscheme U of X_{reg} , $\phi \circ \gamma_m : S^m(U) \rightarrow G$ is a morphism of schemes.

Of course, “ π is a morphism of schemes” stands for “there exists a morphism of schemes whose restriction to closed points coincides with π ”, an abuse of terminology which we will repeat throughout this article.

Proof. Obviously (i) implies (ii). For a given $U \subset X_{\text{reg}}$, the equivalence of (ii), (iii), and (iv) is an easy exercise. In fact, the morphism π in 1.13 (iii) is just $\phi \circ \gamma_1$.

Assume that (iii) holds true for some U . We will show that the corresponding property holds true for X_{reg} itself. To this aim consider the map $\bar{\pi} : X_{\text{reg}} \rightarrow G$ with $\bar{\pi}(x) = \phi(x - p_i)$, for x in the closure of U'_i in X_{reg} , and the graph $\Gamma_{\bar{\pi}}$ of $\bar{\pi}$ in $X_{\text{reg}} \times G$. By definition, $\Gamma_{\bar{\pi}} \cap U \times G$ is the graph Γ_{π} . Let Z be the closure of Γ_{π} in $X_{\text{reg}} \times G$.

$\Gamma_{\bar{\pi}}$ is contained in Z . In fact, given a point $x \in X_{\text{reg}}$ one can find a Cartier curve C through x with $U \cap C$ dense in C and with $B = C \cap X_{\text{reg}}$ non-singular. By lemma 1.12 the morphism $(\pi|_{C \cap U})^{(-)} : \Pi_{C \cap U} \rightarrow G$ factors through a morphism $\text{Pic}^0(C) \rightarrow G$ of algebraic groups and, in particular, it extends to a morphism $\Pi_B \rightarrow G$. Again this implies that the restriction of $\bar{\pi}$ to B is a morphism, hence $\Gamma_{\bar{\pi}} \cap B \times G$ is closed and therefore contained in Z .

By construction the morphism $p_1 : Z \rightarrow X_{\text{reg}}$ induced by the projection is birational and surjective. Let $V \subset X_{\text{reg}}$ be the largest open subscheme with $p_1|_{p_1^{-1}(V)}$ an isomorphism. Then $\bar{\pi}|_V$ is a morphism of schemes and $\text{codim}_{X_{\text{reg}}}(X_{\text{reg}} - V) \geq 2$. By theorem 1 in [BLR], 4.4, $\bar{\pi}|_V$ extends to a morphism $X_{\text{reg}} \rightarrow G$. The graph of this morphism is contained in Z , hence it is equal to Z and $\bar{\pi}$ is a morphism. \square

We end this section by giving the definition of a regular homomorphism, used already in the formulation of the main theorems in the introduction.

Definition 1.14. Let G be a smooth commutative algebraic group. A homomorphism $\phi : CH^n(X)_{\text{deg}0} \rightarrow G$ (of abstract groups) is called a regular homomorphism, if one of the equivalent conditions in 1.13 holds true.

Lemma 1.15. The image of a regular homomorphism $\phi : CH^n(X)_{\text{deg}0} \rightarrow G$ is a connected algebraic subgroup of G .

Proof. Let G' denote the Zariski closure of $\phi(CH^n(X)_{\text{deg}0})$. By 1.12, G' is connected and it is generated by the image of $\Pi_{X_{\text{reg}}} \rightarrow G'$. Hence 1.15 follows from the third part of lemma 1.10. \square

2. The cycle class map

Throughout the next three sections we will assume that the ground field k is the field of complex numbers. \mathcal{O}_X and $\Omega_{X/\mathbb{C}}^m$ will respectively denote the sheaves of holomorphic functions and (analytic Kähler) differential m -forms. As in the introduction consider the Deligne complex

$$\mathcal{D}(n)_X = (0 \rightarrow \mathbb{Z}_X(n) \rightarrow \mathcal{O}_X \rightarrow \Omega_{X/\mathbb{C}}^1 \rightarrow \cdots \rightarrow \Omega_{X/\mathbb{C}}^{n-1} \rightarrow 0) ,$$

and associated cohomology group $D^n(X) = \mathbb{H}^{2n}(X, \mathcal{D}(n)_X)$. In this section we construct the cycle class homomorphism $CH^n(X) \rightarrow D^n(X)$, using Cartier curves C in X .

By the moving lemma 1.3 it will be sufficient to consider reduced Cartier curves C in X . Note, however, that we do not have that C is a local complete intersection in X , in general; this is only given to hold at points of $C \cap X_{\text{sing}}$. This leads to a slight technical difficulty. We will need to define ‘Gysin’ maps for Cartier curves C in X . These are directly defined in case C is a local complete intersection, and in general one has first to make a sequence of point blow ups centered in X_{reg} to reduce to this special case. Indeed, even to show that the cycle homomorphism $Z^n(X) \rightarrow D^n(X)$ respects rational equivalence, a similar procedure needs to be followed.

Note that the exterior derivative yields a map of complexes $\mathcal{D}(n)_X \rightarrow \Omega_{X/\mathbb{C}}^n[-n]$, and there is an obvious map $\mathcal{D}(n)_X \rightarrow \mathbb{Z}(n)_X$.

Lemma 2.1. *For $x \in X_{\text{reg}}$, there is a unique element $[x] \in H_{\{x\}}^{2n}(X, \mathcal{D}(n)_X)$ which maps to the topological cycle class of x in $H_{\{x\}}^{2n}(X, \mathbb{Z}(n))$ as well as to the “Hodge cycle class” of x in $H_{\{x\}}^n(X, \Omega_{X/\mathbb{C}}^n)$.*

This gives rise to a well-defined cycle class homomorphism $Z^n(X) \rightarrow D^n(X)$, whose composition with $D^n(X) \rightarrow H^{2n}(X, \mathbb{Z}(n))$ is the topological cycle class homomorphism.

Proof. The element $[x]$ exists because the topological and Hodge cycle classes both map to the de Rham cycle class of x in $H_{\{x\}}^{2n}(X, \mathbb{C}) = \mathbb{H}_{\{x\}}^{2n}(X, \Omega_{X/\mathbb{C}}^\bullet)$, by a standard local computation. See [EV], §7, for example (though X is singular, the terms in the above computation depend only on a neighbourhood of x in X , and we have $x \in X_{\text{reg}}$; hence [EV], §7 is applicable). □

Lemma 2.2. *In the above situation, if $\dim X = 1$, then there is a natural quasi-isomorphism $\mathcal{D}(1)_X \cong \mathcal{O}_X^*[-1]$, yielding an identification $\text{Pic}(X) \cong D^1(X)$ (and hence also $\text{Pic}^0(X) \cong A^1(X)$). Under the identification, the class of a smooth point $[x] \in D^1(X)$ corresponds to the class of the invertible sheaf $\mathcal{O}_X(x)$.*

Proof. The natural quasi-isomorphism is equivalent to the exactness of the exponential sequence. The description of the class of a point x as the class of the invertible sheaf $\mathcal{O}_X(x)$ is also a standard local computation. □

As we will see in lemma 3.2 the cycle map $X_{\text{reg}} \rightarrow D^n(X)$ in lemma 2.1 is trivially analytic. Our point will be to see that it is algebraic.

First we argue as in [BiS], in order to show that the map $Z^n(X) \rightarrow D^n(X)$ factors through $CH^n(X)$. We follow the convention that the truncated de Rham complex of Kähler differentials

$$\Omega_{X/\mathbb{C}}^{\leq n} = (0 \longrightarrow \mathcal{O}_X \longrightarrow \cdots \longrightarrow \Omega_{X/\mathbb{C}}^{n-1} \rightarrow 0)$$

has \mathcal{O}_X placed in degree 0; thus we have an exact sequence of complexes

$$0 \longrightarrow \Omega_{X/\mathbb{C}}^{\leq n}[-1] \longrightarrow \mathcal{D}(n)_X \longrightarrow \mathbf{Z}(n)_X \longrightarrow 0 .$$

Lemma 2.3. *Let X be a projective variety of dimension n over \mathbb{C} , and $C \subset X$ be a reduced Cartier curve which is a local complete intersection in X . Then there is a commutative diagram*

$$\begin{array}{ccc} Z^1(C) & \longrightarrow & Z^n(X) \\ \downarrow & & \downarrow \\ D^1(C) & \xrightarrow{\text{Gysin}} & D^n(X) \\ \downarrow & & \downarrow \\ H^2(C, \mathbf{Z}(1)) & \xrightarrow{\text{Gysin}} & H^{2n}(X, \mathbf{Z}(n)) \end{array}$$

Proof. Consider the local (hyper) cohomology sheaves $\mathcal{H}_C^j(\mathcal{D}(n)_X)$ of the complex $\mathcal{D}(n)_X$ with support in C . We claim that for any point $x \in C$, the stalks $\mathcal{H}_C^j(\mathcal{D}(n)_X)_x$ vanish for $j \neq 2n - 1$, unless x is a singular point of C . Indeed, if $x \in C$ is a non-singular point (so that $x \in X_{\text{reg}}$ as well), then there is a long exact sequence of stalks

$$\begin{aligned} \cdots &\longrightarrow \mathcal{H}_C^{j-1}(\mathbb{Z}(n)_X)_x \oplus \mathcal{H}_C^{j-1-n}(\Omega_{X/\mathbb{C}}^n)_x \\ &\longrightarrow \mathcal{H}_C^{j-1}(\mathbb{C}_X)_x \longrightarrow \mathcal{H}_C^j(\mathcal{D}(n)_X)_x \\ &\longrightarrow \mathcal{H}_C^j(\mathbb{Z}(n)_X)_x \oplus \mathcal{H}_C^{j-n}(\Omega_{X/\mathbb{C}}^n)_x \longrightarrow \mathcal{H}_C^j(\mathbb{C}_X)_x \longrightarrow \cdots \end{aligned}$$

However $\mathcal{H}_C^i(\mathbb{Z}(n)_X)_x = \mathcal{H}_C^i(\mathbb{C}_X)_x = 0$ for $i \neq 2n - 2$, $\mathcal{H}_C^{2n-2}(\mathbb{Z}(n)_X)$ injects into $\mathcal{H}_C^{2n-2}(\mathbb{C}_X)$, and $\mathcal{H}_C^i(\Omega_{X/\mathbb{C}}^n)_x = 0$ unless $i = n - 1$, for a non-singular point $x \in C$ as above. This implies that $\mathcal{H}_C^j(\mathcal{D}(n)_X)_x = 0$ for $j \neq 2n - 1$, for such x . Also $\mathcal{H}_C^{2n-1}(\mathcal{D}(n)_X)_x$ fits into an exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathcal{H}_C^{2n-2}(\mathbb{C}_X/\mathbb{Z}(n)_X)_x \longrightarrow \mathcal{H}_C^{2n-1}(\mathcal{D}(n)_X)_x \\ &\longrightarrow \mathcal{H}_C^{n-1}(\Omega_{X/\mathbb{C}}^n)_x \longrightarrow 0 \quad , \end{aligned}$$

with $\mathcal{H}_C^{2n-2}(\mathbb{C}_X/\mathbb{Z}(n)_X)_x \cong \mathbb{C}/\mathbb{Z}(1) = \mathbb{C}^*$.

Thus, $\mathcal{H}_C^j(\mathcal{D}(n)_X)$ is supported at a finite set of points, if $j \neq 2n - 1$. Hence in the local-to-global spectral sequence

$$E_2^{p,q} = H^p(C, \mathcal{H}_C^q(\mathcal{D}(n)_X)) \implies \mathbb{H}_C^{p+q}(X, \mathcal{D}(n)_X)$$

we have $E_2^{p,q} = 0$ for $p > 0$, $q \neq 2n - 1$. In particular, there is a well-defined injective map

$$\alpha : H^1(C, \mathcal{H}_C^{2n-1}(\mathcal{D}(n)_X)) \longrightarrow \mathbb{H}_C^{2n}(X, \mathcal{D}(n)_X) \quad .$$

We will next construct a natural map of sheaves on C

$$\mathcal{O}_C^* \longrightarrow \mathcal{H}_C^{2n-1}(\mathcal{D}(n)_X) \quad .$$

The desired Gysin map $D^1(C) \rightarrow D^n(X)$ is then defined to be the composition

$$\begin{aligned} H^1(C, \mathcal{O}_C^*) &\longrightarrow H^1(C, \mathcal{H}_C^{2n-1}(\mathcal{D}(n)_X)) \\ &\xrightarrow{\alpha} \mathbb{H}_C^{2n}(X, \mathcal{D}(n)_X) \longrightarrow \mathbb{H}^{2n}(X, \mathcal{D}(n)_X) = D^n(X) \end{aligned}$$

To construct the map on sheaves $\mathcal{O}_C^* \rightarrow \mathcal{H}_C^{2n-1}(\mathcal{D}(n)_X)$, we argue locally, as follows. Let U be an affine neighbourhood in X of a point $x \in C$, on which the ideal of C is generated by a regular sequence of functions f_1, \dots, f_{n-1} , determining a morphism $f : U \rightarrow \mathbb{A}_{\mathbb{C}}^{n-1}$ such that $f^{-1}(0) = C \cap U$. Note that there are well-defined sections (of the skyscraper sheaves)

$$\alpha \in \Gamma(\mathcal{H}_{\{0\}}^{2n-2}(\mathbb{Z}(n)_{\mathbb{A}_{\mathbb{C}}^{n-1}})) = \mathbb{Z}(1), \quad \beta \in \Gamma(\mathcal{H}_{\{0\}}^{n-1}(\Omega_{\mathbb{A}_{\mathbb{C}}^{n-1}/\mathbb{C}})) \quad ,$$

which have the same image $\gamma \in \Gamma(\mathcal{H}_{\{0\}}^{2n-2}(\Omega_{\mathbb{A}_{\mathbb{C}}^{n-1}/\mathbb{C}}^{\bullet})) = \Gamma(\mathcal{H}_{\{0\}}^{2n-2}(\Omega_{\mathbb{A}_{\mathbb{C}}^{n-1}/\mathbb{C}}^{<n}))$, under the obvious maps, and such that β is annihilated by the ideal of 0 in $\Gamma(\mathcal{O}_{\mathbb{A}_{\mathbb{C}}^{n-1}})$, for the natural module structure on $\Gamma(\mathcal{H}_{\{0\}}^{n-1}(\Omega_{\mathbb{A}_{\mathbb{C}}^{n-1}/\mathbb{C}}))$. In fact, these conditions uniquely determine such a pair of sections (α, β) up to sign, and there is a standard choice, with β determined by $\text{dlog}(z_1) \wedge \cdots \wedge \text{dlog}(z_{n-1})$, where z_j are the coordinate functions, so that β is the cup product of the local divisor classes

$$(2.1) \quad \begin{aligned} \text{dlog}(z_j) &\in \Gamma(\mathbb{A}_{\mathbb{C}}^{n-1}, \mathcal{E}xt_{\mathbb{A}_{\mathbb{C}}^{n-1}}^1(\mathcal{O}_{\{z_j=0\}}, \Omega_{\mathbb{A}_{\mathbb{C}}^{n-1}/\mathbb{C}}^1)) \\ &\subset \Gamma(\mathbb{A}_{\mathbb{C}}^{n-1}, \mathcal{H}_{\{z_j=0\}}^1(\Omega_{\mathbb{A}_{\mathbb{C}}^{n-1}/\mathbb{C}}^1)) . \end{aligned}$$

Hence γ is also determined. Now consider

$$\begin{aligned} f^*\alpha &\in \Gamma(U, \mathcal{H}_C^{2n-2}(\mathbb{Z}(n)_X)), \quad f^*\beta \in \Gamma(U, \mathcal{H}_C^{n-1}(\Omega_{X/\mathbb{C}}^{n-1})), \\ \text{and } f^*\gamma &\in \Gamma(U, \mathcal{H}_C^{2n-1}(\Omega_{X/\mathbb{C}}^{<n})) , \end{aligned}$$

where $f^*\alpha$ and $f^*\beta$ both map to $f^*\gamma$, and $f^*\beta$ is annihilated by any section of the ideal sheaf of $C \cap U$ in U . Thus $f^*\alpha$ and $f^*\beta$ yield maps of sheaves

$$\mathbb{Z}(1)_C|_U \longrightarrow \mathcal{H}_C^{2n-2}(\mathbb{Z}(n)_X)|_U, \quad \mathcal{O}_C|_U \longrightarrow \mathcal{H}_C^{n-1}(\Omega_{X/\mathbb{C}}^{n-1})|_U ,$$

giving rise to a commutative diagram of sheaves

$$\begin{array}{ccc} \mathbb{Z}(1)_C|_U & \longrightarrow & \mathcal{H}_C^{2n-2}(\mathbb{Z}(n)_X)|_U \\ \downarrow & & \downarrow \\ \mathcal{O}_C|_U & \longrightarrow & \mathcal{H}_C^{2n-2}(\Omega_{X/\mathbb{C}}^{<n})|_U . \end{array}$$

There is a long exact sequence of sheaves on C

$$\begin{aligned} \cdots &\longrightarrow \mathcal{H}_C^j(\mathbb{Z}(n)_X) \longrightarrow \mathcal{H}_C^j(\Omega_{X/\mathbb{C}}^{<n}) \longrightarrow \mathcal{H}_C^{j+1}(\mathcal{D}(n)_X) \\ &\longrightarrow \mathcal{H}_C^{j+1}(\mathbb{Z}(n)_X) \longrightarrow \cdots \end{aligned}$$

Hence from the exponential sequence

$$0 \longrightarrow \mathbb{Z}(1)_C \longrightarrow \mathcal{O}_C \xrightarrow{\text{exp}} \mathcal{O}_C^* \longrightarrow 0 ,$$

and the above commutative diagram, we deduce that there is a well-defined map of sheaves

$$\mathcal{O}_C^*|_U \longrightarrow \mathcal{H}_C^{2n-1}(\mathcal{D}(n)_X)|_U .$$

We will now show that these locally defined maps patch together to give well-defined sheaf maps

$$(2.2) \quad \mathcal{O}_C \longrightarrow \mathcal{H}_C^{2n-2}(\Omega_{X/\mathbb{C}}^{n-1}) \quad \text{and} \quad \mathcal{O}_C^* \longrightarrow \mathcal{H}_C^{2n-1}(\mathcal{D}(n)_X) .$$

To do this, it suffices to show that the classes $f^*\alpha$, $f^*\beta$ and $f^*\gamma$ defined above are in fact independent of the map f , i.e., of the choice of generators for the ideal of C in U . This too can be seen “universally”. Since the ideal sheaf of C in X is locally generated by a regular sequence, any two such sets of local generators for \mathcal{I}_C on the affine open set U differ by the operation of an element of $\text{GL}_{n-1}(\mathcal{O}_X(V))$, for some neighbourhood V of $C \cap U$ in U . Hence it suffices to show that if $p : \text{GL}_{n-1}(\mathbb{C}) \times \mathbb{C}^{n-1} \rightarrow \mathbb{C}^{n-1}$ is the projection, and $m : \text{GL}_{n-1}(\mathbb{C}) \times \mathbb{C}^{n-1} \rightarrow \mathbb{C}^{n-1}$ the map given by the operation of $\text{GL}_{n-1}(\mathbb{C})$ on \mathbb{C}^{n-1} by invertible linear transformations, then $p^*\alpha = m^*\alpha$, $p^*\beta = m^*\beta$, and hence also $p^*\gamma = m^*\gamma$. We leave the verification of this to the reader, as a simple application of the Künneth formula.

Finally, note that for $U = X_{\text{reg}}$, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathbb{Z}(1)_C|_U & \longrightarrow & \mathcal{O}_C|_U & \longrightarrow & \mathcal{O}_C^*|_U & \longrightarrow 0 \\ & \cong \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \mathcal{H}_C^{2n-2}(\mathbb{Z}(n)_X)|_U & \longrightarrow & \mathcal{H}_C^{2n-2}(\Omega_{X/\mathbb{C}}^{<n})|_U & \longrightarrow & \mathcal{H}_C^{2n-1}(\mathcal{D}(n)_X)|_U & \longrightarrow 0 \end{array}$$

where the left vertical arrow is an isomorphism. For a smooth point $x \in C$, apply the functors $\mathcal{H}_{\{x\}}^j$ to the rows of the above diagram, and note that $\mathcal{H}_{\{x\}}^j(\mathbb{Z}(1)_C) = 0$ for $j \neq 2$, and $\mathcal{H}_{\{x\}}^j(\mathcal{O}_C) = 0$ for $j \neq 1$. We then obtain another diagram with exact rows

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{H}_{\{x\}}^1(\mathcal{O}_C) & \longrightarrow & \mathcal{H}_{\{x\}}^1(\mathcal{O}_C^*) & \longrightarrow & \mathcal{H}_{\{x\}}^2(\mathbb{Z}(1)_C) & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow \cong & \\ 0 \longrightarrow & \mathcal{H}_{\{x\}}^1(\mathcal{H}_C^{2n-2}(\Omega_{X/\mathbb{C}}^{<n})) & \longrightarrow & \mathcal{H}_{\{x\}}^1(\mathcal{H}_C^{2n-1}(\mathcal{D}(n)_X)) & \longrightarrow & \mathcal{H}_{\{x\}}^2(\mathcal{H}_C^{2n-2}(\mathbb{Z}(n)_X)) & \longrightarrow 0 \end{array}$$

The bottom row may be identified (see [Ha2], III, Ex. 8.7, pg. 161) with the exact sequence

$$0 \longrightarrow \mathcal{H}_{\{x\}}^{2n-1}(\Omega_{X/\mathbb{C}}^{<n}) \longrightarrow \mathcal{H}_{\{x\}}^{2n}(\mathcal{D}(n)_X) \longrightarrow \mathcal{H}_{\{x\}}^{2n}(\mathbb{Z}(n)_X) \longrightarrow 0 .$$

We claim that, under the above identification, the local cycle class of x in $\mathcal{H}_{\{x\}}^2(\mathcal{D}(1)_C) = \mathcal{H}_{\{x\}}^1(\mathcal{O}_C^*)$ maps to the corresponding local cycle class of x in $\mathcal{H}_{\{x\}}^{2n}(\mathcal{D}(n)_X)$. Choosing a suitable regular system of parameters on X at x , we reduce to checking this in the special case when $x \in X$ is the origin $0 \in \mathbb{C}^n$, and the curve C is the z_n -axis, given by the vanishing of the first $n - 1$ coordinates. We again leave this verification to the reader.

This means that, in the commutative diagram

$$\begin{array}{ccc} H^1_{\{x\}}(C, \mathcal{O}_C^*) & \longrightarrow & H^1(C, \mathcal{O}_C^*) = D^1(C) \\ \downarrow & & \downarrow \\ \mathbb{H}_{\{x\}}^{2n}(X, \mathcal{D}(n)_X) & \longrightarrow & D^n(X) \end{array}$$

the cycle class of x in $D^1(C)$ maps to that of x in $D^n(X)$. Hence we have shown that there is a commutative diagram

$$\begin{array}{ccc} Z^1(C) & \longrightarrow & Z^n(X) \\ \downarrow & & \downarrow \\ D^1(C) & \xrightarrow{\text{Gysin}} & D^n(X) \end{array}$$

It remains to show that the Gysin map $\text{Pic}(C) = D^1(C) \rightarrow D^n(X)$ is compatible with the topological Gysin map $H^2(C, \mathbb{Z}(1)) \rightarrow H^{2n}(X, \mathbb{Z}(n))$. Since $Z^1(C) \rightarrow D^1(C)$ is surjective, the compatibility of the two Gysin maps is clear from the fact that each one maps the class of x on C to the corresponding class on X . \square

Remark 2.4. Assume that the local complete intersection curve C lies in the Cohen-Macaulay locus X_{CM} of X . Then the first sheaf map in (2.2) factors as

$$(2.3) \quad \mathcal{O}_C \longrightarrow \mathcal{E}xt_X^{n-1}(\mathcal{O}_C, \Omega_{X/\mathbb{C}}^{n-1}) \longrightarrow \mathcal{H}_C^{2n-2}(\Omega_{X/\mathbb{C}}^{n-1}) .$$

To see this, note that

$$\beta \in \Gamma\left(\mathbb{A}_{\mathbb{C}}^{n-1}, \mathcal{E}xt_{\mathbb{A}_{\mathbb{C}}^{n-1}}^{n-1}\left(\mathcal{O}_{\{0\}}, \Omega_{\mathbb{A}_{\mathbb{C}}^{n-1}/\mathbb{C}}^{n-1}\right)\right) \subset \Gamma\left(\mathbb{A}_{\mathbb{C}}^{n-1}, \mathcal{H}_{\{0\}}^{n-1}\left(\Omega_{\mathbb{A}_{\mathbb{C}}^{n-1}/\mathbb{C}}^{n-1}\right)\right)$$

as it is the product of the classes $\text{dlog}(z_j)$ in (2.1). Further the map $U \rightarrow \mathbb{A}_{\mathbb{C}}^{n-1}$ is flat in a neighbourhood of $C \cap U$, as it is equidimensional. Thus $f^*\beta$ defines a class in $\Gamma(U, \mathcal{E}xt_X^{n-1}(\mathcal{O}_C, \Omega_{X/\mathbb{C}}^{n-1}))$ mapping to $\Gamma(U, \mathcal{H}_C^{n-1}(\Omega_{X/\mathbb{C}}^{n-1}))$. As

$$\begin{aligned} & \text{Ext}_{\text{GL}_{n-1}(\mathbb{C}) \times \mathbb{C}^{n-1}}^{n-1}(\mathcal{O}_{\text{GL}_{n-1}(\mathbb{C}) \times \{0\}}, \Omega_{\text{GL}_{n-1}(\mathbb{C}) \times \mathbb{C}^{n-1}/\mathbb{C}}^{n-1}) \\ & \subset H_{\text{GL}_{n-1}(\mathbb{C}) \times \{0\}}^{n-1}(\text{GL}_{n-1}(\mathbb{C}) \times \mathbb{C}^{n-1}, \Omega_{\text{GL}_{n-1}(\mathbb{C}) \times \mathbb{C}^{n-1}/\mathbb{C}}^{n-1}) \end{aligned} ,$$

the class $f^*\beta$ defines the factorization (2.3).

Lemma 2.5. *Let X be projective of dimension n over \mathbb{C} , $f : Y \rightarrow X$ the blow up of a smooth point $x \in X$. Then the natural maps $f_* : CH^n(Y) \rightarrow CH^n(X)$ and $f^* : D^n(X) \rightarrow D^n(Y)$ are isomorphisms, and there is a commutative diagram*

$$\begin{array}{ccc} Z^n(Y) & \longrightarrow & D^n(Y) \\ f_* \downarrow & & \cong \downarrow f^* \\ Z^n(X) & \longrightarrow & D^n(X) \end{array}$$

Proof. The isomorphism on Chow groups is easy to prove, using the fact that the exceptional divisor E is a projective space (the details are in [BiS]). That $f^* : D^n(X) \rightarrow D^n(Y)$ is an isomorphism is also easy to see, for the same reason, using also the exact sequence

$$0 \longrightarrow f^*\Omega_{X/\mathbb{C}}^1 \longrightarrow \Omega_{Y/\mathbb{C}}^1 \longrightarrow \Omega_{E/\mathbb{C}}^1 \longrightarrow 0 .$$

So we need to prove that if $y \in Y$ is any smooth point, then its class in $D^n(Y)$ is the inverse image of that of $f(y)$ in $D^n(X)$. This is clear if $f(y) \neq x$. If $f(y) = x$, we may argue as follows. There is a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{H}_{\{y\}}^{2n}(Y, \mathcal{D}(n)_Y) & \longrightarrow & H_{\{y\}}^{2n}(Y, \mathbb{Z}(n)) \oplus H_{\{y\}}^n(Y, \Omega_{Y/\mathbb{C}}^n) & \longrightarrow & H_{\{y\}}^{2n}(Y, \mathbb{C}(n)) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{H}_E^{2n}(Y, \mathcal{D}(n)_Y) & \longrightarrow & H_E^{2n}(Y, \mathbb{Z}(n)) \oplus H_E^n(Y, \Omega_{Y/\mathbb{C}}^n) & \longrightarrow & H_E^{2n}(Y, \mathbb{C}(n)) \longrightarrow 0 \\ & & f^* \uparrow & & f^* \uparrow & & f^* \uparrow \\ 0 & \longrightarrow & \mathbb{H}_{\{x\}}^{2n}(X, \mathcal{D}(n)_X) & \longrightarrow & H_{\{x\}}^{2n}(X, \mathbb{Z}(n)) \oplus H_{\{x\}}^n(X, \Omega_{X/\mathbb{C}}^n) & \longrightarrow & H_{\{x\}}^{2n}(X, \mathbb{C}(n)) \longrightarrow 0 \end{array}$$

Here the downward vertical arrows are the natural maps (“increase support”). It is standard that the topological local cycle classes of x and y have the same images in $H_E^{2n}(Y, \mathbb{Z}(n))$. Similarly, the images in $H_E^n(Y, \Omega_{Y/\mathbb{C}}^n)$ of the local cycle classes of x and y in Hodge cohomology are also known to be equal; for example, this follows from the existence of a Gysin map $f_* : H_{\{y\}}^n(Y, \Omega_{Y/\mathbb{C}}^n) \rightarrow H_{\{x\}}^n(X, \Omega_{X/\mathbb{C}}^n)$, which maps the local class of y to that of x , and which factors through

$$H_E^n(Y, \Omega_{Y/\mathbb{C}}^n) \xrightarrow{(f^*)^{-1}} H_{\{x\}}^n(X, \Omega_{X/\mathbb{C}}^n) .$$

Thus $f^*[x] = [y] \in \mathbb{H}_E^{2n}(Y, \mathcal{D}(n)_Y)$, and hence a similar equality is valid in $D^n(Y)$ as claimed. \square

Lemma 2.6. *The map $Z^n(X) \rightarrow D^n(X)$ factors through $CH^n(X)$, and hence determines a homomorphism $\varphi : CH^n(X)_{\text{deg } 0} \rightarrow A^n(X)$.*

Proof. This is similar to the corresponding proof in [BiS]. Let $C \subset X$ be a reduced Cartier curve, and $f \in R(C, X)$. Let $\pi : Y \rightarrow X$ be a composition of blow ups at smooth points so that the strict transform \tilde{C} of C in Y satisfies $\tilde{C}_{\text{sing}} = \tilde{C} \cap Y_{\text{sing}} \cong C \cap X_{\text{sing}}$. Then

$$R(C, X) = R(\tilde{C}, Y) = R(\tilde{C}, \tilde{C}) \quad ,$$

and $\pi_*(f)_{\tilde{C}} = (f)_C \in Z^n(X)$. Now from lemma 2.2, $(f)_{\tilde{C}} \mapsto 0 \in D^n(Y)$, and so from lemma 2.5, $(f)_C = \pi_*(f)_{\tilde{C}} = 0 \in D^n(X)$. \square

Corollary 2.7. *If $f : Y \rightarrow X$ is a composition of blow ups at smooth points, then we have a commutative diagram*

$$\begin{array}{ccc} CH^n(Y) & \longrightarrow & D^n(Y) \\ f_* \downarrow \cong & & \cong \uparrow f^* \\ CH^n(X) & \longrightarrow & D^n(X) \end{array}$$

Corollary 2.8. *For any reduced Cartier curve $C \subset X$, there are commutative diagrams*

$$\begin{array}{ccccc} Z^1(C) & \longrightarrow & Z^n(X) & & \\ \downarrow & & \downarrow & & \\ \text{Pic}(C) & \xrightarrow{\text{Gysin}} & CH^n(X) & \text{and} & \text{Pic}^0(C) \longrightarrow CH^n(X)_{\text{deg } 0} \\ \cong \downarrow & & \downarrow & & \cong \downarrow \\ D^1(C) & \xrightarrow{\text{Gysin}} & D^n(X) & & A^1(C) \longrightarrow A^n(X) \\ \downarrow & & \downarrow & & \\ H^2(C, \mathbb{Z}(1)) & \xrightarrow{\text{Gysin}} & H^{2n}(X, \mathbb{Z}(n)) & & \end{array}$$

Proof. As in the proof of lemma 2.6, by a composition of blow-ups at smooth points, we reduce to the case when C is a local complete intersection in X . Then lemma 2.3 implies the corollary. \square

Considering embedded resolution of singularities one obtains from 2.7 and 2.8 a second construction of the Gysin map in 1.8 over \mathbb{C} . At the same time, it gives the compatibility of this map with the Gysin map for the Deligne cohomology, constructed in 2.3.

3. Some general properties of $A^n(X)$ over \mathbb{C}

It is shown in [BiS] that if X is projective over \mathbb{C} of dimension n , then there is a natural surjection (which is referred to in [BiS] as the *Abel-Jacobi map*)

$$AJ_X^n : CH^n(X)_{\text{deg}0} \longrightarrow J^n(X) \\ := \frac{H^{2n-1}(X, \mathbb{C}(n))}{F^0 H^{2n-1}(X, \mathbb{C}(n)) + \text{image } H^{2n-1}(X, \mathbb{Z}(n))} ,$$

where by results of Deligne, $J^n(X)$ is a semi-abelian variety (since the non-zero Hodge numbers of $H^{2n-1}(X, \mathbb{Z}(n))$ lie in the set $\{(-1, 0), (0, -1), (-1, -1)\}$).

Lemma 3.1. *Let X be projective of dimension n over \mathbb{C} . Then there is a natural surjection $\psi : A^n(X) \rightarrow J^n(X)$, whose kernel is a \mathbb{C} -vector space. $A^n(X)$ has a unique structure as an algebraic group such that ψ is a morphism of algebraic groups, with additive kernel (i.e., with kernel isomorphic to a direct sum of copies of \mathbb{G}_a).*

Proof. By a result of Bloom and Herrera [BH], the natural map

$$H^{2n-1}(X, \mathbb{C}(n)) \longrightarrow H_{DR}^{2n-1}(X/\mathbb{C}) = \mathbb{H}^{2n-1}(X, \Omega_{X/\mathbb{C}}^\bullet)$$

is split injective. As explained in [D] (9.3.2), if $X_\bullet \rightarrow X$ is a suitable hypercovering by a smooth proper simplicial scheme, the splitting may be given by the composition

$$H_{DR}^{2n-1}(X/\mathbb{C}) \longrightarrow H_{DR}^{2n-1}(X_\bullet/\mathbb{C}) \cong H^{2n-1}(X_\bullet, \mathbb{C}(n)) \cong H^{2n-1}(X, \mathbb{C}(n)) .$$

From this description, the splitting is a map of filtered vector spaces, where $H^{2n-1}(X, \mathbb{C}(n))$ has the Hodge filtration for the mixed Hodge structure while $H_{DR}^{2n-1}(X/\mathbb{C})$ has the truncation filtration (i.e., the filtration *bête*).

Hence we obtain a commutative diagram

$$\begin{array}{ccc}
 H^{2n-1}(X, \mathbf{C}(n)) & \longrightarrow & H^{2n-1}(X, \mathbf{C}(n))/F^0 H^{2n-1}(X, \mathbf{C}(n)) \\
 \searrow \alpha & & \nearrow \vartheta \\
 & \mathbb{H}^{2n-1}(X, \Omega_{X/\mathbf{C}}^{<n}) &
 \end{array}$$

The map α comes from $\mathbf{C}(n) \rightarrow \Omega_{X/\mathbf{C}}^\bullet \rightarrow \Omega_{X/\mathbf{C}}^{<n}$, whereas ϑ comes from the fact that $\Omega_{X,\bullet}^i/\mathbf{C}$ is a simplicial sheaf and the definition of $F^0 H^{2n-1}(X, \mathbf{C}(n))$. The map ϑ induces the map ψ taking quotients modulo $H^{2n-1}(X, \mathbf{Z}(n))$. Note that by weight considerations, the natural map

$$H^{2n-1}(X, \mathbf{Z}(n)) \longrightarrow H^{2n-1}(X, \mathbf{C}(n))/F^0 H^{2n-1}(X, \mathbf{C}(n))$$

has a torsion kernel. Hence the kernels of ψ and ϑ are the same (and the latter is a \mathbf{C} -vector space). This represents $A^n(X)$ as an analytic group extension of the semi-abelian variety $J^n(X)$ by an additive group \mathbf{G}_a^r , for some r , and hence as an analytic group extension of an abelian variety by a group $\mathbf{G}_a^r \times \mathbf{G}_m^s$. As noted in [D], (10.1.3.3), for any abelian variety A over \mathbf{C} , the isomorphism classes of analytic and algebraic groups extensions of A by either \mathbf{G}_a or by \mathbf{G}_m coincide (as a consequence of GAGA); hence a similar property is valid for extensions by $\mathbf{G}_a^r \times \mathbf{G}_m^s$. This implies that $A^n(X)$ has a unique algebraic structure such that ψ is a homomorphism of algebraic groups over \mathbf{C} , as claimed. □

The following argument is fairly standard, though we do not have a specific reference for it.

Lemma 3.2. *Considering $D^n(X)$ as an analytic Lie group, with identity component $A^n(X)$, the composite map*

$$\begin{aligned}
 X_{\text{reg}} &\rightarrow Z^n(X) \rightarrow D^n(X), \\
 x &\mapsto \text{cycle class of } x \text{ in } D^n(X) \text{ ,}
 \end{aligned}$$

is analytic.

Proof. More generally, if Y is any irreducible non-singular variety (or connected complex manifold) together with a morphism (holomorphic map) $f : Y \rightarrow X_{\text{reg}}$, we will show that

$$y \mapsto \text{cycle class in } D^n(X) \text{ of } f(y)$$

is analytic. The graph Γ_f of f determines a closed analytic submanifold of $Y \times X$ which is purely of codimension n and contained in $Y \times X_{\text{reg}}$. As such, Γ_f has a cycle class

$$[\Gamma_f] \in D^n(Y \times X) = \mathbb{H}^{2n}(Y \times X, \mathcal{D}(n)_{Y \times X}) ,$$

namely the image of the localized cycle class defined as in [EV], §7 in

$$\mathbb{H}_{\Gamma_f}^{2n}(Y \times X, \mathcal{D}(n)_{Y \times X}) = \mathbb{H}_{\Gamma_f}^{2n}(Y \times X', \mathcal{D}(n)_{Y \times X})$$

where $X' \rightarrow X$ is any desingularization. If $\pi : Y \times X \rightarrow Y$ is the projection, one considers the image of this class $[\Gamma_f]$ under the map

$$\mathbb{H}_{\Gamma_f}^{2n}(Y \times X, \mathcal{D}(n)_{Y \times X}) \longrightarrow H^0(Y, D^n(X)_Y) ,$$

with

$$\begin{aligned} D^n(X)_Y = \mathbf{R}^{2n}\pi_*(0 \longrightarrow \mathcal{Z}_{Y \times X} \longrightarrow \mathcal{O}_{Y \times X} \\ \longrightarrow \Omega_{Y \times X/Y}^1 \longrightarrow \cdots \longrightarrow \Omega_{Y \times X/Y}^{n-1} \longrightarrow 0) . \end{aligned}$$

One has the following properties:

- (i) $\mathbf{R}^i\pi_*\mathcal{Z}(n)$ is the constant sheaf on Y with fibre $H^i(X, \mathcal{Z}(n))$
- (ii) the sheaf

$$\mathbf{R}^i\pi_*(0 \longrightarrow \mathcal{O}_{Y \times X} \longrightarrow \Omega_{Y \times X/Y}^1 \longrightarrow \cdots \longrightarrow \Omega_{Y \times X/Y}^{n-1} \longrightarrow 0)$$

coincides with the sheaf of *holomorphic* sections on Y of the trivial vector bundle with fibre

$$\mathbb{H}^i(X, (0 \longrightarrow \mathcal{O}_X \longrightarrow \Omega_{X/\mathbb{C}}^1 \longrightarrow \cdots \longrightarrow \Omega_{X/\mathbb{C}}^{n-1} \longrightarrow 0))$$

(iii) from (i) and (ii), one deduces that $D^n(X)_Y$ is the sheaf of holomorphic maps from Y to $D^n(X)$, considered as a commutative analytic Lie group.

It now remains to note that, by an obvious functoriality property of the cycle class [EV], §7, the above section of $D^n(X)_Y$, i.e., the holomorphic map $Y \rightarrow D^n(X)$, is just

$$y \mapsto \text{cycle class of } f(y) \text{ in } D^n(X) .$$

□

For X a curve, as in (i) of the next corollary, note that $\text{Pic}^0(X)$ has the natural algebraic structure obtained by representing a suitable Picard

functor. In particular, given an algebraic family of divisors (of degree 0) on X parametrized by a variety (or scheme) T , the induced map $T \rightarrow \text{Pic}^0(X)$ is automatically a morphism. On the other hand, $A^1(X)$ has the algebraic structure given by lemma 3.1. Hence, a priori, the induced map $T \rightarrow A^1(X)$ obtained from such a family is only analytic, since it is essentially given by integration. From (i) of the corollary, it will follow that it is in fact algebraic. The content of (ii) of the corollary is similar.

Corollary 3.3. (i) *If X is a curve, then the natural isomorphism $\text{Pic}^0(X) \cong A^1(X)$ of lemma 2.2 is an isomorphism of algebraic groups.*

(ii) *In general, if $C \subset X$ is a reduced Cartier curve, then the induced homomorphism $A^1(C) \rightarrow A^n(X)$ of corollary 2.8 is algebraic.*

Proof. (i) The identification is certainly analytic, and in both cases, when one represents the algebraic group as an extension of an abelian variety by a commutative affine group, the abelian variety in question is just $\text{Pic}^0(\tilde{X}) = J(\tilde{X}) = D^1(\tilde{X})$, the Jacobian of the normalized curve \tilde{X} (by which we mean the product of the Jacobians of the connected components of \tilde{X}). Now one argues that the identification must be algebraic as well, since one has the one-one correspondence between analytic and algebraic extensions of an abelian variety by $\mathbb{G}_a^r \times \mathbb{G}_m^s$.

(ii) Let $\tilde{X} \rightarrow X^{(n)}$ be a desingularization of $X^{(n)}$ such that the proper transform \tilde{C} of C is the normalization of C . First note that one has a factorization

$$\begin{array}{ccc} A^1(C) & \longrightarrow & A^n(X) \\ \downarrow & & \downarrow \\ J^1(C) & \longrightarrow & J^n(X) \\ \downarrow & & \downarrow \\ A^1(\tilde{C}) & \longrightarrow & A^n(\tilde{X}) \end{array}$$

where all maps are analytic group homomorphisms, and the vertical ones are algebraic (lemma 3.1). Indeed the map $C \rightarrow X$ induces a morphism of mixed Hodge structures $H^1(C) \rightarrow H^{2n-1}(X)$, and therefore an analytic group homomorphism $J^1(C) \rightarrow J^n(X)$, which has to be algebraic as it is compatible with its abelian part $J^1(\tilde{C}) \rightarrow A^n(\tilde{X})$ and all analytic group homomorphisms $\mathbb{G}_m^s \rightarrow \mathbb{G}_m^{s'}$ are algebraic. Similarly, all group homomorphisms $\mathbb{G}_a^r \rightarrow \mathbb{G}_a^{r'}$ are algebraic, and therefore $A^1(C) \rightarrow A^n(X)$ is algebraic as well. \square

Definition 3.4. *For any commutative algebraic group A over \mathbb{C} , let $\Omega(A)$ denote the dual vector space to the Lie algebra $\text{Lie}(A)$. We may then*

identify $\Omega(A)$ with the vector space of (closed) translation invariant regular 1-forms on A .

Our next goal is to give a description of $\Omega(A^n(X))$, generalizing the fact that for a non-singular projective variety X , $\Omega(A^n(X))$ is the space of holomorphic 1-forms on X (since in that case, $A^n(X)$ is the Albanese variety of X).

Lemma 3.5. *Let X be projective of dimension n over \mathbb{C} , and let ω_X denote the dualizing module of X (in the sense of [Ha], Ch. III, §7). Let $X^{(n)}$ be the union of the n -dimensional components of X , and let $\omega_{X^{(n)}}$ denote its dualizing module.*

(i) ω_X is annihilated by the ideal sheaf of $X^{(n)}$ in X . With its natural induced structure as an $\mathcal{O}_{X^{(n)}}$ -module, $\omega_X \cong \omega_{X^{(n)}}$, and is a torsion-free $\mathcal{O}_{X^{(n)}}$ -module. Hence for any coherent \mathcal{O}_X -module \mathcal{F} , the sheaf $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X)$ is also naturally an $\mathcal{O}_{X^{(n)}}$ -module, which is $\mathcal{O}_{X^{(n)}}$ -torsion free, and for any dense open set $U \subset X^{(n)}$, the restriction map

$$\text{Hom}_X(\mathcal{F}, \omega_X) \longrightarrow \text{Hom}_U(\mathcal{F}|_U, \omega_X|_U)$$

is injective. In particular, taking $U = X_{\text{reg}}$, so that $\omega_X|_U = \Omega_{U/\mathbb{C}}^n$, and taking $\mathcal{F} = \Omega_{X/\mathbb{C}}^{n-1}$, we have that $\text{Hom}_X(\Omega_{X/\mathbb{C}}^{n-1}, \omega_X)$ may be identified with a \mathbb{C} -subspace of the vector space of holomorphic 1-forms on X_{reg} which are meromorphic on $X^{(n)}$.

(ii) $\Omega(A^n(X))$ is naturally identified with the subspace of $\text{Hom}_X(\Omega_{X/\mathbb{C}}^{n-1}, \omega_X)$ consisting of closed 1-forms.

(iii) When $n = 1$,

$$\Omega(A^1(X)) = \Omega(\text{Pic}^0(X)) = H^0(X, \omega_X) .$$

(iv) Let $j : X_{\text{CM}} \rightarrow X$ be the inclusion of the open subset of Cohen-Macaulay points. The natural map

$$\Omega(A^n(X)) \longrightarrow (\text{closed 1-forms in } \text{Hom}_X(\Omega_{X/\mathbb{C}}^{n-1}, j_* j^* \omega_X))$$

is an isomorphism, where j_* denotes the meromorphic direct image.

Proof. (i) We note first that $\omega_X \cong \omega_{X^{(n)}}$, and the latter is a torsion-free $\mathcal{O}_{X^{(n)}}$ -module. Indeed, if we fix a projective embedding $X \hookrightarrow \mathbb{P}_{\mathbb{C}}^N$, then

$$\omega_X = \mathcal{E}xt_{\mathbb{P}_{\mathbb{C}}^N}^{N-n}(\mathcal{O}_X, \omega_{\mathbb{P}_{\mathbb{C}}^N}),$$

and there is an analogous formula for $\omega_{X^{(n)}}$. As in [Ha], we see by Serre duality on $\mathbb{P}_{\mathbb{C}}^N$ that $\mathcal{E}xt^i(\mathcal{F}, \omega_{\mathbb{P}_{\mathbb{C}}^N}) = 0$ for all $i \leq N - n$ for any

coherent sheaf \mathcal{F} supported in dimension $< n$. This gives the desired isomorphism, and implies that any local section of $\mathcal{O}_{X^{(n)}}$, which is a non zero-divisor, is also a non zero-divisor on $\mathcal{E}xt_{\mathbb{P}_{\mathbb{C}}^{N-n}}^n(\mathcal{O}_X, \omega_{\mathbb{P}_{\mathbb{C}}^N})$. This means exactly that $\omega_{X^{(n)}}$ is torsion-free.

We conclude that for any coherent \mathcal{O}_X -module \mathcal{F} , the sheaf $\mathcal{H}om_X(\mathcal{F}, \omega_X)$ is a torsion-free $\mathcal{O}_{X^{(n)}}$ -module as well. Applying this to $\mathcal{F} = \Omega_{X/\mathbb{C}}^{n-1}$ gives (i).

(iii) is a special case of (ii). To prove (ii), first note that from the definition of $A^n(X)$, we have

$$(3.1) \quad \text{Lie}(A^n(X)) = \text{coker}(d : H^n(X, \Omega_{X/\mathbb{C}}^{n-2}) \longrightarrow H^n(X, \Omega_{X/\mathbb{C}}^{n-1})) .$$

From Serre duality for H^n and Hom, as in the definition of the dualizing sheaf in [Ha], we have an identification of the dual vector space

$$H^n(X, \Omega_{X/\mathbb{C}}^i)^* = \text{Hom}_{\mathcal{O}_X}(\Omega_{X/\mathbb{C}}^i, \omega_X) ,$$

for any i . Thus $\Omega(A^n(X))$ is identified with the subspace of $\text{Hom}_{\mathcal{O}_X}(\Omega_{X/\mathbb{C}}^{n-1}, \omega_X)$ of elements φ such that the composition

$$\ell : H^n(X, \Omega_{X/\mathbb{C}}^{n-2}) \xrightarrow{d} H^n(X, \Omega_{X/\mathbb{C}}^{n-1}) \xrightarrow{\varphi} H^n(X, \omega_X) \cong \mathbb{C}$$

is 0. It remains to show that, identifying elements $\varphi \in \text{Hom}_{\mathcal{O}_X}(\Omega_{X/\mathbb{C}}^{n-1}, \omega_X)$ with certain holomorphic 1-forms on X_{reg} , $\Omega(A^n(X))$ is just the subspace of closed 1-forms.

To see this, since we may consider φ as a meromorphic 1-form on X which is holomorphic on X_{reg} , we can find a coherent sheaf of ideals \mathcal{J} , defining the Zariski closed subset $X_{\text{sing}} \subset X$ (i.e., the subscheme determined by \mathcal{J} has X_{sing} as its underlying reduced scheme), such that

(i) $\eta \mapsto \eta \wedge \varphi$ defines an element of $\text{Hom}_{\mathcal{O}_X}(\mathcal{J}\Omega_{X/\mathbb{C}}^{n-2}, \Omega_{X/\mathbb{C}}^{n-1})$

(ii) $\eta \mapsto \eta \wedge d\varphi$ defines an element of $\text{Hom}_{\mathcal{O}_X}(\mathcal{J}\Omega_{X/\mathbb{C}}^{n-2}, \omega_X)$, where we view ω_X as a certain coherent extension of $\Omega_{X_{\text{reg}}/\mathbb{C}}^n$ to X .

(Here $\mathcal{J}\mathcal{F}$ denotes image $(\mathcal{J} \otimes \mathcal{F} \rightarrow \mathcal{F})$, for any ideal sheaf \mathcal{J} and coherent sheaf \mathcal{F}). Since \mathcal{J} defines X_{sing} within X , the natural map

$$H^n(X, \mathcal{J}\Omega_{X/\mathbb{C}}^{n-2}) \longrightarrow H^n(X, \Omega_{X/\mathbb{C}}^{n-2})$$

is surjective, and for any $\varphi \in \text{Hom}_{\mathcal{O}_X}(\Omega_{X/\mathbb{C}}^{n-1}, \omega_X)$, the composition

$$\ell_1 : H^n(X, \mathcal{J}\Omega_{X/\mathbb{C}}^{n-2}) \xrightarrow{d} H^n(X, \Omega_{X/\mathbb{C}}^{n-1}) \xrightarrow{\varphi} H^n(X, \omega_X) \cong \mathbb{C}$$

factors through ℓ . Thus

$$\varphi \in \Omega(A^n(X)) \iff \ell_1 = 0 .$$

We have 2 other related linear functionals

$$\ell_2 : H^n(X, \mathcal{J}\Omega_{X/\mathbb{C}}^{n-2}) \longrightarrow \mathbb{C}, \quad \ell_3 : H^n(X, \mathcal{J}\Omega_{X/\mathbb{C}}^{n-2}) \longrightarrow \mathbb{C} ,$$

defined by

$$\begin{aligned} \ell_2 : H^n(X, \mathcal{J}\Omega_{X/\mathbb{C}}^{n-2}) &\xrightarrow{\wedge d\varphi} H^n(X, \omega_X) \cong \mathbb{C}, \\ \ell_3 : H^n(X, \mathcal{J}\Omega_{X/\mathbb{C}}^{n-2}) &\xrightarrow{\wedge d\varphi} H^n(X, \Omega_{X/\mathbb{C}}^{n-1}) \xrightarrow{d} H^n(X, \omega_X) \cong \mathbb{C} , \end{aligned}$$

where in the definition of ℓ_3 , we have let d also denote the composite of the exterior derivative $\Omega_{X/\mathbb{C}}^{n-1} \rightarrow \Omega_{X/\mathbb{C}}^n$ with the natural map $\Omega_{X/\mathbb{C}}^n \rightarrow \omega_X$. The formula

$$d(\eta \wedge \varphi) = d\eta \wedge \varphi + (-1)^{n-2} \eta \wedge d\varphi ,$$

for any $n - 2$ form η , implies that $\ell_3 = \ell_1 + (-1)^{n-2} \ell_2$.

Now by Serre duality and the $\mathcal{O}_{X(n)}$ -torsion freeness of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}\Omega_{X/\mathbb{C}}^{n-2}, \omega_X)$ (see (i)), ℓ_2 vanishes precisely when $d\varphi = 0$ as a 2-form on X_{reg} . On the other hand, we claim that for any $\varphi \in \text{Hom}_{\mathcal{O}_X}(\Omega_{X/\mathbb{C}}^{n-1}, \omega_X)$, the map ℓ_3 constructed as above is always 0. This will imply that $\ell_1 = 0 \iff \varphi$ is a closed meromorphic 1-form.

To prove that ℓ_3 vanishes, it suffices to prove that the map

$$H^n(X, \Omega_{X/\mathbb{C}}^{n-1}) \xrightarrow{d} H^n(X, \omega_X)$$

vanishes. One way to understand this is to note that if $\pi : Y \rightarrow X$ is a resolution of singularities, then there is a commutative diagram

$$\begin{array}{ccc} H^n(X, \Omega_{X/\mathbb{C}}^{n-1}) & \xrightarrow{d} & H^n(X, \omega_X) \\ \pi^* \downarrow & & \uparrow \pi_* \\ H^n(Y, \Omega_{Y/\mathbb{C}}^{n-1}) & \xrightarrow{d} & H^n(Y, \omega_Y) \end{array}$$

which reduces us to proving that $H^n(Y, \Omega_{Y/\mathbb{C}}^{n-1}) \xrightarrow{d} H^n(Y, \omega_Y)$ vanishes. This follows from Hodge theory, or alternately may be proved as in [Ha2], III, lemma 8.4.

Proof of (iv): We begin by recalling that since X is reduced, it is Cohen-Macaulay in codimension 1, so that $Z = X - j(X_{\text{CM}})$ has

codimension ≥ 2 in X . Let \mathcal{I} denote the ideal sheaf of Z in X . Let \mathcal{D}_m be the complex of sheaves

$$\mathcal{D}_m = (0 \longrightarrow j_! \mathbb{Z}_{X_{\text{CM}}}(n) \longrightarrow \mathcal{I}^{m+n-1} \xrightarrow{d} \mathcal{I}^{m+n-2} \Omega_{X/\mathbb{C}}^1 \xrightarrow{d} \dots \xrightarrow{d} \mathcal{I}^m \Omega_{X/\mathbb{C}}^{n-1}) .$$

Then \mathcal{D}_m is a subcomplex of $\mathcal{D}(n)_X$, whose cokernel complex consists of sheaves supported on Z ; the 0-th term of the cokernel is $\mathbb{Z}(n)_Z$, while the other terms are coherent sheaves supported on Z . Since $\dim Z \leq n - 2$, we see that \mathbb{H}^i of this cokernel complex vanishes for $i \geq 2n - 1$. Hence $\mathbb{H}^{2n}(X, \mathcal{D}_m) \rightarrow \mathbb{H}^{2n}(X, \mathcal{D}(n)_X)$ is an isomorphism, for all m . Now as in the proof of (i), one uses duality, to conclude that for all m , there are isomorphisms

$$\begin{aligned} \text{Hom}(\Omega_{X/\mathbb{C}}^{n-1}, \omega_X) &\longrightarrow \text{Hom}(\mathcal{I}^m \Omega_{X/\mathbb{C}}^{n-1}, \omega_X), \\ \text{Hom}(\Omega_{X/\mathbb{C}}^{n-2}, \omega_X) &\longrightarrow \text{Hom}(\mathcal{I}^m \Omega_{X/\mathbb{C}}^{n-2}, \omega_X) , \end{aligned}$$

and taking the direct limit over all m , we obtain (iv). □

Our next goal is the proof of proposition 3.8, which gives us another useful way to recognize elements of the vector space $\Omega(A^n(X))$. We make use of two lemmas.

Lemma 3.6. *Let $X \subset \mathbb{P}_{\mathbb{C}}^N$ be a reduced projective variety of dimension n . Then we can find a finite number of linear projections $\pi_i : X \rightarrow \mathbb{P}_{\mathbb{C}}^n$, each of which is a finite morphism, such that the induced sheaf map*

$$\bigoplus_i \pi_i^* \Omega_{\mathbb{P}_{\mathbb{C}}^n/\mathbb{C}}^{n-1} \longrightarrow \Omega_{X/\mathbb{C}}^{n-1}$$

is surjective.

Proof. For any linear projection $\pi : X \rightarrow \mathbb{P}_{\mathbb{C}}^n$, there is a factorization

$$\pi^* \Omega_{\mathbb{P}_{\mathbb{C}}^n/\mathbb{C}}^{n-1} \longrightarrow \Omega_{\mathbb{P}_{\mathbb{C}}^N/\mathbb{C}}^{n-1} \otimes \mathcal{O}_X \xrightarrow{\psi} \Omega_{X/\mathbb{C}}^{n-1} ,$$

where the natural map ψ is surjective.

So it suffices to prove the stronger assertion that there are projections π_i as above such that the induced sheaf map

$$\bigoplus_i \pi_i^* \Omega_{\mathbb{P}_{\mathbb{C}}^n/\mathbb{C}}^{n-1} \longrightarrow \Omega_{\mathbb{P}_{\mathbb{C}}^N/\mathbb{C}}^{n-1} \otimes \mathcal{O}_X$$

is surjective.

We claim that for any $x \in X$, we can find a finite set of such projections $\pi_i : X \rightarrow \mathbb{P}_{\mathbb{C}}^n$ such that the map of \mathbb{C} -vector spaces

$$\bigoplus_i \pi_i^* \Omega_{\mathbb{P}_{\mathbb{C}}^n/\mathbb{C}}^{n-1} \otimes \mathbb{C}(\pi_i(x)) \longrightarrow \Omega_{\mathbb{P}_{\mathbb{C}}^n/\mathbb{C}}^{n-1} \otimes \mathbb{C}(x)$$

is surjective. Indeed, the Grassmannian $\mathbf{G}_{\mathbb{C}}(n + 1, N + 1)$ (of $n + 1$ dimensional subspaces of \mathbb{C}^{N+1}) parametrizes linear projections from $\mathbb{P}_{\mathbb{C}}^N$ to $\mathbb{P}_{\mathbb{C}}^n$, and it contains a dense Zariski open subset corresponding to projections which are finite morphisms on X . Hence the n -dimensional vector subspaces

$$\pi^* \Omega_{\mathbb{P}_{\mathbb{C}}^n/\mathbb{C}}^1 \otimes \mathbb{C}(\pi(x)) \subset \Omega_{\mathbb{P}_{\mathbb{C}}^N/\mathbb{C}}^1 \otimes \mathbb{C}(x)$$

also range over a Zariski open subset of the Grassmannian of n -dimensional subspaces of the cotangent space of $\mathbb{P}_{\mathbb{C}}^N$ at x . In particular, we can find a finite number of them whose $(n - 1)$ -th exterior powers span the $(n - 1)$ -th exterior power of this cotangent space, namely $\Omega_{\mathbb{P}_{\mathbb{C}}^N/\mathbb{C}}^{n-1} \otimes \mathbb{C}(x)$.

Now suppose π_1, \dots, π_r are chosen finite linear projections $X \rightarrow \mathbb{P}_{\mathbb{C}}^n$, and that

$$\bigoplus_{i=1}^r \pi_i^* \Omega_{\mathbb{P}_{\mathbb{C}}^n/\mathbb{C}}^{n-1} \longrightarrow \Omega_{\mathbb{P}_{\mathbb{C}}^n/\mathbb{C}}^{n-1} \otimes \mathcal{O}_X$$

is not surjective. We can then find a point $x \in X$ at which the cokernel is non-zero. By the above claim, we can augment the set of projections to $\pi_1, \dots, \pi_r, \pi_{r+1}, \dots, \pi_s$ so that the cokernel of the new map

$$\bigoplus_{i=1}^{r+s} \pi_i^* \Omega_{\mathbb{P}_{\mathbb{C}}^n/\mathbb{C}}^{n-1} \longrightarrow \Omega_{\mathbb{P}_{\mathbb{C}}^n/\mathbb{C}}^{n-1} \otimes \mathcal{O}_X$$

does not have x in its support. Thus the support of the cokernel has strictly decreased. Now the lemma follows by Noetherian induction. \square

Lemma 3.7. *Let \mathcal{F} be a reflexive coherent sheaf on $\mathbb{P}_{\mathbb{C}}^n$, and ω a meromorphic section of $\mathcal{F} \otimes \Omega_{\mathbb{P}_{\mathbb{C}}^n/\mathbb{C}}^1$, which is regular on some given (non-empty) Zariski open subset $W \subset \mathbb{P}_{\mathbb{C}}^n$. Suppose there is a non-empty open set V in $\mathbf{G}_{\mathbb{C}}(2, n + 1)$, the Grassmannian of lines in $\mathbb{P}_{\mathbb{C}}^n$, such that*

(i) *each line $L \in V$ meets W , and is disjoint from the non-locally free locus of \mathcal{F}*

(ii) *for each $L \in V$, the image of ω in $(\mathcal{F} \otimes \Omega_{L/\mathbb{C}}^1) |_{L \cap W}$ extends to a regular section of $\mathcal{F} \otimes \Omega_{L/\mathbb{C}}^1$ on L .*

Then ω extends (uniquely) to a regular section on $\mathbb{P}_{\mathbb{C}}^n$ of $\mathcal{F} \otimes \Omega_{\mathbb{P}_{\mathbb{C}}^n/\mathbb{C}}^1$.

Proof. Since \mathcal{F} is reflexive, it is locally free outside a Zariski closed set A (of codimension ≥ 3), and any section of $\mathcal{F} \otimes \Omega_{\mathbb{P}_{\mathbb{C}}^n/\mathbb{C}}^1$ defined in the complement of A extends uniquely to a section on all of $\mathbb{P}_{\mathbb{C}}^n$. Since ω is a meromorphic section, it determines a (unique) regular section of some twist $\mathcal{F} \otimes \Omega_{\mathbb{P}_{\mathbb{C}}^n/\mathbb{C}}^1(D)$, for an effective divisor D ; there is a unique such twist D which is minimal with respect to the partial order on effective divisors (determined by inclusion of subschemes). Our goal is to show that $D = 0$.

If F is an irreducible component of $\text{supp } D$ which appears in D with multiplicity $r > 0$, then we can find a point $x \in F$ such that

(i) x is a non-singular point of F , and does not lie on any other component of D ; further, \mathcal{F} is locally free near x

(ii) V contains a line through x

(iii) there is a regular parameter $t \in \mathcal{O}_{x, \mathbb{P}_{\mathbb{C}}^n}$ (i.e., t is part of a regular system of parameters) such that t defines the ideal of F at x , and such that $t^r \omega$ determines a regular, non-vanishing section of $\mathcal{F} \otimes \Omega_{\mathbb{P}_{\mathbb{C}}^n/\mathbb{C}}^1$ in a neighbourhood of x .

It then follows that for a non-empty Zariski open set of lines L through x , we have $L \in V$, and $t^r \omega$ maps to a regular, non-vanishing section of $\mathcal{F} \otimes \Omega_{L/\mathbb{C}}^1$ near x , while ω itself maps to a regular section of $\mathcal{F} \otimes \Omega_{L/\mathbb{C}}^1$. However, t vanishes at x . This is a contradiction. \square

If $C \subset X$ is a reduced, local complete intersection Cartier curve, then in fact $C \subset X_{\text{CM}} \cap X^{(n)}$ (recall that X_{CM} denotes the (dense) Zariski open subset of Cohen-Macaulay points of X). The sheaf map $\mathcal{O}_C \rightarrow \mathcal{H}_C^{2n-1}(\Omega_{X/\mathbb{C}}^{n-1})$ in (2.2) induces a composite map

$$\begin{aligned} \alpha_C : H^1(C, \mathcal{O}_C) &\longrightarrow H^1(C, \mathcal{H}_C^{2n-1}(\Omega_{X/\mathbb{C}}^{n-1})) \\ &\longrightarrow H_C^n(X, \Omega_{X/\mathbb{C}}^{n-1}) \longrightarrow \mathbb{H}^{2n-1}(X, \Omega_X^{<n}_{/\mathbb{C}}) . \end{aligned}$$

This is just the map $\text{Lie}(\text{Pic}^0(C)) \rightarrow \text{Lie}(A^n(X))$ on Lie algebras induced by the composition of the group homomorphisms $\text{Pic}^0(C) \rightarrow A^1(C)$ and the Gysin map $A^1(C) \rightarrow A^n(X)$.

Proposition 3.8. (a) *Let $C \subset X$ be a reduced, local complete intersection Cartier curve, and let $U \subset X_{\text{reg}}$ be a dense open subset such that $U \cap C$ is dense in C_{reg} . Then the dual α_C^\vee of $\alpha_C : H^1(C, \mathcal{O}_C) \rightarrow \mathbb{H}^{2n-1}(X, \Omega_X^{<n}_{/\mathbb{C}})$ (i.e., of $\text{Lie}(\text{Pic}^0(C)) \rightarrow \text{Lie}(A^n(X))$) fits into a commutative diagram*

$$\begin{array}{ccc} \Omega(A^n(X)) & \xrightarrow{\subset} & H^0(X_{\text{reg}}, \Omega_{X_{\text{reg}}/\mathbb{C}}^1) \\ \downarrow \alpha_C^\vee & & \downarrow \text{restriction} \\ H^0(C, \omega_C) & \xrightarrow{\subset} & H^0(C \cap U, \Omega_{C \cap U/\mathbb{C}}^1) . \end{array}$$

(Here the right hand vertical arrow is given by restriction of 1- forms.)

(b) Let $U \subset X_{\text{reg}}$ be a dense Zariski open set, and let $\omega \in \Gamma(U, \Omega_{U/\mathbb{C}}^1)$ be closed. Then $\omega \in \Omega(A^n(X))$ if and only if

- (i) ω yields a meromorphic section on X of $\Omega_{X/\mathbb{C}}^1$
- (ii) for any reduced, local complete intersection Cartier curve $C \subset X$ such that $C \cap U$ is dense in C , the restriction of ω to $B = C_{\text{reg}} \cap U$ is in the image of the natural injective map

$$H^0(C, \omega_C) \longrightarrow H^0(B, \Omega_{B/\mathbb{C}}^1) .$$

Proof. First we prove (a). From lemma 3.5, it suffices to prove that if $\beta_C : H^1(C, \mathcal{O}_C) \rightarrow H^n(X, \Omega_{X/\mathbb{C}}^{n-1})$ is the obvious map through which α_C factors, then the dual map β_C^\vee fits into a commutative diagram

$$\begin{array}{ccc} H^0(X, \mathcal{H}om_X(\Omega_{X/\mathbb{C}}^{n-1}, \omega_X)) & \xrightarrow{\subset} & H^0(X_{\text{reg}}, \Omega_{X_{\text{reg}}/\mathbb{C}}^1) \\ \downarrow \beta_C^\vee & & \downarrow \text{restriction} \\ H^0(C, \omega_C) & \xrightarrow{\subset} & H^0(C \cap X_{\text{reg}}, \omega_{C \cap X_{\text{reg}}}) . \end{array}$$

Here we have used Serre duality on X and C to make the identifications

$$\begin{aligned} H^n(X, \Omega_{X/\mathbb{C}}^{n-1})^\vee &= H^0(X, \mathcal{H}om_X(\Omega_{X/\mathbb{C}}^{n-1}, \omega_X)), \\ H^1(C, \mathcal{O}_C)^\vee &= H^0(C, \omega_C) . \end{aligned}$$

Since C is a reduced, local complete intersection Cartier curve in X (so that $C \subset X_{\text{CM}} \cap X^{(n)}$), we have the adjunction formula

$$\omega_C = \mathcal{H}om_C \left(\bigwedge^{n-1} \mathcal{I}_C / \mathcal{I}_C^2, \omega_X \otimes \mathcal{O}_C \right) .$$

Hence there is a natural sheaf map

$$\psi_C : \mathcal{H}om_X(\Omega_{X/\mathbb{C}}^{n-1}, \omega_X) \longrightarrow \mathcal{H}om_C \left(\bigwedge^{n-1} \mathcal{I}_C / \mathcal{I}_C^2, \omega_X \otimes \mathcal{O}_C \right) = \omega_C$$

induced by restriction to C , and composition with the natural map

$$\bigwedge^{n-1} \mathcal{I}_C / \mathcal{I}_C^2 \rightarrow \Omega_{X/\mathbb{C}}^{n-1} \otimes \mathcal{O}_C,$$

$$f_1 \wedge \cdots \wedge f_{n-1} \mapsto df_1 \wedge \cdots \wedge df_{n-1} .$$

On any open set $U \subset X_{\text{reg}}$ with $U \cap C \subset C_{\text{reg}}$, one verifies at once, from the explicit description, that the map $\psi_C|_U$ is just the restriction map on 1-forms $\Omega_{U/\mathbb{C}}^1 \rightarrow \Omega_{C \cap U/\mathbb{C}}^1$.

Hence the desired commutativity (which implies (a)) follows from:

Claim 3.9. β^\vee is the map induced by ψ_C on global sections.

To prove the claim, first note that for the local complete intersection curve C in X_{CM} , one also has

$$\mathcal{E}xt_X^{n-a}(\mathcal{O}_C, \omega_X) = \begin{cases} \omega_C & \text{for } a = 1 \\ 0 & \text{for } a \neq 1 \end{cases} ,$$

Hence there is a Gysin map given as the composite

$$\begin{aligned} H^1(C, \omega_C) &= H^1(X, \mathcal{E}xt_X^{n-1}(\mathcal{O}_C, \omega_X)) \xrightarrow{\epsilon} \text{Ext}_X^n(\mathcal{O}_C, \omega_X) \\ &\longrightarrow H_C^n(X, \omega_X) \longrightarrow H^n(X, \omega_X) \end{aligned}$$

where ϵ is the isomorphism resulting from the (degenerate) spectral sequence

$$E_2^{a,b-a} = H^a(X, \mathcal{E}xt_X^{b-a}(\mathcal{O}_C, \omega_X)) \implies \text{Ext}_X^b(\mathcal{O}_C, \omega_X) .$$

The trace map $\text{Tr}_C : H^1(C, \omega_C) \rightarrow \mathbb{C}$ (of Serre duality on C) factors as

$$\text{Tr}_C : H^1(C, \omega_C) \xrightarrow{\text{Gysin}} H^n(X, \omega_X) \xrightarrow{\text{Tr}_X} \mathbb{C}$$

(one way to verify this is to show that the composite $\text{Tr}_X \circ \text{Gysin}$ has the universal property of Tr_C).

Now the claim 3.9 amounts to the assertion that the following diagram commutes:

$$\begin{array}{ccc} H^1(C, \mathcal{O}_C) & \xrightarrow{\text{Gysin}} & H^n(X, \Omega_{X/\mathbb{C}}^{n-1}) \\ \psi_C(\varphi) \downarrow & & \downarrow \varphi \\ H^1(C, \omega_C) & \xrightarrow{\text{Gysin}} & H^n(X, \omega_X) \end{array}$$

From remark 2.4, this will follow if we prove the commutativity of the diagram of \mathcal{O}_X -linear maps

$$\begin{array}{ccc} \mathcal{O}_C & \longrightarrow & \mathcal{E}xt_X^{n-1}(\mathcal{O}_C, \Omega_X^{n-1}/\mathbb{C}) \\ \psi_C(\varphi) \downarrow & & \downarrow \varphi \\ \omega_C & \xrightarrow{\cong} & \mathcal{E}xt_X^{n-1}(\mathcal{O}_C, \omega_X) . \end{array}$$

As ω_C is torsion-free, it is enough to check this commutativity on a suitable open subset of the regular locus of C , where it is easily verified.

We now show the “if” part of (b) (note that the other direction follows directly from (a)). By lemma 3.6, it suffices to prove that for each finite, linear projection $\pi : X \rightarrow \mathbb{P}_{\mathbb{C}}^n$, the meromorphic 1-form ω determines a section of

$$\mathcal{H}om_X(\pi^* \Omega_{\mathbb{P}_{\mathbb{C}}^n}^{n-1}, \omega_X).$$

Since π is a finite morphism,

$$\pi_* \omega_X = \mathcal{H}om_{\mathbb{P}_{\mathbb{C}}^n}(\pi_* \mathcal{O}_X, \omega_{\mathbb{P}_{\mathbb{C}}^n}) ,$$

and we have a sequence of natural identifications of sheaves

$$\begin{aligned} \pi_* \mathcal{H}om_X(\pi^* \Omega_{\mathbb{P}_{\mathbb{C}}^n}^{n-1}, \omega_X) &\cong \mathcal{H}om_{\mathbb{P}_{\mathbb{C}}^n}(\Omega_{\mathbb{P}_{\mathbb{C}}^n}^{n-1} \otimes \pi_* \mathcal{O}_X, \omega_{\mathbb{P}_{\mathbb{C}}^n}) \\ &\cong \mathcal{H}om_{\mathbb{P}_{\mathbb{C}}^n}(\pi_* \mathcal{O}_X, \mathcal{H}om_{\mathbb{P}_{\mathbb{C}}^n}(\Omega_{\mathbb{P}_{\mathbb{C}}^n}^{n-1}, \omega_{\mathbb{P}_{\mathbb{C}}^n})) \cong \mathcal{F} \otimes \Omega_{\mathbb{P}_{\mathbb{C}}^n}^1, \end{aligned}$$

where $\mathcal{F} = \mathcal{H}om_{\mathbb{P}_{\mathbb{C}}^n}(\pi_* \mathcal{O}_X, \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^n})$ is a (non-zero) coherent reflexive sheaf on $\mathbb{P}_{\mathbb{C}}^n$.

Let $W \subset \mathbb{P}_{\mathbb{C}}^n$ be a dense open subset such that $\pi^{-1}(W) \subset U$. Then ω determines a section of $\mathcal{F} \otimes \Omega_{\mathbb{P}_{\mathbb{C}}^n}^1$ on W , and we want to show it extends to a global section of this sheaf. We do this by verifying that the hypotheses of lemma 3.7 are satisfied.

Let L be a line in $\mathbb{P}_{\mathbb{C}}^n$, disjoint from the non-flat locus of $\pi : X^{(n)} \rightarrow \mathbb{P}_{\mathbb{C}}^n$ (which is a subset of $\mathbb{P}_{\mathbb{C}}^n$ of codimension ≥ 2 , since $X^{(n)}$ is reduced and purely of dimension n). Then the scheme-theoretic inverse image of L in $X^{(n)}$ is a closed, local complete intersection subscheme of $X^{(n)}$, purely of dimension 1, which is contained in the Cohen-Macaulay locus of $X^{(n)}$ (since $X^{(n)}$ is Cohen-Macaulay precisely at all points $x \in X$ where π is flat). If further L is not contained in the branch locus of π on $X^{(n)}$ (i.e., π is étale over all but finitely many points of L), then $\pi^{-1}(L) = D$ is non-singular outside a finite set. Thus D is a reduced, complete intersection curve in $X^{(n)}$. Further, if $D \cap X^{<n} = \emptyset$, then D is a reduced local complete intersection curve in X , whose non-singular locus is contained in X_{reg} . In particular D is a reduced Cartier curve in X . Finally, if L is not contained in the

image of $X - U$, then D has finite intersection with $X - U$, and hence $D \cap U$ is dense in D . Clearly the set of all such lines L contains a non-empty open subset of the Grassmannian of lines.

For a line L as above, we have

$$\begin{aligned} \pi_* \omega_D &\cong \mathcal{H}om_L(\pi_* \mathcal{O}_D, \omega_L) \cong \mathcal{H}om_L(\pi_* \mathcal{O}_X \otimes \mathcal{O}_L, \omega_L) \\ &\cong \mathcal{H}om_L(\pi_* \mathcal{O}_X \otimes \mathcal{O}_L, \Omega_{L/\mathbb{C}}^1) \cong \mathcal{F} \otimes \Omega_{L/\mathbb{C}}^1, \end{aligned}$$

since $\mathcal{F} \otimes \mathcal{O}_L \cong \mathcal{H}om_L(\pi_* \mathcal{O}_D, \mathcal{O}_L)$ (as π is flat over L). Since we are given that the image of ω in $\Omega_{D \cap U/\mathbb{C}}^1$ extends to a global section of ω_D , it follows that the corresponding section of $\mathcal{F} \otimes \Omega_{L/\mathbb{C}}^1|_{L \cap W}$ extends to a global section of $\mathcal{F} \otimes \Omega_{L/\mathbb{C}}^1$. Thus we have verified the hypotheses of lemma 3.7. □

Remark 3.10. Two properties of $A^n(Y)$, which are true for smooth projective varieties Y , do not carry over to the general case: the compatibility with products, and its dimension being constant in a flat family. We give examples to illustrate these pathologies.

Let X and Y be projective varieties of dimension n and m , respectively, and let $r(X)$ and $r(Y)$ denote the number of irreducible components of dimensions n and m respectively. By [D] the Künneth decomposition

$$\begin{aligned} &H^{2(n+m)-1}(X \times Y, \mathbb{Z}) / (\text{torsion}) \\ &= \left[H^{2n-1}(X, \mathbb{Z})^{r(Y)} \oplus H^{2m-1}(Y, \mathbb{Z})^{r(X)} \right] / (\text{torsion}) \end{aligned}$$

is compatible with the Hodge structure. Thus

$$J^{n+m}(X \times Y) = J^n(X)^{r(Y)} \times J^m(Y)^{r(X)}. \tag{3.2}$$

For $A^{n+m}(X \times Y)$ the picture is wilder. By (3.1) in the proof of 3.5, we have

$$\begin{aligned} &\text{Lie}(A^{n+m}(X \times Y)) \\ &= \frac{H^n(\Omega_{X/\mathbb{C}}^{n-1}) \otimes H^m(\Omega_{Y/\mathbb{C}}^m) \oplus H^n(\Omega_{X/\mathbb{C}}^n) \otimes H^m(\Omega_{Y/\mathbb{C}}^{m-1})}{H^n(\Omega_{X/\mathbb{C}}^{n-2}) \otimes H^m(\Omega_{Y/\mathbb{C}}^m) \oplus H^n(\Omega_{X/\mathbb{C}}^{n-1}) \otimes H^m(\Omega_{Y/\mathbb{C}}^{m-1}) \oplus H^n(\Omega_{X/\mathbb{C}}^n) \otimes H^m(\Omega_{Y/\mathbb{C}}^{m-2})} \end{aligned}$$

where the maps from the denominator are

$$d_X \otimes \text{id}_Y, \quad d_X \otimes \text{id}_Y + (-1)^{n-1} \text{id}_X \otimes d_Y \text{ and } \text{id}_X \otimes d_Y.$$

Consider an elliptic curve E , the rational curve $\Gamma = (x^3 - y^2z) \subset \mathbb{P}_{\mathbb{C}}^2$, with a cusp, and the union of three rational curves $C = (xyz) \subset \mathbb{P}_{\mathbb{C}}^2$. They all are fibres of the family $\mathcal{C} \rightarrow \mathbb{P} = \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)))$ of curves of degree three in \mathbb{P}^2 .

Hodge theory implies that

$$d : H^1(E, \mathcal{O}_E) \longrightarrow H^1(E, \Omega_E^1) \text{ and } d : H^1(C, \mathcal{O}_C) \longrightarrow H^1(C, \Omega_C^1) \cong \mathbb{C}^3$$

are both zero. Using (3.3) this shows

$$\begin{aligned} A^2(C \times E) &= J^2(C \times E) = A^1(C) \times A^1(E)^3 = \mathbb{G}_m \times E^3 \\ A^2(C \times C) &= J^2(C \times C) = A^1(C)^3 \times A^1(C)^3 = \mathbb{G}_m^6 . \end{aligned}$$

On the other hand, $\Gamma - (0 : 1 : 0) = \text{Spec}(\mathbb{C}[t^2, t^3])$ and, if $\pi : \tilde{\Gamma} \rightarrow \Gamma$ denotes the normalization, one has exact equations

$$0 \longrightarrow \mathcal{O}_{\Gamma} \longrightarrow \pi_*(\mathcal{O}_{\tilde{\Gamma}}) \longrightarrow \mathbb{C}t \longrightarrow 0$$

and

$$0 \longrightarrow \Omega_{\Gamma/\mathbb{C}}^1 \longrightarrow \pi_*(\Omega_{\tilde{\Gamma}/\mathbb{C}}^1) \longrightarrow \mathbb{C}dt \longrightarrow 0 .$$

Thus $\mathbb{C}t = H^1(\Gamma, \mathcal{O}_{\Gamma}) \xrightarrow[\cong]{d} \mathbb{C}dt \xrightarrow[\mathbb{C}]{d} H^1(\Gamma, \Omega_{\Gamma/\mathbb{C}}^1) \cong \mathbb{C}^2$ and one obtains by 3.3

$$\begin{aligned} A^2(\Gamma \times E) &= \mathbb{G}_a \times E = A^1(\Gamma) \times A^1(E) \\ A^2(\Gamma \times \Gamma) &= \frac{\mathbb{C}^2 \times \mathbb{C}^2}{\mathbb{C}} = \mathbb{G}_a^3 \text{ whereas} \\ A^1(\Gamma) \times A^1(\Gamma) &= \mathbb{G}_a \times \mathbb{G}_a . \end{aligned}$$

In particular, a product formula as (3.2) fails for A^n instead of J^n , and the dimension of J^n and A^n are not constant for the fibres $\mathcal{C} \times \mathcal{C} \rightarrow \mathbb{P} \times \mathbb{P}$.

It is amusing to write down the cycle map for the last example. Writing

$$\begin{aligned} \Gamma_{\text{reg}} \times \Gamma_{\text{reg}} &= (\Gamma - (0 : 0 : 1)) \times (\Gamma - (0 : 0 : 1)) \\ &= \text{Spec}(\mathbb{C}[u] \otimes_{\mathbb{C}} \mathbb{C}[v]) , \end{aligned}$$

$\Omega(\Gamma \times \Gamma) = \text{Hom}_{\Gamma \times \Gamma}(\Omega_{\Gamma \times \Gamma/\mathbb{C}}^1, \omega_{\Gamma \times \Gamma})_{\text{cl}}$ decomposes as

$$\begin{aligned}
 & (H^0(\Gamma, \mathcal{H}om(\Omega_{\Gamma/\mathbb{C}}^1, \omega_\Gamma)) \otimes H^0(\Gamma, \omega_\Gamma) \\
 & \quad \oplus H^0(\Gamma, \omega_\Gamma) \otimes H^0(\Gamma, \mathcal{H}om(\Omega_{\Gamma/\mathbb{C}}^1, \omega_\Gamma)))_{\text{cl}} \\
 & = (\mathbb{C}dv \oplus \mathbb{C}udv \oplus \mathbb{C}du \oplus \mathbb{C}vdu)_{\text{cl}} \\
 & = \mathbb{C}dv \oplus \mathbb{C}du \oplus \mathbb{C}(udv + vdu).
 \end{aligned}$$

The cycle map is

$$\begin{aligned}
 \Pi_{\Gamma_{\text{reg}} \times \Gamma_{\text{reg}}} &= \mathbb{A}_{\mathbb{C}}^2 \times \mathbb{A}_{\mathbb{C}}^2 \longrightarrow \mathbb{G}_a^3 \\
 ((x_1, x_2), (y_1, y_2)) &\mapsto \begin{cases} du \mapsto y_1 - x_1 \\ dv \mapsto y_2 - x_2 \\ udv + vdu \mapsto y_1y_2 - x_1x_2 \end{cases} .
 \end{aligned}$$

4. The universal property over \mathbb{C}

Let U_1, \dots, U_r be the connected components of X_{reg} , and for each i , let $p_i \in U_i$ be a base point.

Let G be a commutative algebraic group. By 1.14 and 1.13 a homomorphism (of abstract groups) $\phi : CH^n(X)_{\text{deg } 0} \rightarrow G$ is regular, if and only if $\phi \circ \gamma_m : S^m(X_{\text{reg}}) \rightarrow G$ is a morphism of varieties, for some $m > 0$.

Theorem 4.1. (i) *The homomorphism $\varphi : CH^n(X)_{\text{deg } 0} \rightarrow A^n(X)$ constructed in lemma 2.6 is regular and surjective.*

(ii) *The cokernel of the map $H_1(X_{\text{reg}}, \mathbb{Z}) \rightarrow \text{Lie}(A^n(X))$, defined by integration of 1-forms over homology classes, is naturally isomorphic to $A^n(X)$ and the composite $(\varphi \circ \gamma)^{(-)} : \Pi_{X_{\text{reg}}} \rightarrow CH^n(X)_{\text{deg } 0} \rightarrow A^n(X)$ is given by*

$$(x, y) \mapsto \left\{ \omega \mapsto \int_x^y \omega \right\}$$

(iii) (Universality) *φ satisfies the following universal property: for any regular homomorphism $\phi : CH^n(X)_{\text{deg } 0} \rightarrow G$ to a commutative algebraic group there exists a unique homomorphism $h : A^n(X) \rightarrow G$ of algebraic groups with $\phi = h \circ \varphi$.*

Proof of (i). It suffices to prove that $\varphi \circ \gamma_1 : U = X_{\text{reg}} \rightarrow A^n(X)$ is a morphism. Note that by lemma 3.2 it is analytic. Further, we have the following.

(a) The composition $U \rightarrow A^n(X) \rightarrow \text{Alb}(\tilde{X})$ is a morphism, where \tilde{X} is a resolution of singularities of $X^{(n)}$, since we may then regard U as an open subset of \tilde{X} , and the map $U \rightarrow \text{Alb}(\tilde{X})$ is the restriction of the Albanese mapping for \tilde{X} , with appropriate base-points. Here $\text{Alb}(\tilde{X})$ is the product of the Albanese varieties of the connected components of \tilde{X} , and $A^n(X)$ is an extension of $\text{Alb}(\tilde{X})$ by a group $\mathbf{G}_a^r \times \mathbf{G}_m^s$, so that in particular $A^n(X) \rightarrow \text{Alb}(\tilde{X})$ is a Zariski locally trivial fibre bundle.

(b) For each reduced Cartier curve $C \subset X$, the composite

$$C_{\text{reg}} \longrightarrow U \longrightarrow A^n(X)$$

is a morphism. Indeed, for each component B_0 of C_{reg} , the composition

$$B_0 \longrightarrow U \xrightarrow{\gamma_1} CH^n(X)_{\text{deg } 0} \xrightarrow{\varphi} A^n(X)$$

agrees with

$$B_0 \longrightarrow \text{Pic}^0(C) \longrightarrow CH^n(X)_{\text{deg } 0} \xrightarrow{\varphi} A^n(X)$$

up to a translation, and by corollary 3.3, the latter is algebraic.

Now we may argue as in [BiS]: we are reduced to proving that if V is a non-singular affine variety, a holomorphic function on V which is algebraic when restricted to “almost all” algebraic curves in V , is in fact an algebraic regular function. This may be proved using Noether normalization and power series expansions for holomorphic functions on \mathbf{C}^n , or deduced from [Si], (1.1).

Since $\Omega(A^n(X))$ is a finite dimensional subspace of 1-forms on U , there exist reduced local complete intersection Cartier curves $C_i \subset X$, for $i = 1, \dots, s$, such that

$$\Omega(A^n(X)) \longrightarrow \bigoplus_{i=1}^s H^0(C_i, \omega_{C_i})$$

is injective. Hence

$$\bigoplus_{i=1}^s \text{Pic}^0(C_i) \xrightarrow{\Sigma \psi_i} A^n(X)$$

is surjective.

Proof of (ii) and (iii): Let $\phi : CH^n(X)_{\text{deg } 0} \rightarrow G$ be a regular homomorphism to a commutative algebraic group G . By lemma 1.12 the image of ϕ is contained in the connected component of the identity of G . Hence we may assume without loss of generality that G is connected.

Now $\Omega(G)$ consists of closed, translation-invariant 1-forms. Thus if

$$h = \phi \circ \gamma_1 : U \longrightarrow G ,$$

then the image of $h^* : \Omega(G) \rightarrow \Gamma(U, \Omega_{X/\mathbb{C}}^1)$ is contained in the subspace of closed 1-forms. We claim that in fact $h^*(\Omega(G)) \subset \Omega(A^n(X))$.

This is deduced from the criterion of proposition 3.8, (b), since we know that for any reduced Cartier curve C in X , the composition

$$\text{Pic}^0(C) \longrightarrow CH^n(X)_{\text{deg}0} \xrightarrow{\phi} G$$

is a homomorphism of algebraic groups. Now we observe that if B_0 is any component of C_{reg} , then

$$B_0 \longrightarrow \text{Pic}^0(C) = \text{Lie}(A^1(C))/\text{image } H_1(C_{\text{reg}}, \mathbb{Z})$$

is given by integration of 1-forms in $H^0(C, \omega_C)$. Moreover the composite

$$B_0 \longrightarrow U \xrightarrow{h} G$$

agrees with

$$B_0 \longrightarrow \text{Pic}^0(C) \longrightarrow G ,$$

up to a translation by an element of G (and elements of $\Omega(G)$ are translation invariant). Dualizing the above inclusion on 1-forms, we thus obtain a map on Lie algebras $\text{Lie}(A^n(X)) \rightarrow \text{Lie}(G)$. This fits into a commutative diagram

$$\begin{array}{ccc} H_1(U, \mathbb{Z}) & \longrightarrow & \text{Lie}(A^n(X)) \\ \downarrow & & \downarrow \\ H_1(G, \mathbb{Z}) & \longrightarrow & \text{Lie}(G) \end{array}$$

where the horizontal arrows are given by integration of 1-forms over homology classes. Further there is a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & \text{Lie}(A^n(X))/\text{image } H_1(U, \mathbb{Z}) \\ \gamma_1 \downarrow & & \downarrow \tilde{\phi} \\ CH^n(X)_{\text{deg}0} & \xrightarrow{\phi} & G = \text{Lie}(G)/\text{image } H_1(G, \mathbb{Z}) \end{array}$$

where $\tilde{\phi}$ is a homomorphism of analytic groups, and where the upper horizontal arrow is given by integration of 1-forms in $\Omega(A^n(X))$.

We claim that the map $H_1(U, \mathbb{Z}) \rightarrow \text{Lie}(A^n(X)) = \Omega(A^n(X))^*$ factors through the (surjective) composition

$$H_1(U, \mathbb{Z}) \xrightarrow{\cong} H_c^{2n-1}(U, \mathbb{Z}(n)) \longrightarrow H^{2n-1}(X, \mathbb{Z}(n)) ,$$

where H_c^* denotes compactly supported cohomology, and the isomorphism is by Poincaré duality. Indeed, let $C \subset X^{(n)}$ be a sufficiently general reduced complete intersection curve in $X^{(n)}$. Then

$$C \cap X^{<n} = \emptyset, \quad C_{\text{sing}} = C \cap X_{\text{sing}}^{(n)} = C \cap X_{\text{sing}},$$

and one has a Gysin homomorphism

$$H^1(C, \mathbb{Z}(1)) \longrightarrow H^{2n-1}(X, \mathbb{Z}(n)) \cong H^{2n-1}(X^{(n)}, \mathbb{Z}(n))$$

which fits into a commutative diagram with exact rows

$$\begin{array}{ccccc} H^0(C_{\text{sing}}, \mathbb{Z}(1)) & \longrightarrow & H_c^1(C \cap U, \mathbb{Z}(1)) & \longrightarrow & H^1(C, \mathbb{Z}(1)) \\ \downarrow & & \downarrow & & \downarrow \text{Gysin} \\ H^{2n-2}(X_{\text{sing}}^{(n)}, \mathbb{Z}(n)) & \longrightarrow & H_c^{2n-1}(U, \mathbb{Z}(n)) & \longrightarrow & H^{2n-1}(X^{(n)}, \mathbb{Z}(n)) \end{array}$$

The left hand vertical arrow is in fact surjective, since $H^{2n-2}(X_{\text{sing}}^{(n)}, \mathbb{Z}(n-1))$ is the free abelian group on the $(n-1)$ -dimensional components of $X_{\text{sing}}^{(n)}$, and (since C is a general complete intersection) C_{sing} has non-empty intersection (which is supported at smooth points, and is transverse) with each such component of $X_{\text{sing}}^{(n)}$. Now we note that the composite $H_1(C \cap U, \mathbb{Z}) \rightarrow H_1(U, \mathbb{Z}) \rightarrow \text{Lie}(A^n(X))$ factors through the surjective composite $H_1(U, \mathbb{Z}) \cong H_c^1(U, \mathbb{Z}(1)) \rightarrow H^1(C, \mathbb{Z}(1))$, since $C \cap U \rightarrow U \rightarrow A^n(X)$ is compatible with a homomorphism $\text{Pic}^0(C) \rightarrow A^n(X)$ (here “compatible” means that for any component B_0 of $C \cap U$, the composites $B_0 \rightarrow U \rightarrow A^n(X)$ and $B_0 \rightarrow \text{Pic}^0(C) \rightarrow A^n(X)$ agree up to translation by an element of $A^n(X)$). Now a diagram chase implies the claim made at the beginning of the paragraph.

Thus in the diagram (4.1) we see that $\text{Lie}(A^n(X))/\text{image } H_1(U, \mathbb{Z})$ is identified with

$$\text{Lie}(A^n(X))/\text{image } H^{2n-1}(X, \mathbb{Z}(n)) = A^n(X) .$$

Hence there is a homomorphism $\tilde{\phi} : A^n(X) \rightarrow G$, such that $\gamma_1 \circ \phi : U \rightarrow G$ factors through $A^n(X)$. Since $\gamma_1^* : \Omega(A^n(X)) \rightarrow \Gamma(U, \Omega_{U/\mathbb{C}}^1)$ is injective, the induced map $A^n(X) \rightarrow G$ with this property is unique, since the corresponding map on Lie algebras is uniquely determined. Since image γ_1 generates $CH^n(X)_{\text{deg } 0}$, the two homomorphisms

$$\phi : CH^n(X)_{\text{deg } 0} \longrightarrow G, \quad CH^n(X)_{\text{deg } 0} \xrightarrow{\varphi} A^n(X) \xrightarrow{\tilde{\phi}} G$$

must coincide. This proves the universal property of φ , except that we need to note that $\tilde{\phi}$ is a morphism. By lemma 1.13, ϕ induces an algebraic group homomorphism $\text{Pic}^0(C) \rightarrow G$ for all admissible pairs (C', ι) , with $C = \iota(C')$. As above, we can choose reduced complete intersection curves $C_i, i = 1, \dots, s$, such that

$$\bigoplus_{i=1}^s \text{Pic}^0(C_i) \xrightarrow{\oplus \psi_i} A^n(X)$$

is surjective. As $\tilde{\phi} \circ (\oplus \psi_i)$ is an algebraic group homomorphism, $\tilde{\phi}$ is an algebraic group homomorphism as well. □

Remark 4.2. Lemma 3.1, combined with the Roitman Theorem proved in [BiS], imply that $\varphi : CH^n(X)_{\text{deg } 0} \rightarrow A^n(X)$ is an isomorphism on torsion subgroups. In other words, the Roitman Theorem is valid for $\varphi : CH^n(X)_{\text{deg } 0} \rightarrow A^n(X)$, over \mathbb{C} . This is another similarity with the Albanese mapping for a non-singular projective variety.

Remark 4.3. The proof of theorem 4.1 is close in spirit to the construction of a “generalized Albanese variety” in [FW]. There Faltings and Wüstholz consider a finite dimensional subspace $V \subset H^0(X_{\text{reg}}, \Omega^1_{X_{\text{reg}}})$, containing the 1-forms with logarithmic poles on some desingularization of $X^{(n)}$, and they construct a commutative algebraic group G_V together with a morphism $X_{\text{reg}} \rightarrow G_V$, which is universal among the morphisms $\tau : X_{\text{reg}} \rightarrow H$ to commutative algebraic groups H , with $\tau^*(\Omega(H)) \subset V$.

5. Picard groups of Cartier curves

In the next section, we give an algebraic construction of $A^n(X)$ for a reduced projective n -dimensional variety X , defined over an algebraically closed field k . As in the analytic case, we will use the Picard scheme for Cartier curves in X and for families of such curves. In this section, we discuss some properties of such families of curves, and the corresponding Picard schemes. In particular, we establish the technical results 5.6 and 5.8, which are important steps in the algebraic construction of $A^n(X)$.

Let S be an irreducible non-singular variety, and let $f : \mathcal{C} \rightarrow S$ be a flat proper family of projective curves with reduced geometric fibres $C_s = f^{-1}(s)$. By [G], 7.8.6, the morphism f is cohomologically flat in dimension zero. Hence $f_*\mathcal{O}_{\mathcal{C}}$ and $R^1f_*\mathcal{O}_{\mathcal{C}}$ are both locally free and compatible with base change and

$$g(C_s) := \dim_{k(s)} H^1(C_s, \mathcal{O}_{C_s}) \text{ and } \#C_s := \dim_{k(s)} H^0(C_s, \mathcal{O}_{C_s})$$

are both constant on S . Moreover, the relative Picard functor $\text{Pic}_{\mathcal{C}/S}$ is represented by an algebraic space $\text{Pic}(\mathcal{C}/S)$ (see [BLR], 8.3, theorem 1).

If $S' \rightarrow S$ is a finite covering such that $f' : \mathcal{C}' = \mathcal{C} \times_S S' \rightarrow S'$ is the disjoint union of families of curves $\mathcal{C}'_i \rightarrow S'$, for $i = 1, \dots, r$ with connected fibres, then

$$\text{Pic}(\mathcal{C}/S) \times_S S' = \text{Pic}(\mathcal{C}'/S') = \text{Pic}(\mathcal{C}'_1/S') \times_{S'} \cdots \times_{S'} \text{Pic}(\mathcal{C}'_r/S') .$$

For the smooth locus \mathcal{C}_{sm} of f consider the g -th symmetric product

$$f'^g : S^g(\mathcal{C}'_{\text{sm}}/S') \longrightarrow S'$$

over S' . For any open subscheme $W' \subset S^g(\mathcal{C}'_{\text{sm}}/S')$ there is a natural map $\vartheta_{W'} : W' \rightarrow \text{Pic}(\mathcal{C}'/S')$. By [BLR], 9.3, lemmas 5 and 6, one has the following generalization of 1.11:

Lemma 5.1. *After replacing S' by an étale covering, there exists an open subscheme $W' \subset S^g(\mathcal{C}'_{\text{sm}}/S')$ with irreducible fibres over S' , such that $\vartheta_{W'} : W' \rightarrow \text{Pic}(\mathcal{C}'/S')$ is an open embedding.*

Recall that $X^{(n)}$ denotes the union of the n -dimensional irreducible components of X , and $X^{<n}$ is the union of the smaller dimensional components.

Notations 5.2. For a very ample invertible sheaf \mathcal{L} on $X^{(n)}$ we write

$$|\mathcal{L}|^{n-1} = \mathbb{P}(H^0(X^{(n)}, \mathcal{L})) \times \cdots \times \mathbb{P}(H^0(X^{(n)}, \mathcal{L})) \quad (n-1)\text{-times}$$

and $|\mathcal{L}|_0^{n-1}$ for the open subscheme defined by $n-1$ -tuples D_1, \dots, D_{n-1} of divisors such that

- (i) $C = D_1 \cap \cdots \cap D_{n-1}$ is a reduced complete intersection curve in $X^{(n)}$,
- (ii) $C \cap X^{<n} = \emptyset$, and
- (iii) $X_{\text{reg}} \cap C$ is non-singular and dense in C .

Note that by (ii), C is a reduced Cartier curve in X which is a local complete intersection. By abuse of notation we will sometimes write $C \in |\mathcal{L}|^{n-1}$ instead of $(D_1, \dots, D_{n-1}) \in |\mathcal{L}|_0^{n-1}$.

The normalization $\pi : \tilde{C} \rightarrow C$ induces a surjection $\pi^* : \text{Pic}^0(C) \rightarrow \text{Pic}^0(\tilde{C})$. By [BLR], 9.2, the kernel of π^* is the largest linear subgroup $H(C)$ of $\text{Pic}(C)$. One has

$$\begin{aligned}
 \dim(H(C)) &= \dim(\text{Pic}^0(C)) - \dim(\text{Pic}^0(\tilde{C})) \\
 &= \dim_k(H^1(C, \mathcal{O}_C)) - \dim_k(H^1(\tilde{C}, \mathcal{O}_{\tilde{C}})) \\
 (5.1) \qquad &= \chi(\tilde{C}, \mathcal{O}_{\tilde{C}}) - \chi(C, \mathcal{O}_C) - (\#\tilde{C} - \#C),
 \end{aligned}$$

where again $\#C$ and $\#\tilde{C}$ denote the numbers of connected components of C and \tilde{C} , respectively.

Given a flat family of projective curves $f : \mathcal{C} \rightarrow S$ over an irreducible variety S with reduced geometric fibres C_s , there exist a finite covering $S' \rightarrow S$ and an open dense subscheme $S'_0 \subset S'$ such that the normalization of $\mathcal{C} \times_S S'_0$ is smooth over S'_0 . Hence $\#\tilde{C}_s$, and the dimension of the linear part $H(C_s)$ of $\text{Pic}^0(C_s)$, are both constant on the image of S'_0 .

Definition 5.3. For a reduced projective curve C we define $r(C)$ to be the number of irreducible components of C , and $\mu(C)$ to be the dimension of the largest linear subgroup of $\text{Pic}^0(C)$.

By [F], Satz 5.2, for a very ample invertible sheaf \mathcal{L} the open subscheme $|\mathcal{L}|_0^{n-1}$ is not empty. Let $r(\mathcal{L})$ and $\mu(\mathcal{L})$ denote the values of $r(C)$ and of $\mu(C)$ for $C \in |\mathcal{L}|_0^{n-1}$ in general position.

By the equality (5.1) one has:

$$\chi(\tilde{C}, \mathcal{O}_{\tilde{C}}) - \chi(C, \mathcal{O}_C) \geq \mu(C) \geq \chi(\tilde{C}, \mathcal{O}_{\tilde{C}}) - \chi(C, \mathcal{O}_C) - r(C) + 1 . \tag{5.2}$$

Lemma 5.4. For a very ample invertible sheaf \mathcal{L} and for a positive integer N ,

$$\mu(\mathcal{L}^N) \leq N^{n-1} \cdot (\mu(\mathcal{L}) + r(\mathcal{L}) - 1) .$$

Proof. Given $D_j^{(i)} \in |\mathcal{L}|$, for $i = 1, \dots, N$ and $j = 1, \dots, n - 1$, we write

$$\begin{aligned}
 I &= \{1, \dots, N\}^{n-1} \\
 C^{(\underline{i})} &= D_1^{(i_1)} \cap \dots \cap D_{n-1}^{(i_{n-1})} \quad \text{for } \underline{i} = (i_1, \dots, i_{n-1}) \in I \\
 \text{and } C &= \bigcup_{\underline{i} \in I} C^{(\underline{i})} = \bigcap_{j=1}^{n-1} (D_j^{(1)} \cup \dots \cup D_j^{(N)}) .
 \end{aligned}$$

Claim 5.5. There exists a choice of the divisors $D_j^{(i)} \in |\mathcal{L}|$ such that
 (a) $C^{(\underline{i})} \in |\mathcal{L}|_0^{n-1}$, $\mu(C^{(\underline{i})}) = \mu(\mathcal{L})$ and $r(C^{(\underline{i})}) = r(\mathcal{L})$

(b) $C^{(\underline{i})} \cap C^{(\underline{i}')} \cap X_{\text{sing}} = \emptyset$ for $\underline{i} \neq \underline{i}'$

(c) each point $x \in C_{\text{sing}} \cap X_{\text{reg}}$ lies on exactly two components $C^{(\underline{i})}$ and $C^{(\underline{i}')}$. In this case, there exists one v with $i_j = i'_j$ for all $j \neq v$. Locally in x the surface

$$Y = D_1^{(i_1)} \cap \dots \cap \widehat{D_v^{(i_v)}} \cap \dots \cap D_{n-1}^{(i_{n-1})} \cap X_{\text{reg}}$$

is nonsingular and contains $C^{(\underline{i})}$ and $C^{(\underline{i}')}$ as two smooth divisors intersecting transversally.

(d) C is a reduced complete intersection curve in $|\mathcal{L}^N|^{n-1}$.

Proof. (d) follows from (a), (b) and (c). Since $|\mathcal{L}|_0^{n-1}$ is open and dense in $|\mathcal{L}|^{n-1}$, (a) holds true for sufficiently general divisors. Counting dimensions one finds that for $\underline{i} \neq \underline{i}'$ the intersection $C^{(\underline{i})} \cap C^{(\underline{i}')}$ is either empty or consists of finitely many points. The latter can only happen, if all but one entry in \underline{i} and \underline{i}' are the same, and obviously one may assume that the intersection points all avoid X_{sing} . Moreover

$$C^{(\underline{i})} \cap C^{(\underline{i}')} \cap C^{(\underline{i}'')} = \emptyset$$

for pairwise different $\underline{i}, \underline{i}', \underline{i}'' \in I$. Now (c) follows from the Bertini theorem [F], Satz 5.2, saying that for sufficiently general divisors $D_j^{(i)}$

$$Y = D_1^{(i_1)} \cap \dots \cap \widehat{D_v^{(i_v)}} \cap \dots \cap D_{n-1}^{(i_{n-1})} \cap X_{\text{reg}}$$

$$C^{(\underline{i})} = Y \cap D_v^{(i_v)} \quad \text{and} \quad C^{(\underline{i}')} = Y \cap D_v^{(i'_v)}$$

are non-singular and that $C^{(\underline{i})}$ and $C^{(\underline{i}')}$ meet transversally on Y . \square

Let $\mathbb{A}^{M+1} \subset |\mathcal{L}^N|^{n-1}$ be an affine open subspace containing the point s_0 which corresponds to the tuple $\{D_j^{(1)} \cup \dots \cup D_j^{(N)}\}_{j=1, \dots, n-1}$, and let \mathbb{P}^M be the projective space parametrizing lines in \mathbb{A}^{M+1} , passing through s_0 . There is a line $S \in \mathbb{P}^M$ such that

(i) the total space \mathcal{C} of the restriction

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\tau} & X \\ f \downarrow & & \\ S & \xrightarrow{\subset} & |\mathcal{L}^N|^{n-1} \end{array}$$

of the universal family to S is non-singular in a neighbourhood of each point $x \in C_{\text{sing}} \cap X_{\text{reg}}$

(ii) the intersection of X_{reg} with the general fibre of $f : \mathcal{C} \rightarrow S$ is non-singular.

In fact, using the notation from 5.5 (c), we can choose for a point $x \in C_{\text{sing}} \cap X_{\text{reg}}$ a line S connecting s_0 with a point $(D'_1, \dots, D'_{n-1}) \in \mathbb{A}^{M+1}$, where

$$D'_j = D_j^{(1)} \cup \dots \cup D_j^{(N)} \text{ for } j \neq v ,$$

where $D'_v \cap Y_{\text{reg}}$ is non singular, and where $x \notin D'_v$. By this choice, in a neighbourhood of x the restriction of the universal family \mathcal{C} to S is just a fibering of Y over S . Hence the condition (i) is valid for the chosen point x .

However, for each point $x \in C_{\text{sing}} \cap X_{\text{reg}}$, the condition (i) is an open condition in \mathbb{P}^M , and hence for a general line S , (i) holds for all points in $C_{\text{sing}} \cap X_{\text{reg}}$; clearly the second condition (ii) holds as well. The family $f : \mathcal{C} \rightarrow S$ has only finitely many non-reduced fibres and outside of them $U = \tau^{-1}(X_{\text{reg}})$ contains only finitely many points, which are singularities of the fibres.

Replacing S by an open neighbourhood of s_0 , we may assume thereby, that for $s \neq s_0$ the fibre $C_s = f^{-1}(s)$ is reduced, that $C_s \cap X_{\text{reg}}$ is non-singular and dense in C_s and that $\mu(C_s) = \mu(\mathcal{L}^N)$. In particular U is non-singular outside of the points $C_{\text{sing}} \cap X_{\text{reg}}$, and by condition (i), U is non singular. Moreover, $f|_U : U \rightarrow S$ is semi-stable; hence $f|_U$ is a local complete intersection morphism, smooth outside a finite subset of U . Let L be a finite extension of the function field $k(S)$ such that the normalization of $\mathcal{C} \times_S \text{Spec}(L)$ is smooth over L , and let S' be the normalization of S in L . Consider

$$\begin{array}{ccccc} \mathcal{C}' & \xrightarrow{\sigma} & \mathcal{C} \times_S S' & \xrightarrow{\eta'} & \mathcal{C} \\ & \searrow f' & \downarrow p_2 & & \downarrow f \\ & & S' & \xrightarrow{\eta'} & S \end{array}$$

where σ denotes the normalization. Since $U \times_S S' \rightarrow S'$ is a local complete intersection morphism, smooth outside a finite subset of the domain, $U \times_S S'$ is normal and σ restricted to $U' = \sigma^{-1}(U \times_S S')$ is an isomorphism. By construction the general fibre of f' is smooth and \mathcal{C}' is normal. Since for all $s' \in S'$ the fibres $C'_s = f'^{-1}(s')$ of f' are reduced on the open dense subvariety U' , they are reduced everywhere. Note also that $\mathcal{C} - U \rightarrow S$ is finite, and hence so is $\mathcal{C}' - U' \rightarrow S'$.

Let $s', s'_0 \in S'$ be points, with $s_0 = \eta(s'_0)$, and with $s = \eta(s')$ in general position. The inequality (5.2) implies that

$$\begin{aligned} \mu(\mathcal{L}^N) &= \mu(C_s) \leq \chi(C'_s, \mathcal{O}_{C'_s}) - \chi(C_s, \mathcal{O}_{C_s}) \\ &= \chi(C'_{s'_0}, \mathcal{O}_{C'_{s'_0}}) - \chi(C_{s_0}, \mathcal{O}_{C_{s_0}}) . \end{aligned}$$

Since $C'_{s'_0} \cap U'$ is isomorphic to $C_{s_0} \cap U$ the curve $C'_{s'_0}$ is finite over and birational to $C = C_{s_0}$. Moreover, the fibres $C'_{s'_0} \cap U'$ and $C \cap U$ have the same number δ of double points. Writing $C^{(i)}$ for the preimage of $C^{(i)}$ in $C'_{s'_0}$ one obtains

$$\begin{aligned} \chi(C, \mathcal{O}_C) + \delta &= \sum_{i \in I} \chi(C^{(i)}, \mathcal{O}_{C^{(i)}}), \\ \chi(C'_{s'_0}, \mathcal{O}_{C'_{s'_0}}) + \delta &= \sum_{i \in I} \chi(C'^{(i)}, \mathcal{O}_{C'^{(i)}}) \\ \text{and } \mu(\mathcal{L}^N) &\leq \sum_{i \in I} (\chi(C'^{(i)}, \mathcal{O}_{C'^{(i)}}) - \chi(C^{(i)}, \mathcal{O}_{C^{(i)}})) . \end{aligned}$$

Finally, $C'^{(i)}$ is finite over and birational to $C^{(i)}$, thus it is dominated by the normalization of $C^{(i)}$, and (5.2) implies

$$\begin{aligned} &\sum_{i \in I} (\chi(C'^{(i)}, \mathcal{O}_{C'^{(i)}}) - \chi(C^{(i)}, \mathcal{O}_{C^{(i)}})) \\ &\leq \sum_{i \in I} (\mu(C^{(i)}) + r(C^{(i)}) - 1) \\ &= N^{n-1} \cdot (\mu(\mathcal{L}) + r(\mathcal{L}) - 1) . \end{aligned} \quad \square$$

Replacing \mathcal{L} by its N -th power one obtains by lemma 5.4 ample invertible sheaves on $X^{(n)}$ with many more linearly independent sections than $\mu(\mathcal{L})$. For example, if X_1, \dots, X_r are the irreducible components of $X^{(n)}$, then

$$\text{image } (H^0(X^{(n)}, \mathcal{L}^N) \rightarrow H^0(X_i, \mathcal{L}^N|_{X_i})) = H^0(X_i, \mathcal{L}^N|_{X_i}) ,$$

for sufficiently large N , and its dimension is bounded below by a non-zero multiple of N^n , whereas by 5.4, $\mu(\mathcal{L}^N)$ is bounded above by $(\mu(\mathcal{L}) + r(\mathcal{L}) - 1) \cdot N^{n-1}$. One obtains:

Corollary 5.6. *There exists a very ample sheaf \mathcal{L} on $X^{(n)}$ with*

$$\dim_k(\text{image } (H^0(X^{(n)}, \mathcal{L}) \rightarrow H^0(X_i, \mathcal{L}|_{X_i}))) \geq 2 \cdot \mu(\mathcal{L}) + r + 2 ,$$

for $i = 1, \dots, r$.

Over a field k of positive characteristic we will need a stronger technical condition. Recall that $\Pi_{X_{\text{reg}}} = \bigcup_{i=1}^r (U_i \times U_i)$, where $U_i = X_i \cap X_{\text{reg}}$ are the irreducible components of X_{reg} .

Assumption 5.7. Let $Z \subset S^d(\Pi_{X_{\text{reg}}}) \times S^d(\Pi_{X_{\text{reg}}}) \times |\mathcal{L}|_0^{n-1}$ be the incidence variety of points

$$((x_1, x'_1), \dots, (x_d, x'_d)), ((x_{d+1}, x'_{d+1}), \dots, (x_{2d}, x'_{2d})), (D_1, \dots, D_{n-1}) \in S^d(\Pi_{X_{\text{reg}}}) \times S^d(\Pi_{X_{\text{reg}}}) \times |\mathcal{L}|_0^{n-1}$$

with $x_1, \dots, x_{2d}, x'_1, \dots, x'_{2d} \in C = D_1 \cap \dots \cap D_{n-1}$. Then the projection

$$pr'_{12} = pr_{12}|_Z : Z \longrightarrow S^d(\Pi_{X_{\text{reg}}}) \times S^d(\Pi_{X_{\text{reg}}})$$

is dominant.

Proposition 5.8. There exists a very ample sheaf \mathcal{L} on $X^{(n)}$ which satisfies the assumption 5.7, for all $d \leq \mu(\mathcal{L})$.

Proof. Let \mathcal{I}_i be the ideal sheaf of $\bigcup_{j \neq i} X_j$ on $X^{(n)}$. Then \mathcal{I}_i is annihilated by the ideal sheaf of X_i in $X^{(n)}$. In particular, one can talk on its cohomology on X_i , as being the same as its cohomology on $X^{(n)}$. Hence if \mathcal{F} is a torsion-free coherent sheaf on $X^{(n)}$ and $\mathcal{I}_i \setminus \mathcal{F} := \text{image}(\mathcal{F} \otimes \mathcal{I}_i \rightarrow \mathcal{F})$ then

$$H^0(X_i, \mathcal{I}_i \setminus \mathcal{F}) = H^0(X^{(n)}, \mathcal{I}_i \setminus \mathcal{F})$$

and $\bigoplus_{i=1}^r H^0(X_i, \mathcal{I}_i \setminus \mathcal{F}) \subset H^0(X^{(n)}, \mathcal{F})$.

Claim 5.9. There exists a very ample invertible sheaf \mathcal{L} on $X^{(n)}$ such that

$$\dim_k(\text{image}(H^0(X_i, \mathcal{L} \otimes \mathcal{I}_i|_{X_i}) \rightarrow H^0(C, \mathcal{L}|_C))) \geq 4 \cdot \mu(\mathcal{L}) \quad , \quad (5.3)$$

for $i = 1, \dots, r$, and for all $C \in |\mathcal{L}|_0^{n-1}$.

Proof. Given a very ample invertible sheaf \mathcal{L} it suffices to find a lower bound for the dimension of the image of the composite map

$$\tau : H^0(X_i, \mathcal{L}^N \otimes \mathcal{I}_i|_{X_i}) \longrightarrow H^0(X^{(n)}, \mathcal{L}^N) \longrightarrow H^0(C, \mathcal{L}^N|_C)$$

which is independent of $C \in |\mathcal{L}^N|_0^{n-1}$ and grows like N^n . If \mathcal{I}_C denotes the ideal sheaf of C on $X^{(n)}$, then

$$\ker(\tau) = H^0(X_i, \mathcal{I}_i \setminus (\mathcal{L}^N \otimes \mathcal{I}_C)) \subset H^0(X^{(n)}, \mathcal{L}^N \otimes \mathcal{I}_C) \quad .$$

Hence it is sufficient to give an upper bound for $\dim(H^0(X^{(n)}, \mathcal{L}^N \otimes \mathcal{I}_C))$ by some polynomial in N of degree $n - 1$, independent of C .

For $j < n$ the dimension of $H^j(X^{(n)}, \mathcal{L}^{-N})$ is bounded by a polynomial of degree $n - 2$. In fact, $X^{(n)}$ is a subscheme of $\mathbb{P}^M = \mathbb{P}(H^0(X^{(n)}, \mathcal{L}))$ and by [Ha], III.7.1 and III.6.9 one has, for N sufficiently large,

$$\begin{aligned} H^j(X^{(n)}, \mathcal{L}^{-N}) &\cong \text{Ext}^{M-j}(\mathcal{O}_{X^{(n)}} \otimes \mathcal{L}^{-N}, \omega_{\mathbb{P}^M}) \\ &\cong H^0(\mathbb{P}^M, \mathcal{E}xt^{M-j}(\mathcal{O}_{X^{(n)}}, \omega_{\mathbb{P}^M} \otimes \mathcal{O}_{\mathbb{P}^M}(N))) \\ &\cong H^0(\mathbb{P}^M, \mathcal{E}xt^{M-j}(\mathcal{O}_{X^{(n)}}, \omega_{\mathbb{P}^M}) \otimes \mathcal{O}_{\mathbb{P}^M}(N)) . \end{aligned}$$

Since X is Cohen-Macaulay outside of a subscheme T of codimension 2, the support of $\mathcal{E}xt^{M-j}(\mathcal{O}_{X^{(n)}}, \omega_{\mathbb{P}^M})$ lies in T for $M - j > M - n$.

The curve C being a complete intersection of divisors in $|\mathcal{L}^N|$, a resolution of the ideal sheaf \mathcal{I}_C on $X^{(n)}$ is given by the Koszul complex

$$\begin{aligned} 0 \rightarrow \mathcal{L}^{-(n-1)N} = \bigwedge^{n-1} \left(\bigoplus \mathcal{L}^{-N} \right) \rightarrow \dots \rightarrow \bigwedge^2 \left(\bigoplus \mathcal{L}^{-N} \right) \\ \rightarrow \bigoplus^{n-1} \mathcal{L}^{-N} \rightarrow \mathcal{I}_C \rightarrow 0 . \end{aligned}$$

Therefore $\dim(H^0(X^{(n)}, \mathcal{L}^N \otimes \mathcal{I}_C))$ is bounded from above by

$$\begin{aligned} \sum_{j=0}^{n-2} \dim_k \left(H^j \left(X^{(n)}, \mathcal{L}^N \otimes \bigwedge^{j+1} \left(\bigoplus \mathcal{L}^{-N} \right) \right) \right) \\ = \sum_{j=0}^{n-2} \dim_k \left(H^j \left(X^{(n)}, \mathcal{L}^{-jN} \right) \right) \binom{n-1}{j+1} . \end{aligned}$$

□

Let \mathcal{L} be a very ample invertible sheaf on $X^{(n)}$ which satisfies the inequality (5.3) in 5.9. We fix some curve $C \in |\mathcal{L}|_0^{n-1}$ and some natural number $d \leq \mu(\mathcal{L})$.

Each irreducible component of $S^d(\Pi_{X_{\text{reg}}}) \times S^d(\Pi_{X_{\text{reg}}})$ is of the form

$$\begin{aligned} S_{\underline{d}} = (S^{d_1}(U_1 \times U_1) \times \dots \times S^{d_r}(U_r \times U_r)) \\ \times (S^{d_{r+1}}(U_1 \times U_1) \times \dots \times S^{d_{2r}}(U_r \times U_r)) , \end{aligned}$$

for some tuple $\underline{d} = (d_1, \dots, d_{2r})$ of non-negative integers with

$$d_1 + \dots + d_r = d_{r+1} + \dots + d_{2r} = d .$$

Given such a tuple \underline{d} , we claim that there are (pairwise distinct) points $x_1, \dots, x_{2d}, x'_1, \dots, x'_{2d}$, with $x_v, x'_v \in C \cap U_i$, for

$$\sum_{j=1}^{i-1} d_j < v \leq \sum_{j=1}^i d_j \quad \text{and for} \quad \sum_{j=1}^{r+i-1} d_j < v \leq \sum_{j=1}^{r+i} d_j ,$$

such that the restriction map

$$(5.4) \quad H^0(X^{(n)}, \mathcal{L}) \longrightarrow \bigoplus_{i=1}^{2d} k_{x_i} \oplus k_{x'_i}$$

is surjective. In fact, by the inequality (5.3) the dimension of the image of

$$H^0(X_i, \mathcal{L} \otimes \mathcal{I}_i|_{X_i}) \longrightarrow H^0(C \cap X_i, \mathcal{L} \otimes \mathcal{I}_i|_{C \cap X_i}) \longrightarrow H^0(C, \mathcal{L}|_C)$$

is at least $4 \cdot \mu(\mathcal{L}) \geq 4 \cdot d \geq 2 \cdot (d_i + d_{r+i})$ and for sufficiently general points $x_1, \dots, x_{2d}, x'_1, \dots, x'_{2d} \in C$ the composite

$$\bigoplus_{i=1}^r H^0(X_i, \mathcal{L} \otimes \mathcal{I}_i|_{X_i}) \subset H^0(X^{(n)}, \mathcal{L}) \longrightarrow \bigoplus_{i=1}^{2d} k_{x_i} \oplus k_{x'_i}$$

is surjective.

By construction

$$w := (((x_1, x'_1), \dots, (x_d, x'_d)), ((x_{d+1}, x'_{d+1}), \dots, (x_{2d}, x'_{2d}))) \in \underline{S}_d .$$

Let V denote the subspace of divisors $D \in |\mathcal{L}|$ with

$$x_1, \dots, x_{2d}, x'_1, \dots, x'_{2d} \in D .$$

The fibre $pr'^{-1}_{12}(w)$ of the morphism $pr'_{12} : Z \rightarrow S^d(\Pi_{X_{\text{reg}}}) \times S^d(\Pi_{X_{\text{reg}}})$ is the intersection of V^{n-1} with $|\mathcal{L}|_0^{n-1}$. In particular, since $(w, C) \in Z$, this intersection is non-empty.

If $\delta = \dim(|\mathcal{L}|)$, the surjectivity of the restriction map (5.4) implies that $\dim(V) = \delta - 4 \cdot d$ and $\dim(pr'^{-1}_{12}(w)) = (n - 1) \cdot (\delta - 4 \cdot d)$. The fibres of $pr'_3 : Z \rightarrow |\mathcal{L}|_0^{n-1}$ are equidimensional of dimension $4 \cdot d$ and hence Z is equidimensional of dimension $(n - 1) \cdot \delta + 4 \cdot d$. Therefore the dimension of $pr'_{12}(Z) \cap \underline{S}_d$ can not be smaller than

$$(n - 1) \cdot \delta + 4 \cdot d - (n - 1) \cdot (\delta - 4 \cdot d) = n \cdot 4 \cdot d = \dim(\underline{S}_d) . \quad \square$$

6. The algebraic construction of $A^n(X)$

Let X be a projective variety of dimension n , defined over an algebraically closed field k . As a first step towards the construction of $A^n(X)$ we need to bound the dimension of the image of a regular homomorphism

$$\phi : CH^n(X)_{\text{deg } 0} \longrightarrow G$$

to a smooth connected commutative algebraic group G .

By the theorem of Chevalley and Rosenlicht (theorems 1 and 2 in [BLR], 9.2) there exists a unique smooth linear subgroup L of G such that $G/L = A$ is an abelian variety. In addition, L is canonically isomorphic to a product of a unipotent group and a torus. Let us write

$$0 \longrightarrow L \longrightarrow G \xrightarrow{\delta} A \longrightarrow 0$$

for the extension.

Lemma 6.1. *There exists a unique smooth connected algebraic subgroup H of G , with $\delta(H) = A$, such that every smooth connected algebraic subgroup J of G with $\delta(J) = A$ contains H . Moreover, the quotient group G/H is linear.*

Proof. Given a smooth algebraic subgroup J of G , one has the commutative diagram of exact sequences

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & L \cap J & \longrightarrow & J & \longrightarrow & \delta(J) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & L & \longrightarrow & G & \longrightarrow & A \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & L/(L \cap J) & \longrightarrow & G/J & \longrightarrow & A/\delta(J) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Since $A/\delta(J)$ is an abelian variety and $L/(L \cap J)$ a linear algebraic group, $\delta(J) = A$ if and only if G/J is linear. Observe further, that $\delta(J) = A$ if and only if $\delta(J') = A$ for the connected component J' of J containing the identity.

Choose H to be any smooth connected algebraic subgroup of G with $\delta(H) = A$ and such that $\delta(H') \neq A$ for all proper algebraic subgroups H' of H . For J as in 6.1 consider the commutative diagram of exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & J \cap H & \longrightarrow & G & \longrightarrow & G/(J \cap H) \longrightarrow 0 \\
 & & \downarrow & & \Delta \downarrow & & \downarrow \iota \\
 0 & \longrightarrow & J \oplus H & \longrightarrow & G \oplus G & \longrightarrow & G/J \oplus G/H \longrightarrow 0
 \end{array}$$

where Δ is the diagonal embedding. Since $J \cap H = \Delta^{-1}(J \oplus H)$ the morphism ι is injective on closed points, and hence $G/(J \cap H)$ is a linear algebraic group. By the choice of H one obtains $J \cap H = H$. \square

Recall that X has n -dimensional irreducible components X_1, \dots, X_r , whose union is denoted $X^{(n)}$, and $U_i = X_{\text{reg}} \cap X_i$. Also $X^{<n}$ is the union of the lower dimensional components of X .

Proposition 6.2. *Let \mathcal{L} be a very ample invertible sheaf on $X^{(n)}$ which satisfies the assumption 5.7. Let $g = \dim_k(H^1(C, \mathcal{O}_C))$, for $C \in |\mathcal{L}|_0^{n-1}$.*

Let $\phi : CH^n(X)_{\text{deg } 0} \rightarrow G$ be a surjective regular homomorphism to a smooth connected commutative algebraic group G . Then the induced morphism (see 1.9)

$$\pi^{(-)} : S^{g+v \cdot \mu(\mathcal{L})}(\Pi_{X_{\text{reg}}}) \longrightarrow CH^n(X)_{\text{deg } 0} \xrightarrow{\phi} G$$

is dominant, for $v > 0$, and surjective, for $v > 1$. In particular the dimension of G is bounded by $2 \cdot n \cdot (g + \mu(\mathcal{L}))$.

Probably the bound for the dimension of G is far from being optimal. We will indicate in 6.4 how to obtain $\dim(G) \leq g$ in characteristic zero, under a weaker assumption on \mathcal{L} .

Proof of 6.2. Let again L be the largest smooth linear algebraic subgroup and $\delta : G \rightarrow A = G/L$ the projective quotient group. Recall that $|\mathcal{L}|_0^{n-1}$ denotes the set of tuples (D_1, \dots, D_{n-1}) of divisors in the linear system $|\mathcal{L}|$ for which $C = D_1 \cap \dots \cap D_{n-1}$ is a reduced complete intersection curve (in $X^{(n)}$), $C \cap X^{<n} = \emptyset$, and $C \cap X_{\text{reg}}$ non-singular and dense in C .

Claim 6.3. *There exists an open dense subscheme $S \subset |\mathcal{L}|_0^{n-1}$ such that*

$$\text{Pic}^0(C) \xrightarrow{\psi} G \xrightarrow{\delta} A$$

is surjective and such that the dimension of $\text{image}(\psi : \text{Pic}^0(C) \rightarrow G)$ is constant, for $C \in S$.

Proof. Returning to the notation introduced in 5.2 let $S \subset |\mathcal{L}|_0^{n-1}$ be an open subvariety, and let

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\sigma} & X \\ f \downarrow & & \\ S & & \end{array}$$

denote the restriction of the universal complete intersection to S . The smooth locus of f is $\mathcal{C}_{\text{sm}} = \sigma^{-1}(X_{\text{reg}})$ and \mathcal{C}_{sm} is dominant over X_{reg} . Let $S' \rightarrow S$ be the finite morphism, and let $W' \subset S^g(\mathcal{C}_{\text{sm}} \times_S S'/S')$ be the open subscheme considered in lemma 5.1, with irreducible fibres over S' . By 5.1 the morphism

$$\vartheta_{W'} : W' \longrightarrow \text{Pic}(\mathcal{C}/S) \times_S S' = \text{Pic}(\mathcal{C} \times_S S'/S')$$

is an open embedding. On the other hand, one has a morphism of schemes

$$h : W' \longrightarrow S^g(\mathcal{C}_{\text{sm}} \times_S S'/S') \longrightarrow S^g(X_{\text{reg}}) ,$$

and the image of the connected scheme W' lies in some connected component, say $S_g = S^{g_1}(U_1) \times \cdots \times S^{g_r}(U_r)$. Since $\phi : CH^n(X)_{\text{deg } 0} \rightarrow G$ is regular, the composite

$$h^{(-)} : W' \times_{S'} W' \longrightarrow S_g \times S_g \xrightarrow{\theta} G$$

is a morphism, where

$$\theta(\underline{x}, \underline{x}') = \phi \left(\sum_{i=1}^g \gamma(x_i) - \sum_{i=1}^g \gamma(x'_i) \right).$$

The morphism $h^{(-)}$ induces S' -morphisms

$$h_{S'}^{(-)} : W' \times_{S'} W' \longrightarrow G \times S' \quad \text{and} \quad h_{S'}^{(-)} \circ \delta : W' \times_{S'} W' \longrightarrow A \times S' .$$

Since $W' \times_{S'} W'$ is irreducible one can choose irreducible locally closed subschemes W_G and W_A of the images

$$h_{S'}^{(-)}(W' \times_{S'} W') \quad \text{and} \quad h_{S'}^{(-)} \circ \delta(W' \times_{S'} W')$$

respectively, dense in the closure of the images. Choosing S' and S small enough, one may assume that $S' \rightarrow S$ is surjective and that W_G and W_A are both equidimensional over S' . For $C \in S$ choose a point $s' \in S'$ mapping to $C \in S$ and let $W'_{s'}$ denote the fibre of W' over s' . Then the image of $W'_{s'} \times W'_{s'}$ in $\text{Pic}^0(C)$ is dense and thereby $\dim(\psi(\text{Pic}^0(C)))$ and $\dim(\delta(\psi(\text{Pic}^0(C)))) = d'$ are both constant on S .

Assume that $d' < \dim(A)$. The closure of $\delta(h^{(-)}(W'_C \times W'_C))$ is the image of $\text{Pic}^0(C)$, hence $\delta(h^{(-)}(W'_C \times W'_C))$ lies in some abelian subvariety B of A of dimension $d' < \dim(A)$. Since S' and W' are connected, and since an abelian variety A does not contain non-trivial families of abelian subvarieties, B is independent of the curve C chosen.

\mathcal{C}_{sm} being dominant over X_{reg} this implies that the image $\delta\phi(CH^n(X)_{\text{deg}0})$ lies in B , contradicting the assumptions made. \square

In general, a commutative algebraic group G can contain non-trivial families of subgroups and the argument used above does not extend to G instead of A .

Let $H \subset G$ be the smallest connected algebraic subgroup with $\delta(H) = A$, as constructed in 6.1. By 6.3 and by the universal property in 6.1, for $C \in S$ the image of $\psi(\text{Pic}^0(C))$ contains H .

By 1.10 the image of $S^g(\Pi_{X_{\text{reg}}})$ in G is $\psi(\text{Pic}^0(C))$ and hence H is contained in the image of $S^g(\Pi_{X_{\text{reg}}})$. In order to show that

$$\pi^{(-)} : S^{g+\mu(\mathcal{L})}(\Pi_{X_{\text{reg}}}) \longrightarrow CH^n(X)_{\text{deg}0} \xrightarrow{\phi} G$$

is dominant, it suffices to verify that the image Y_0 of the composite

$$\tau^{(-)} : S^\mu(\Pi_{X_{\text{reg}}}) \longrightarrow CH^n(X)_{\text{deg}0} \xrightarrow{\phi} G \longrightarrow G/H$$

is dense, for some $\mu \leq \mu(\mathcal{L})$. Applying claim 6.3 to G/H instead of G one finds a non-empty open subscheme $S \subset |\mathcal{L}|_0^{n-1}$ such that the dimension d of $\psi'(\text{Pic}^0(C))$ is constant on S , where $\psi' : \text{Pic}^0(C) \rightarrow G/H$ is the natural map (see 1.12). Since G/H is a linear algebraic group, we must have $d \leq \mu(\mathcal{L})$, and choosing S small enough, we may assume that

$$(6.1) \quad d = \dim(\psi'(\text{Pic}^0(C))) \leq \mu(C) = \mu(\mathcal{L}), \quad \text{for all } C \in S .$$

Since Y_0 generates the group G/H , it is dense in G/H if and only if its closure Y is a group. By assumption the image of the incidence variety

$$Z \xrightarrow{p'_{12}} S^d(\Pi_{X_{\text{reg}}}) \times S^d(\Pi_{X_{\text{reg}}}) \xrightarrow{\tau^{(-)} \times \tau^{(-)}} Y \times Y$$

defined in 5.7 contains some open dense subscheme T . By definition, for each $t \in T$ there exist divisors D_1, \dots, D_{n-1} with $C = D_1 \cap \dots \cap D_{n-1} \in S$ and with

$$t \in \text{image} (S^d(\Pi_B) \times S^d(\Pi_B) \xrightarrow{\vartheta^{(-)} \times \vartheta^{(-)}} Y \times Y)$$

for $B = C \cap X_{\text{reg}}$ and for the induced map $\vartheta^{(-)}$ from Π_B to G/H . By 1.10 $\psi'(\text{Pic}^0(C)) = \vartheta^{(-)}(S^d(\Pi_B)) \subset Y$. Since $\psi'(\text{Pic}^0(C))$ is an algebraic subgroup of G/H , the image of t under the morphism

$$\text{diff} : G/H \times G/H \longrightarrow G/H \quad \text{with } (g, g') \mapsto g - g'$$

is contained in $\psi'(\text{Pic}^0(C))$, hence in Y .

Thereby T is a subset of $\text{diff}^{-1}(Y)$, and the same is true of its closure $Y \times Y$. One obtains that $\text{diff}(Y \times Y) \subset Y$ and Y is a subgroup of G/H .

Since Y_0 is dense in G/H , by lemma 1.10 (ii) the image of $S^{2d}(\Pi_{X_{\text{reg}}})$ is G/H . □

As indicated already, the proposition 6.2 can be improved in characteristic zero.

Variante 6.4. *Assume that $\text{char}(k) = 0$. Let \mathcal{L} be a very ample invertible sheaf on $X^{(n)}$ with*

$$\dim_k(\text{image}(H^0(X^{(n)}, \mathcal{L}) \rightarrow H^0(X_i, \mathcal{L}|_{X_i})) \geq 2 \cdot \mu(\mathcal{L}) + r + 2 \quad , \tag{6.2}$$

for $i = 1, \dots, r$. Let G be a smooth connected commutative algebraic group, and let $\phi : CH^n(X)_{\text{deg } 0} \rightarrow G$ be a surjective regular homomorphism. Then there exists an open dense subvariety $S \subset |\mathcal{L}|_0^{n-1}$ such that for each $C \in S$ the induced homomorphism (see 1.12)

$$\psi : \text{Pic}^0(C) \longrightarrow CH^n(X)_{\text{deg } 0} \xrightarrow{\phi} G$$

is surjective. In particular the dimension of G is bounded by $g = \dim_k H^1(C, \mathcal{O}_C)$.

Proof. The first part of the proof is the same as the one for 6.2. In particular we may assume claim 6.3 to hold true.

Let $H \subset G$ be the smallest subgroup with $\delta(H) = A$, as constructed in 6.1. By 6.3 and by the universal property in 6.1, for all $C \in S$ the image of $\psi(\text{Pic}^0(C))$ contains H . Hence $\psi : \text{Pic}^0(C) \longrightarrow G$ is surjective if and only if

$$\text{Pic}^0(C) \xrightarrow{\psi} G \longrightarrow G/H$$

is surjective. In order to prove 6.4 we may assume thereby that G is linear and $A = 0$. By claim 6.3 we may assume that for all $C \in S$ the dimension of $\text{image}(\psi : \text{Pic}^0(C) \rightarrow G)$ is the same.

For $C \in S$, let $\gamma_B : B = C \cap X_{\text{reg}} \rightarrow CH^n(X)$ denote the natural map and let Γ_B be the image of the composite

$$\vartheta^{(-)} : \Pi_B \xrightarrow{\gamma_B^{(-)}} CH^n(X)_{\text{deg } 0} \xrightarrow{\phi} G .$$

For any subset $M \subset G$ we will denote by $G(M)$ the smallest algebraic subgroup of G which contains M . If M contains a point of infinite order, then $\dim(G(M)) > 0$. In characteristic zero the converse holds true, as well. In fact, if $\dim(G(M)) > 0$ then $G(M)$ contains a subgroup isomorphic either to \mathbb{G}_a or to \mathbb{G}_m . In characteristic zero, both contain points of infinite order.

Hence if the dimension of $G(\Gamma_B) = \psi(\text{Pic}^0(C))$ is larger than zero, the constructible set Γ_B contains a point α_1 of infinite order and $\dim(G(\alpha_1)) > 0$. Repeating this for $G/G(\alpha_1, \dots, \alpha_v)$ instead of G , we find recursively points $\alpha_1, \dots, \alpha_d \in \Gamma_B$ with $G(\Gamma_B) = G(\alpha_1, \dots, \alpha_d)$.

Let us choose points $x_1, \dots, x_d, x'_1, \dots, x'_d \in B$ with $\alpha_j = \vartheta^{(-)}((x_j, x'_j))$, and moreover, for each component X_i of $X^{(n)}$, choose a base point $q_i \in B \cap X_i$.

Claim 6.5. *There exists a closed subscheme $Z \subset S$ such that the restriction*

$$\begin{array}{ccccc} \mathcal{C}' = \mathcal{C} \times_S Z & \xrightarrow{\sigma' = \sigma|_{\mathcal{C}'}} & X^{(n)} & \xrightarrow{\subset} & X \\ f' = f|_{\mathcal{C}'} \downarrow & & & & \\ Z & \xrightarrow{\subset} & S & \xrightarrow{\subset} & |\mathcal{L}|_0^{n-1} \end{array}$$

of the universal family satisfies:

- (a) For each point $z \in Z$ the curve $C_z = f'^{-1}(z)$ contains the points

$$x_1, \dots, x_d, x'_1, \dots, x'_d, q_1, \dots, q_r .$$

- (b) $\sigma' : \mathcal{C}' \rightarrow X^{(n)}$ is dominant.

Proof. For

$$V_i = (\text{image}(H^0(X^{(n)}, \mathcal{L}) \rightarrow H^0(X_i, \mathcal{L}|_{X_i})) - 0) / k^* \subset |(\mathcal{L}|_{X_i})|$$

consider the rational map $\tilde{p}_i : |\mathcal{L}|_0^{n-1} \rightarrow V_i^{n-1}$. Since each $C \in S$ is a complete intersection curve, the restriction $p_i : S \rightarrow V_i^{n-1}$ of \tilde{p}_i is a morphism.

For $x \in X_i \cap \sigma(\mathcal{C})$ the condition “ $x \in C_s$ ” defines a multilinear subspace Δ_x^i of V_i^{n-1} of codimension $n - 1$. Let $I_i \subset \{1, \dots, d\}$ denote the set of all the v with $x_v, x'_v \in X_i$. Then the codimension of

$$\Delta^i = \Delta_{q_i}^i \cap \bigcap_{v \in I_i} (\Delta_{x_v}^i \cap \Delta_{x'_v}^i)$$

is at most $(n - 1) \cdot (2 \cdot \#I_i + 1)$.

Let $\mathcal{C}^i \rightarrow \Delta^i$ be the intersection on X_i of the divisors in $\Delta^i \subset V_i^{n-1}$. Then the general fibre of $\mathcal{C}^i \rightarrow X_i$ has dimension at least

$$\begin{aligned} \dim(\Delta^i) + 1 - \dim(X) &= \dim(V_i^{n-1}) + 1 - n - \text{codim}(\Delta^i) \\ &\geq \dim(V_i^{n-1}) + 1 - n - (n - 1) \cdot (2 \cdot \#I_i + 1) \\ &\geq (n - 1) \cdot (2 \cdot (\mu(\mathcal{L}) - \#I_i) + r - 1) . \end{aligned}$$

Since some $C \in S$ contains all the points x_j, x'_j and q_i , the intersection

$$Z = S \cap \bigcap_{i=1}^r p_i^{-1}(\Delta^i)$$

is non-empty. For the restriction \mathcal{C}' of the universal curve \mathcal{C} to Z the dimension of the general fibre of $\sigma' : \mathcal{C}' \rightarrow X$ over X_i has dimension larger than or equal to

$$\begin{aligned} &(n - 1) \cdot (2 \cdot (\mu(\mathcal{L}) - \#I_i) + r - 1) - \sum_{j \neq i} (n - 1) \cdot (2 \cdot \#I_j + 1) \\ &= (n - 1) \cdot 2 \cdot (\mu(\mathcal{L}) - \sum_{j=1}^r \#I_j) = (n - 1) \cdot 2 \cdot (\mu(\mathcal{L}) - d) . \end{aligned}$$

By the inequality (6.1) the last expression is larger than or equal to 0 and σ' is dominant. □

Let $G(C_z)$ denote the image of $\text{Pic}^0(C_z)$ in G . By the choice of Z the intersection $B_z = C_z \cap X_{\text{reg}}$ is non-singular and the dimension of $G(C_z) = G(\Gamma_{B_z})$ is the same as the dimension of $G(\Gamma_B) = G(\alpha_1, \dots, \alpha_d)$.

By 6.5 the points $\alpha_i = \phi(\sigma'(x_i) - \sigma'(x'_i))$ are contained in Γ_{B_z} , hence

$$G(C_z) = G(\Gamma_{B_z}) = G(\Gamma_B) = G(C)$$

for all $z \in Z$.

As σ' is dominant, $\sigma'(\mathcal{C}')$ contains some V , open and dense in X_{reg} (and hence in $X^{(n)}$). For $q \in V \cap X_i$ one finds some $z \in Z$ with $q \in C_z$. By 5.7 C_z contains the chosen base point q_i and

$$\phi(\gamma(q_i) - \gamma(q)) \in G(C_z) = G(C).$$

By 1.4 (i), the points $\gamma(q_i) - \gamma(q)$ (for $q \in V$) generate $CH^n(X)_{\text{deg } 0}$. Since ϕ was assumed to be surjective, we obtain $G = G(C)$, as claimed. \square

Using proposition 6.2 or its variant 6.4 the construction of $A^n(X)$ proceeds now along the lines of Lang’s construction in [La] of the Albanese variety of a smooth projective variety.

Theorem 6.6. *There exists a smooth connected commutative algebraic group $A^n(X)$ and a surjective regular homomorphism $\varphi : CH^n(X)_{\text{deg } 0} \rightarrow A^n(X)$ satisfying the following universal property: For any smooth commutative algebraic group G and for any regular homomorphism $\phi : CH^n(X)_{\text{deg } 0} \rightarrow G$ there exists a unique homomorphism $h : A^n(X) \rightarrow G$ of algebraic groups with $\phi = h \circ \varphi$.*

Moreover, if $k \subset K$ is an extension of algebraically closed fields, then

$$A^n(X \times_k K) = A^n(X) \times_k K .$$

Proof. By lemma 1.15 it is sufficient to consider connected groups G , and surjective regular homomorphisms ϕ .

By 5.8 there exists a very ample invertible sheaf \mathcal{L} which satisfies the assumption 5.7 and we can apply 6.2. (As we have seen in 5.6 the inequality (6.2) in 6.4 holds true for some \mathcal{L} , and if $\text{char}(k) = 0$ we can use the variant 6.4, as well.)

Let $g = \dim_k(H^1(C, \mathcal{O}_C))$, for some curve $C \in |\mathcal{L}|_0^{n-1}$ in general position. Then for all regular homomorphisms $\phi : CH^n(X)_{\text{deg } 0} \rightarrow G$ to smooth connected commutative algebraic groups G the induced morphism

$$\pi^{(-)} : S^{g+\mu(\mathcal{L})}(\Pi_{X_{\text{reg}}}) \xrightarrow{\gamma_{g+\mu(\mathcal{L})}^{(-)}} CH^n(X)_{\text{deg } 0} \xrightarrow{\phi} G$$

has a dense image in G . Hence for the product Π of all the different connected components of $S^{g+\mu(\mathcal{L})}(\Pi_{X_{\text{reg}}})$ the induced morphism $\pi' : \Pi \rightarrow G$ is dominant and π' induces a unique embedding of function fields $k(G) \subset k(\Pi)$.

If $\phi_v : CH^n(X)_{\text{deg } 0} \rightarrow G_v$, for $v = 1, 2$ are two surjective regular homomorphisms to smooth connected commutative algebraic groups, then

$$\phi_3 : CH^n(X)_{\text{deg } 0} \longrightarrow G_1 \times G_2$$

is regular. Let G_3 be the image of ϕ_3 . Then ϕ_v factors through the regular homomorphism $\phi_3 : CH^n(X)_{\text{deg } 0} \rightarrow G_3$ and $k(G_v) \subset k(G_3) \subset k(\Pi)$, for $v = 1, 2$.

Hence among the smooth connected commutative algebraic groups G with a regular surjective homomorphisms from $\phi : CH^n(X)_{\text{deg } 0} \rightarrow G$, there is one, $A^n(X)$, for which the subfield $k(A^n(X))$ is maximal in $k(\Pi)$ and $A^n(X)$ dominates all the other G in a unique way.

It remains to show that $A^n(X)$ satisfies base-change for algebraically closed fields. Let us write $Z_K = Z \times_k K$, for a variety Z defined over k . We first show:

Claim 6.7. *Let $K \supset k$ be an algebraically closed extension field of k . The cycle map $X_K^{(-)} : (\Pi_{X_{\text{reg}}})_K \rightarrow A^n(X)_K$ factors through a surjective homomorphism $u_{K,k} : A^n(X_K) \rightarrow A^n(X)_K$ of algebraic groups.*

Proof. Let $U = X_{\text{reg}}$, and let (C', ι) be an admissible pair defined over K , with $B = \iota^{-1}(U_K)_{\text{reg}}$. Choose a rational function $f \in R(C', X_K)$ such that

$$\text{div } f = \sum a_i - \sum b_i$$

for $p = (a_1, b_1, \dots, a_m, b_m) \in S^m(\Pi_B)(K)$. Choose a smooth k -variety S with $k(S) \subset K$, such that C', B, p, a_i, b_i, f come by base-change from $k(S)$ to K from

$$\mathcal{C}' \rightarrow S, \mathcal{B} \rightarrow S, \pi : S \rightarrow S^m(\Pi_{\mathcal{B}/S}), \alpha_i, \beta_i : S \rightarrow \mathcal{B}, \varphi \in k(\mathcal{B})^\times$$

with $\text{div } \varphi = \sum \alpha_i - \sum \beta_i$. Since $f \in R(C', X_K)$, we can replace S by a dense open subscheme, so that we can arrange that for each $s \in S(k)$, if we specialize to $\mathcal{C}'_s = \mathcal{C}' \times_S s$, then $\pi(s)$ maps to zero in $\text{Pic}^0(\mathcal{C}'_s)$. As

$$S^m(\Pi_{\mathcal{B}_s}) \longrightarrow S^m(\Pi_U) \times s \longrightarrow A^n(X) \times s$$

factors through $\text{Pic}^0(\mathcal{C}'_s)$, the composite morphism

$$S \xrightarrow{\pi} S^m(\Pi_{\mathcal{B}/S}) \longrightarrow S^m(\Pi_U) \times S \longrightarrow A^n(X) \times S$$

maps all k -points of S to the zero section. Thus it is the zero section, and therefore $S^m(\Pi_U)_K \rightarrow A^n(X)_K$ factors through $CH^n(X_K)$, inducing $u_{K,k}$ by lemma 1.12. □

Since $d_K := \dim A^n(X_K)$ is bounded by $2n(g + \mu(\mathcal{L}))$ (proposition 6.2), there is an algebraically closed field K_1 with $d_{K_1} = d_L$ for all $L \supset K_1$ algebraically closed. For any ascending chain $K_i \subset K_{i+1}$ of algebraically closed fields with $K_i \supset K_1$ one has $\text{deg } u_{K_i, K_1} \leq \text{deg } u_{K_{i+1}, K_1}$. Since the latter is bounded by the degree of the algebraic closure of $K_1(A^n(X_{K_1}))$ in $K_1(\Pi_{X_{\text{reg}}})$ one concludes that there is an

algebraically closed field $E \supset K_1$ such that $u_{E,L}$ is an isomorphism for all algebraically closed fields $L \supset E$.

We will make use of the following lemma.

Lemma 6.8. *Let K be a field, W, Y, Z be geometrically integral K -varieties, such that there are K -morphisms $\alpha : W \rightarrow Y, \beta : W \rightarrow Z$, such that α has dense image. Then:*

- (i) *there is at most one K -morphism $f : Y \rightarrow Z$ such that $\beta = f \circ \alpha$*
- (ii) *suppose that for some extension field L of K , there is an L -morphism $h : Y_L \rightarrow Z_L$ such that $\beta_L = h \circ \gamma_L : W_L \rightarrow Z_L$; then there is a K -morphism $f : Y \rightarrow Z$ as in (i), and we have $h = f_L$.*

Proof. Let $\Gamma \subset W \times_K Z$ be the graph of β , and let $\bar{\Gamma} \subset W \times_K Z$ be the closure of $(\alpha \times 1_Z)(\Gamma)$. The projection $\bar{\Gamma} \rightarrow Y$ has dense image. If there is a K -morphism $f : Y \rightarrow Z$ as in (i), then $\bar{\Gamma}$ must be its graph, and so there is at most one such morphism, which exists precisely when $\bar{\Gamma} \rightarrow Y$ is an isomorphism. Clearly if this is an isomorphism after base change to L , it is an isomorphism to begin with. □

There is a smooth k variety S , with $k(S) \subset E$, together with a smooth commutative S -group scheme $\mathcal{A} \rightarrow S$ with connected fibres, such that $A^n(X_E) = \mathcal{A} \times_S \text{Spec } E$. Choosing S small enough one also has a natural surjective S -morphism $u_{S,k} : \mathcal{A} \rightarrow A^n(X) \times S$ and a natural morphism $\Pi \times S \rightarrow \mathcal{A}$ which is fibrewise dominant for the irreducible variety Π constructed in the first part of the proof.

Let F be an algebraic closure of the quotient field of $E \otimes_k E$, $p : k(S \times_k S) \hookrightarrow F$ the natural inclusion, and let

$$p_i^* : k(S) \hookrightarrow k(S \times_k S), \quad i = 1, 2$$

be the inclusions defined by the two projections $p_i : S \times_k S \rightarrow S$. Set $q_i = p \circ p_i^*$, and for any S -scheme T , let q_i^*T be the F -scheme obtained by the base change to F determined by q_i .

The S -morphism $\Pi \times S \rightarrow \mathcal{A}$ gives rise to the fibrewise dominant morphism

$$\alpha'_i : q_i^*(\Pi \times_k S) = \Pi_F \longrightarrow q_i^*\mathcal{A} .$$

By the assumption on E the two F -varieties $q_i^*\mathcal{A}$ are isomorphic via

$$u' = u_{F,E_1} \circ u_{F,E_2}^{-1} : q_1^*A^n(X_E) \longrightarrow q_2^*A^n(X_E) .$$

where $E_i \subset F$ are the images of the two embeddings $E \hookrightarrow E \otimes_k E \hookrightarrow F$, $x \mapsto x \otimes 1$ and $x \mapsto 1 \otimes x$. By construction, u' satisfies $\alpha'_2 = u' \circ \alpha'_1$.

Hence by lemma 6.8, applied to the extension of fields $k(S \times_k S) \hookrightarrow F$, the isomorphism u' comes from an isomorphism

$$u : (p_1^* \mathcal{A})_{k(S \times_k S)} \longrightarrow (p_2^* \mathcal{A})_{k(S \times_k S)} ,$$

Then u in fact extends uniquely to an isomorphism of groups schemes (again denoted u)

$$u : (p_1^* \mathcal{A})_U \longrightarrow (p_2^* \mathcal{A})_U$$

over an open dense subset $U \subset S \times_k S$. Replacing S by some open dense subscheme, we may assume that $p_i : U \rightarrow S$ is surjective, for $i = 1, 2$. Further, if $\alpha_i : \Pi \times U \rightarrow (p_i^* \mathcal{A})_U$, $i = 1, 2$ are the natural fibrewise dominant maps, then $\alpha_2 = u \circ \alpha_1$.

The uniqueness statement in lemma 6.8 (i) similarly implies that u satisfies the ‘‘cocycle condition’’

$$u_{23} \circ u_{12} = u_{13} : \pi_1^* \mathcal{A} \longrightarrow \pi_3^* \mathcal{A}$$

on the fibres over the generic point of $S \times_k S \times_k S$, and hence (by continuity) over the open dense subset

$$\pi_{12}^{-1}(U) \cap \pi_{23}^{-1}(U) \cap \pi_{13}^{-1}(U) \subset S \times_k S \times_k S .$$

Here $\pi_j : S \times_k S \times_k S \rightarrow S$ are the 3 projections, and $u_{ij} = \pi_{ij}^* u$, for the 3 projections $\pi_{ij} : S \times_k S \times_k S \rightarrow S \times_k S$.

Given two points $s_i \in S(k)$, one finds a third one $s \in S(k)$ such that $(s_1, s) \in U(k)$ and $(s, s_2) \in U(k)$. The cocycle condition implies that the induced composite isomorphism

$$\theta_{s_1 s_2} : \mathcal{A}|_{s_1} \xrightarrow{u|_{(s_1, s)}} \mathcal{A}|_s \xrightarrow{u|_{(s, s_2)}} \mathcal{A}|_{s_2}$$

does not depend on the point $s \in S(k)$ chosen. Also u is compatible with the surjective morphisms $\Pi \times U \rightarrow p_i^* \mathcal{A}$.

We claim that for each closed point $s \in S(k)$, the morphism $\Pi \times s \rightarrow \mathcal{A}|_s$ induces a regular homomorphism $CH^n(X)_{\text{deg } 0} \rightarrow \mathcal{A}|_s$. Let (C', ι) be an admissible pair on X , defined over k , with $B = \iota^{-1}(U)_{\text{reg}}$. The morphism $(\Pi_B)_E \rightarrow A^n(X_E) = \mathcal{A}_E$, and the resulting morphism $S^g(\Pi_B)_E \rightarrow \mathcal{A}_E$ (with $g := \dim \text{Pic}^0(C')$) induces a homomorphism $\text{Pic}^0(C')_E \rightarrow \mathcal{A}_E$, since by the defining property of $A^n(X_E)$, we have a factorization through $CH^n(X_E)_{\text{deg } 0}$. Since $S^g(\Pi_B) \rightarrow \text{Pic}^0(C')$, lemma 6.8 gives a map $\text{Pic}^0(C') \times_k k(S) \rightarrow \mathcal{A}_{k(S)}$, compatible with the maps from $(\Pi_B)_{k(S)}$. This then induces a map $\text{Pic}^0(C') \times S^0 \rightarrow \mathcal{A}_{S^0}$ for some open dense subscheme $S^0 \subset S$.

Choosing a k -point $s_1 \in S^0(k)$, we get that the map $\Pi_B \times s_1 \rightarrow \mathcal{A}|_{s_1}$ is compatible with a homomorphism $\text{Pic}^0(C') \rightarrow \mathcal{A}|_{s_1}$. The isomorphism u is compatible with the morphisms $\Pi_B \times U \rightarrow p_i^* \mathcal{A}$. Hence, the isomorphisms θ_{ss_1} are compatible with the maps $\Pi_B \cong \Pi_B \times s \rightarrow \mathcal{A}|_s$ and $\Pi_B \cong \Pi_B \times s_1 \rightarrow \mathcal{A}|_{s_1}$, for all $s \in S(k)$. We deduce that for any $s \in S(k)$, the map $\Pi_B \times s \rightarrow \mathcal{A}|_s$ gives rise to a compatible morphism $\text{Pic}^0(C') \times_k s \rightarrow \mathcal{A}|_s$. This implies that there is an induced regular homomorphism $CH^n(X)_{\text{deg } 0} \rightarrow \mathcal{A}|_s$ for each $s \in S(k)$.

Hence, one obtains morphisms $v_s : A^n(X) \rightarrow \mathcal{A}|_s$, verifying $v_t = \theta_{st} \circ v_s$ for all $s, t \in S(k)$. Choosing now $s \in S(k)$, we set $G = \mathcal{A}|_s$, and $v = v_s$. The fibrewise dominant morphism $\Pi \times_k S \rightarrow \mathcal{A}$ induces a dominant map from $\Pi \times_k s$ onto G , hence v is surjective. Since the composite

$$u_{s,k} \circ v : A^n(X) \longrightarrow G \longrightarrow A^n(X)$$

is an isomorphism, v is an isomorphism. Thus $u_{S,k} : \mathcal{A} \rightarrow A^n(X) \times S$ is an isomorphism when restricted to each $t \in S(k)$, and is hence an isomorphism. By base change to E one finds that $u_{E,k} : A^n(X_E) \rightarrow A^n(X)_E$ is an isomorphism.

Now if $K \supset k$ is any algebraically closed field, we choose an algebraically closed field F with $F \supset K \supset k$ and $F \supset E \supset k$, hence

$$u_{K,k} \otimes \text{id}_F \circ u_{F,K} = u_{F,k} = u_{E,k} \otimes \text{id}_F \circ u_{F,E} ,$$

and $u_{K,k}$ is an isomorphism as well. □

7. Finite dimensional Chow groups of zero cycles

The definition of finite dimensionality for the Chow group of 0-cycles is a natural generalization of the definition in the non-singular (and normal) case (see [M], [S]).

Definition 7.1. *$CH^n(X)$ is said to be finite dimensional if for some $m > 0$, the map*

$$\gamma_m : S^m(X_{\text{reg}}) \longrightarrow CH^n(X)_{\text{deg } 0}$$

(introduced in 1.9) is surjective.

One can see that this is also equivalent to the statement that for some integer $m' > 0$, depending only on X , any element of $CH^n(X)_{\text{deg } 0}$ is represented by a 0-cycle $\sum_{i=1}^{m'} \delta_i$, where for each i , the cycle δ_i is a difference of two effective 0-cycles of degree m' supported in X_i .

In the proof of the next theorem we will use the notion of a *regular map* $f : Z \rightarrow CH^n(X)_{\text{deg } 0}$ from a variety Z . This is a map of sets such that

- (i) the composition $Z \rightarrow CH^n(X)_{\text{deg } 0} \rightarrow A^n(X)$ is a morphism
- (ii) there is a surjective morphism $W \rightarrow Z$ such that

$$W \longrightarrow Z \xrightarrow{f} CH^n(X)_{\text{deg } 0}$$

factors as $W \xrightarrow{h} S^m(X_{\text{reg}}) \xrightarrow{\gamma_m} CH^n(X)_{\text{deg } 0}$, for some morphism h .

For example, let C' be a reduced Cartier curve in X or, more generally, let (C', ι) be an admissible pair. Then the homomorphism $\eta : \text{Pic}^0(C') \rightarrow CH^n(X)_{\text{deg } 0}$ constructed in lemma 1.8 is regular. In fact, the first condition holds true by 1.12 whereas the second one follows from the dominance of $S^g(C'_{\text{reg}}) \rightarrow \text{Pic}^0(C')$, for $g = \dim_k(H^1(C', \mathcal{O}_{C'}))$.

Recall that k is called a universal domain, if its transcendence degree over the prime field is uncountable.

Theorem 7.2. *Let X be a projective variety of dimension n over a universal domain k . Then $CH^n(X)$ is finite dimensional if and only if*

$$\varphi : CH^n(X)_{\text{deg } 0} \longrightarrow A^n(X)$$

defines an isomorphism between $CH^n(X)_{\text{deg } 0}$ and (the closed points of) $A^n(X)$.

Proof. Let us write $U = X_{\text{reg}}$. By lemma 1.10 (ii) the composite

$$S^m(U) \longrightarrow CH^n(X)_{\text{deg } 0} \longrightarrow A^n(X)$$

is always surjective for $m = 2 \cdot \dim(A^n(X))$. Hence, if $CH^n(X)_{\text{deg } 0} \rightarrow A^n(X)$ is an isomorphism, then $CH^n(X)$ is finite dimensional.

So the main thrust of the theorem is the converse, that if $CH^n(X)_{\text{deg } 0}$ is finite dimensional, then $CH^n(X)_{\text{deg } 0} \rightarrow A^n(X)$ is an isomorphism. We imitate Roitman’s proof of this result in the non-singular case, and the analogous argument for the normal case in [S]; however there are extra refinements needed here, particularly in characteristic $p > 0$.

First, we note that by [LW], proposition 4.2, the “graphs of rational equivalence”

$$\Gamma_{r,s} = S^r(U) \times_{CH^n(X)_{\text{deg}0}} S^s(U)$$

decompose as a countable union of locally closed subvarieties, for each r, s . This immediately implies that if $f_j : Z_j \rightarrow CH^n(X)_{\text{deg}0}$, $j = 1, 2$, are regular maps, then

$$Z_1 \times_{CH^n(X)_{\text{deg}0}} Z_2 = \{(z_1, z_2) \in Z_1 \times Z_2; f_1(z_1) = f_2(z_2)\}$$

is a countable union of locally closed subvarieties of $Z_1 \times Z_2$.

Now arguing as in [S], lemma (1.3), (where one uses the requirement that k be a universal domain), we first see that if G is a smooth connected commutative algebraic group, and $f : G \rightarrow CH^n(X)_{\text{deg}0}$ is any regular map which is a group homomorphism, then there is a well-defined connected component of the identity $G^0 \subset \ker f$, which is a connected algebraic subgroup of G , and which has countable index in $\ker f$. Then the induced homomorphism

$$G/G^0 \longrightarrow CH^n(X)_{\text{deg}0}$$

has a countable kernel. Hence, for any such homomorphism $G \rightarrow CH^n(X)_{\text{deg}0}$, we can define the *dimension of the image of G* to be the dimension of G/G^0 .

Next, notice that if $G_1 \rightarrow CH^n(X)_{\text{deg}0}$ and $G_2 \rightarrow CH^n(X)_{\text{deg}0}$ are two regular homomorphisms from smooth connected commutative algebraic groups G_i such that image G_1 is properly contained in image G_2 , then in fact

$$\dim \text{image } G_1 < \dim \text{image } G_2 .$$

Indeed, we may assume the maps $G_i \rightarrow CH^n(X)_{\text{deg}0}$ have countable kernel, so that we wish to assert that $\dim G_1 < \dim G_2$. Now $G_3 = G_1 \times_{CH^n(X)_{\text{deg}0}} G_2$ is a subgroup of $G_1 \times G_2$ which is a countable union of locally closed subvarieties, and hence has a connected component of the identity which is a connected algebraic group, say H . Then $H \rightarrow G_i$ are homomorphisms of algebraic groups with countable, hence finite kernels, such that $H \rightarrow G_1$ is surjective, and the image of H in G_2 is a strictly smaller subgroup. Thus $\dim G_1 = \dim H < \dim G_2$.

Now suppose γ_m is surjective. We claim that for any homomorphism

$$G \longrightarrow CH^n(X)_{\text{deg}0}$$

as above, with countable kernel, we have $\dim G \leq \dim S^m(U)$. Indeed,

$$G \times_{CH^n(X)_{\text{deg } 0}} S^m(U)$$

is a countable union of subvarieties of $G \times S^m(U)$ which projects onto G , and maps to $S^m(U)$ with countable fibres. Hence some irreducible component of this fibre product dominates G under the projection, and maps to $S^m(U)$ with finite fibres.

We now claim that we can find a finite number of reduced complete intersection curves C_1, \dots, C_s such that the induced homomorphism from $\bigoplus \text{Pic}^0(C_j)$ to $CH^n(X)_{\text{deg } 0}$ is surjective. Indeed, given a finite collection of such curves, if

$$P = \bigoplus \text{Pic}^0(C_j) \longrightarrow CH^n(X)_{\text{deg } 0}$$

is not surjective, we can find a curve C of the same sort such that

$$\text{image}(\text{Pic}^0(C) \longrightarrow CH^n(X)_{\text{deg } 0})$$

is not contained in the image of P . Then the induced map

$$P \times \text{Pic}^0(C) \longrightarrow CH^n(X)_{\text{deg } 0}$$

has strictly larger dimensional image than that of P . Since the dimension of the image is bounded above by $\dim S^m(U) = mn$, this process can be repeated at most a finite number of times.

So we may assume given a surjective regular homomorphism

$$f : A \longrightarrow CH^n(X)_{\text{deg } 0}$$

with countable kernel, where A is a connected smooth commutative algebraic group, and for some Cartier curves C_1, \dots, C_s a surjective homomorphism

$$(7.1) \quad \bigoplus_{j=1}^s \text{Pic}^0(C_j) \xrightarrow{\oplus \rho_j} A .$$

Note that the composition $h : A \rightarrow CH^n(X)_{\text{deg } 0} \rightarrow A^n(X)$ is then a surjective homomorphism of algebraic groups.

We now distinguish between the case $k = \mathbb{C}$, and that of a general universal domain k .

Proof of 7.2 for $k = \mathbb{C}$. We first show that the surjective homomorphism $h : A \rightarrow A^n(X)$ is an *isogeny*. Clearly h induces an injective homomorphism

$$h^* : \Omega(A^n(X)) \longrightarrow \Omega(A) .$$

We will use proposition 3.8 to show that $h^* : \Omega(A^n(X)) \rightarrow \Omega(A)$ is an isomorphism. Since h^* is injective, it suffices to prove that $\dim \Omega(A) \leq \dim \Omega(A^n(X))$.

Consider the set $\Gamma = U \times_{CH^n(X)_{\text{deg}0}} A$. This is a countable union of algebraic subvarieties, and maps surjectively to U under the projection. Recalling that $U = \cup_j U_j$, we can then find irreducible varieties $\Gamma_j \subset \Gamma$ such that Γ_j dominates U_j under the projection $\Gamma \rightarrow U$. Then $\pi_j : \Gamma_j \rightarrow U_j$ has countable, and hence finite, fibres. Let d_j be the degree of π_j , and let $V_j \subset U_j$ be a dense open subset such that $\pi_j : \pi_j^{-1}(V_j) \rightarrow V_j$ is an étale covering of degree d_j . Let c be the l.c.m. of the d_j , and let $c = d_j c_j$. If $q : \Gamma \rightarrow A$ is the second projection, then consider the morphism

$$\begin{aligned} \mu : V = \bigcup_j V_j &\longrightarrow A, \\ \mu(x) &= c_j \sum_{y \in \pi_j^{-1}(x)} q(y) \quad \text{for } x \in V_j . \end{aligned}$$

One verifies at once that the diagram

$$(7.2) \quad \begin{array}{ccc} \bigcup_j V_j = V & \xrightarrow{\mu} & A \\ \downarrow & & \downarrow f \\ \bigcup_j U_j = U & \xrightarrow{c \cdot \gamma_1} & CH^n(X)_{\text{deg}0} \end{array}$$

commutes.

The image of $\mu(V)$ in $CH^n(X)_{\text{deg}0}$ generates $CH^n(X)_{\text{deg}0}$ as a group, since $CH^n(X)_{\text{deg}0}$ is c -divisible, and any 0-cycle on X is rationally equivalent to a cycle supported on V . Hence the subgroup of A generated by $\mu(V)$ has countable index, and is also a countable increasing union of constructible subsets, namely the images of $\mu(V)^{2m}$ under the maps

$$\begin{aligned} \sigma_m : A^{2m} &\longrightarrow A, \quad m \geq 1, \\ (a_1, \dots, a_{2m}) &\mapsto a_1 + \dots + a_m - a_{m+1} - \dots - a_{2m} . \end{aligned}$$

By dimension considerations, one of the subsets $\sigma_m(\mu(V)^{2m})$ must be dense in A , and then $\sigma_{2m}(\mu(V)^{4m}) = A$. Hence the induced map on 1-forms

$$\Omega(A) \longrightarrow H^0(V^{4m}, \Omega_{V^{4m}/\mathbb{C}}^1)$$

is injective. Now the action of σ_{2m} on 1-forms is given by

$$\sigma_{2m}^*(\omega) = (\omega, \dots, \omega, -\omega, -\omega, \dots, -\omega) .$$

This means that the map on 1-forms $\Omega(A) \rightarrow H^0(V, \Omega_{V/\mathbb{C}}^1)$, induced by the morphism $V \rightarrow \mu(V) \hookrightarrow A$, is injective.

We claim that $\text{image } \Omega(A) \subset \Omega(A^n(X))$, so that $\dim \Omega(A) \leq \dim \Omega(A^n(X))$. To see this, it suffices by proposition 3.8 to show that for any reduced local complete intersection Cartier curve $C \subset X$ with $B = (C_{\text{reg}}) \cap V$ dense in C , the image of any element of $\Omega(A)$ in $H^0(B, \Omega_{B/\mathbb{C}}^1)$ lies in the image of $H^0(C, \omega_C)$. Fixing base points in each component of B , we obtain a morphism $\vartheta : C_{\text{reg}} \rightarrow \text{Pic}^0(C)$. If C_i is any component of C_{reg} , then the two induced maps

$$\begin{aligned} C_i &\longrightarrow \text{Pic}^0(C) \longrightarrow CH^n(X)_{\text{deg } 0}, \\ C_i &\hookrightarrow U \xrightarrow{\gamma_1} CH^n(X)_{\text{deg } 0} \end{aligned}$$

agree up to translation by a fixed element of $CH^n(X)_{\text{deg } 0}$.

Now consider the subgroup $\Gamma_C = \text{Pic}^0(C) \times_{CH^n(X)_{\text{deg } 0}} A$. As before, this is a countable union of subvarieties of $\text{Pic}^0(C) \times A$. Hence there is a connected algebraic subgroup $\Gamma_C^0 \subset \Gamma_C$ such that Γ_C/Γ_C^0 is a countable group. Further, $\Gamma_C \rightarrow \text{Pic}^0(C)$ is surjective with countable fibres. Hence $\Gamma_C^0 \rightarrow \text{Pic}^0(C)$ is an isogeny. Restricting (7.2) one obtains a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\mu} & A \\ \downarrow & & \downarrow f \\ C_{\text{reg}} & \xrightarrow{c\gamma_1} & CH^n(X)_{\text{deg } 0} \end{array}$$

and hence a morphism $B \rightarrow \Gamma_C^0$ such that

- (i) for each component C_i of C_{reg} , the composite

$$C_i \cap B \longrightarrow \Gamma_C^0 \longrightarrow \text{Pic}^0(C)$$

equals the restriction of the composite

$$C_i \longrightarrow \text{Pic}^0(C) \xrightarrow{c \cdot} \text{Pic}^0(C) ,$$

up to a translation (here $c \cdot$ denotes multiplication by c)

- (ii) $C_i \cap B \rightarrow \Gamma_C^0 \rightarrow A$ agrees with μ , up to a translation.

Hence, by (ii),

$$\text{image } (\Omega(A) \xrightarrow{\mu^*} \Gamma(B, \Omega_{C/\mathbb{C}}^1)) \subset \text{image } (\Omega(\Gamma_C^0) \longrightarrow \Gamma(B, \Omega_{C/\mathbb{C}}^1))$$

while by (i),

$$\text{image } \Omega(\Gamma_C^0) = \text{image } \Omega(\text{Pic}^0(C)) = \text{image } \Gamma(C, \omega_C) .$$

Since C was arbitrary, we have verified the hypotheses of proposition 3.8. This completes the proof that the composite $h : A \rightarrow CH^n(X)_{\text{deg}0} \rightarrow A^n(X)$ is an isogeny.

In particular, $f : A \rightarrow CH^n(X)_{\text{deg}0}$ has a finite kernel. Replacing A by $A/(\ker f)$, we may assume given a regular homomorphism $f : A \rightarrow CH^n(X)_{\text{deg}0}$ which is an *isomorphism* of groups. Now repeating the above arguments once more, we obtain (7.2) with $c = 1$. By corollary 1.13, this means the group isomorphism $f^{-1} : CH^n(X)_{\text{deg}0} \rightarrow A$ is a regular homomorphism, which must factor through $\varphi : CH^n(X)_{\text{deg}0} \rightarrow A^n(X)$. This forces φ to be an isomorphism of groups, as well. □

Remark 7.3. Over the field of complex numbers the last part of the proof of 7.2 is consistent with the Roitman theorem proved in [BiS]. In fact, if

$$A \cong CH^n(X)_{\text{deg}0} \longrightarrow A^n(X)$$

is surjective with finite kernel the generalization of Roitman’s theorem implies that the composite

$$A \xrightarrow{\cong} CH^n(X)_{\text{deg}0} \longrightarrow A^n(X) \longrightarrow J^n(X)$$

is an isomorphism on torsion subgroups, so that $CH^n(X)_{\text{deg}0} \rightarrow A^n(X)$ is an injection on torsion subgroups. Hence the isogeny $A \cong CH^n(X)_{\text{deg}0} \rightarrow A^n(X)$ must be an isomorphism.

In the algebraic case we have to modify the arguments, in particular since the lower horizontal morphism in the diagram (7.2) need not to be surjective in characteristic $p > 0$.

Proof of 7.2 for k a universal domain. Let us write B for the kernel of $h : A \rightarrow A^n(X)$, a closed subgroup scheme of A , not necessarily reduced. We may replace A by A/κ , for any (zero dimensional) closed subgroup scheme κ of B such that $\kappa(k) \subset \ker f$.

The group B acts on $U \times_{A^n(X)} A$ with quotient $U \times_{A^n(X)} A^n(X) = U$. The kernel \mathcal{K} of the map $A(k) \rightarrow CH^n(X)_{\text{deg}0}$

consists of countably many closed points, the induced action on $U \times_{CH^n(X)_{\text{deg}_0}} A$ is free, and the induced map on the quotient

$$(U \times_{CH^n(X)_{\text{deg}_0}} A) / \mathcal{K} = U \times_{CH^n(X)_{\text{deg}_0}} (A / \mathcal{K}) \longrightarrow U$$

is a bijection on the closed points.

Let $V \subset U$ be an open dense subscheme, and let Γ_j be a locally closed irreducible subscheme of $V \times_{A^n(X)} A$, contained in $V \times_{CH^n(X)_{\text{deg}_0}} A$, and dominant over the component $V_j = V \cap U_j$ of V under the first projection. For V small enough, we may assume that $\Gamma_j \rightarrow V_j$ is finite. Let $\kappa_j \subset \mathcal{K}$ be the subgroup of elements g with $g(\Gamma_j) = \Gamma_j$. Then κ_j is a finite group and $\Gamma_j / \kappa_j \rightarrow V_j$ is an isomorphism on the closed points. Replacing Γ_j by its image in $U \times_{A^n(X)} (A / \kappa_j)$ and A by A / κ_j we may assume that κ_j is trivial, and thereby that $\Gamma_j \rightarrow V_j$ is purely inseparable.

Repeating this construction for the different components of U we finally reduce to the situation, where U has an open dense subscheme V , and where $V \times_{CH^n(X)_{\text{deg}_0}} A$ has a closed subscheme Γ which is finite, surjective and purely inseparable over V .

Assume that $\Gamma \rightarrow V$ is not an isomorphism, in particular, that the characteristic of k is $p > 0$. The restriction of the group action to $B \times \Gamma$ factors as

$$B \times \Gamma \xrightarrow{\cong} (V \times_{A^n(X)} A) \times_V \Gamma \xrightarrow{pr_1} V \times_{A^n(X)} A$$

and the preimage $S(\Gamma)$ of $\Gamma \subset V \times_{A^n(X)} A$ is isomorphic to $\Gamma \times_V \Gamma$. Thus $S(\Gamma)$ is a subscheme of $B \times \Gamma$, supported in the zero section $\{e\} \times \Gamma$. Hence $S(\Gamma)$ is contained in the v -th infinitesimal neighbourhood $\{e\}_v \times \Gamma$ of the zero section, for some $v > 0$.

The kernel $\kappa^{(v')}$ of the v' -th geometric Frobenius $F^{(v')} : B \rightarrow B^{(v')}$ is defined by the sheaf of ideals in \mathcal{O}_B , generated by the $p^{v'}$ -th powers of the generators of the sheaf of ideals \mathfrak{m} defining $\{e\} \subset B$. For some $v' > 0$ it is contained in \mathfrak{m}^v and $\{e\}_v$ is a subscheme of $\kappa^{(v')}$.

Dividing A by $\kappa^{(v')}$, we may assume that $S(\Gamma) = \Gamma \times_V \Gamma$ is isomorphic to Γ , and thereby that Γ is isomorphic to V .

Independent of the characteristic of k , we have thus reduced to the situation where U has an open dense subscheme V , for which

$$pr_1 : V \times_{CH^n(X)_{\text{deg}_0}} A \longrightarrow V$$

has a section, such that by projecting to A we obtain a morphism $\mu : V \rightarrow A$ and (using the notation introduced in 1.9) a commutative diagram

$$(7.3) \quad \begin{array}{ccc} \Pi_Y & \xrightarrow{\mu^{(-)}} & A \\ \subseteq \downarrow & & \downarrow f \\ \Pi_U & \xrightarrow{\gamma^{(-)}} & CH^n(X)_{\text{deg } 0} . \end{array}$$

Claim 7.4. *There exists a surjective homomorphism $\phi : CH^n(X)_{\text{deg } 0} \rightarrow A$ with $\mu^{(-)} = \phi \circ \gamma^{(-)}|_{\Pi_Y}$. In particular, ϕ is regular.*

Proof. Let (C', ι) be an admissible pair with $B = (\iota^{-1}(V))_{\text{reg}}$ dense in C' . By restriction (7.3) gives rise to a commutative diagram

$$(7.4) \quad \begin{array}{ccc} \Pi_B & \xrightarrow{\mu'^{(-)}} & A \\ = \downarrow & & \downarrow f \\ \Pi_B & \xrightarrow{\gamma_B^{(-)}} & CH^n(X)_{\text{deg } 0} \end{array}$$

where $\mu' = \mu|_B$. By lemma 1.12 the lower horizontal map in the diagram (7.4) factors as

$$(7.5) \quad \begin{array}{ccc} \Pi_B & \xrightarrow{\gamma_B^{(-)}} & CH^n(X)_{\text{deg } 0} \\ & \searrow \vartheta^{(-)} & \uparrow \eta \\ & & \text{Pic}^0(C') . \end{array}$$

Let $\Gamma_{C'}^0$ be the connected component of $\Gamma_{C'} = \text{Pic}^0(C') \times_{CH^n(X)_{\text{deg } 0}} A$ containing the origin. $\Gamma_{C'}/\Gamma_{C'}^0$ is a countable group and $\Gamma_{C'}^0 \rightarrow \text{Pic}^0(C')$ is an isogeny. Since the diagrams (7.4) and (7.5) are commutative, the image of

$$\Pi_B \xrightarrow{(\vartheta^{(-)}, \mu'^{(-)})} \text{Pic}^0(C') \times A$$

is contained in $\Gamma_{C'}^0$. This implies that $\Gamma_{C'}^0 \rightarrow \text{Pic}^0(C')$ must be an isomorphism. In fact, by 1.11 there is an open connected subscheme W of $S^g(B)$ such that the morphism $\vartheta_W : W \rightarrow \text{Pic}^0(C)$ is an open embedding. On the other hand, ϑ_W factors through the isogeny $\Gamma_{C'}^0 \rightarrow \text{Pic}^0(C')$.

Hence the morphism $\mu'^{(-)}$ in the diagram (7.4) is the composite

$$\Pi_B \xrightarrow{\vartheta^{(-)}} \text{Pic}^0(C') \cong \Gamma_{C'}^0 \xrightarrow{pr_2} A ,$$

and the condition (b) in lemma 1.12 holds true.

Thereby the homomorphism ϕ in 7.4 exists, and it remains to show that ϕ is surjective. Equivalently, it suffices to show that the image of ϕ generates A as a group, which will follow if we show that $\mu^{(-)}(\Pi_V)$ generates A . But we know that $\gamma^{(-)}(\Pi_V)$ generates $CH^n(X)_{\text{deg } 0}$, and so $\mu^{(-)}(\Pi_V)$ generates a subgroup of countable index in A . Since k is a universal domain, $\mu^{(-)}(\Pi_V)$ generates A . \square

By claim 7.4 and by the universal property for $A^n(X)$ the regular homomorphism $\phi : CH^n(X)_{\text{deg } 0} \rightarrow A$ factors through a homomorphism of algebraic groups $\chi : A^n(X) \rightarrow A$. Since ϕ is surjective, the induced morphism χ is surjective as well. Further, the composite

$$CH^n(X)_{\text{deg } 0} \xrightarrow{\phi} A \xrightarrow{f} CH^n(X)_{\text{deg } 0}$$

is clearly the identity, since it is so on the image of Π_V , which is a set of generators. By the universal property of $\varphi : CH^n(X)_{\text{deg } 0} \rightarrow A^n(X)$, we deduce that the composite

$$A^n(X) \xrightarrow{\chi} A \xrightarrow{h} A^n(X)$$

is the identity. Hence χ and h are inverse isomorphisms of algebraic groups, and $f : A \rightarrow CH^n(X)_{\text{deg } 0}$ and $\varphi : CH^n(X)_{\text{deg } 0} \rightarrow A^n(X)$ are both isomorphisms (of groups) as well. \square

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