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ON A CHARACTERIZATION OF AN ABELIAN VARIETY IN THE CLASSIFICATION THEORY OF ALGEBRAIC VARIETIES

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In this paper we shall prove the following theorem which was conjectured by S. Iitaka (B_n in p. 131 in [1]) and proven by K. Ueno for $n = 3$ [2]. In this paper everything is defined over the complex number field $\mathbb{C}$.

MAIN THEOREM: *Let X be an algebraic variety and let $f: X \to A$ be a dominant generically finite morphism to an abelian variety. If the Kodaira dimension $\kappa(X) = 0$, then f is birationally equivalent to an étale morphism and X is birationally equivalent to an abelian variety.*

To prove the main theorem we shall reduce it to the following theorem 1.

THEOREM 1: *Let A be abelian variety of dimension n, let X be a reduced irreducible divisor on A and let $\bar{X}$ be a resolution of X. If X is an algebraic variety of general type, then $q_k(\bar{X}) \equiv \dim H^0(\bar{X}, \Omega^k_{\bar{X}}) \geq \binom{n}{k}$, for $k = 1, \ldots, n-1$. Moreover, if $p_g(\bar{X}) \equiv q_{n-1}(\bar{X}) = n$, then $q_k(\bar{X}) = \binom{n}{k}$, for $k = 1, \ldots, n-2$, and in particular $|\chi(\mathcal{O}_{\bar{X}})| = 1$.*

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The following lemma is just a special case of a theorem of Ueno (3.3 of [2]).

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LEMMA 2: *Let the notations and assumptions be as in the Main Theorem. Then* $\dim(H^0(X, \Omega_X^{n-1})) \leq n$.

PROOF: We want to show that $f^*(dz_1 \wedge \cdots \wedge dz_{i-1} \wedge dz_{i+1} \wedge \cdots \wedge dz_n) = \omega_i$ are generators of $H^0(X, \Omega_X^{n-1})$, for a global coordinate system $(z_1, \ldots, z_n)$ of A. Take $\omega \in H^0(X, \Omega_X^{n-1})$ and $a_i \in \mathbb{C}$, such that $\omega \wedge f^*(dz_i) = a_i \cdot f^*(dz_1 \wedge \cdots \wedge dz_n)$. This is always possible, since $H^0(X, \Omega_X^n)$ is generated by $f^*(dz_1 \wedge \cdots \wedge dz_n)$. Replacing ω by $\omega - \sum_{i=1}^{n} (-1)^{n-1} a_i\omega_i$ we may assume that $a_i = 0$ for $i = 1, \ldots, n$. Choose a small open subset $U \subseteq X$, such that $f|_U$ is an embedding. $(z_1, \ldots, z_n)$ is a local coordinate system of U. Since $\omega \wedge dz_i = 0$ for $i = 1, \ldots, n$, ω must be 0 on U and hence on X.

LEMMA 3: *Let the notations and assumptions be as in the main theorem. Let* $f_0 : X_0 \to A$ *be the normalisation of* A *in* $\mathbb{C}(X)$. *Let* $D_1, \ldots, D_m$ *be the irreducible components of the discriminant* $\Delta(X_0/A)$ *and let* $\bar{D}_1, \ldots, \bar{D}_m$ *be their desingularisations. Then*

$$\sum_{i=1}^{m} p_g(\bar{D}_i) \leq \dim(A).$$

PROOF: Choose Δ_i to be one irreducible component of $f^{-1}(D_i)$, such that Δ_i is ramified over A. We may assume, that X is projective and that $\Delta_1 \cup \cdots \cup \Delta_m$ is a regular subvariety of X. Let $\omega_X = \Omega_X^n$. Then $\omega_X \otimes_{O_X} O_X\left(\sum_{i=1}^{m} \Delta_i\right) \subseteq \omega_X^2$ and, since $H^0(X, \omega_X) = H^0(X, \omega_X^2) = \mathbb{C}$, we know that $\bigoplus_{i=1}^{m} H^0(\Delta_i, \omega_{\Delta_i})$ is a subspace of $H^1(X, \omega_X) = H^{n-1}(X, O_X) = H^0(X, \Omega_X^{n-1})$. However, $H^0(\bar{D}_i, \omega_{\bar{D}_i}) \subseteq H^0(\Delta_i, \omega_{\Delta_i})$.

Now we recall the following Theorem of Ueno (p. 120 in [1]):

THEOREM 4: *Let* B *be a subvariety of an abelian variety* A. *Then there exist an abelian subvariety* A_1 *of* A *and an algebraic variety* W *which is a subvariety of an abelian variety such that*

(1) *B is an analytic fibre bundle over W whose fibre is A_1,*

(2) $\kappa(W) = \dim W = \kappa(B)$.

A_1 is characterized as the maximal connected subgroup of A such that $A_1 + B \subseteq B$.

PROOF OF "THEOREM 1 $\Rightarrow$ MAIN THEOREM": Let $\eta : A' \to A$ be any étale covering and $X_\eta = X \times_A A'$. Then $X_\eta \to A'$ also satisfies the conditions of the main theorem. Let $X_{\eta,0}$ be the normalisation of A' in $\mathbb{C}(X_\eta)$. Then $\Delta(X_{\eta,0}/A')$ is the pullback of $\Delta(X_0/A)$ by η. Suppose $\Delta(X_0/A)$ is not empty. Any abelian variety has étale coverings of arbitrary high degree (for example "multiplication with $r \gg 0$"). Every subvariety of an abelian variety has $p_g > 0$. Hence, replacing A by some étale covering we may assume, that for every étale covering $\eta: A' \to A$ the number of irreducible components of $\Delta(X_0/A)$ and $\Delta(X_{\eta,0}/A')$ is the same (Lemma 3).

Put $B = \{x \in A; x + \Delta(X_0/A) \subseteq \Delta(X_0/A)\}^0$. Again replacing A by an étale covering, we may assume that $A = B' \times B$. Let Y_0 be the Stein factorisation of $X_0 \to A \to B'$ and Y any desingularisation of Y_0. Since X_0 is a finite covering of $Y_0 \times B$ we have $\kappa(X) = 0 \geq \kappa(Y) + \kappa(B) \geq 0$ and $\kappa(Y) = 0$.

Assume that $\Delta(Y_0/B') = \emptyset$. Since $\Delta(X_0/A) \simeq \Delta' \times B$ for some positive divisor $\Delta' \subseteq B'$, the ramification divisor of $X_0 \to Y_0 \times B$ must be a (rational) multiple of the pullback of some divisor $\tilde{\Delta}$ of Y_0. Then $\kappa(X) \geq \kappa(Y_0, O(\tilde{\Delta})) > 0$, in contradiction to our assumptions. Therefore $\Delta(Y_0/B') \neq \emptyset$ and, repeating this step if necessary, we may assume $B = 0$.

Let $B_i = \{x \in A; x + D_i \subseteq D_i\}^0$ for $i = 1, \ldots, m$. We have $\cap \mathrm{B}_i = 0$. By Theorem 4 each D_i is a fibre bundle over a certain E_i with fibre B_i, for $i = 1, \ldots, m$. We have $p_g(\bar{D}_i) \geq p_g(\bar{E}_i)$ for a desingularisation $\bar{E}_i$ of E_i and by Theorem 1 $p_g(\bar{E}_i) \geq \operatorname{codim}_A B_i$.
Since the equalities must be true by Lemma 3, we have $|\mathbf{X}(O_{\bar{E}_i})| = 1$ by theorem 1, for $i = 1, \ldots, m$.

Let r be a natural number, with $r \geq 2$, and let $r: A \to A$ be the multiplication with r. Using the notation introduced above, $\Delta(X_{r,0}/A)$ must have components $D_{r,i}$, $i = 1, \ldots, m$ such that the corresponding base space $E_{r,i}$ satisfies $|\mathbf{X}(O_{\bar{E}_{r,i}})| = \text{degree}(r) \cdot |\mathbf{X}(O_{\bar{E}_i})| \geq 2$. This is a contradiction.

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Proof of Theorem 1: Let $\{x_1, \ldots, x_n\}$ be a global coordinate system on A such that the set $\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n\}$ gives a basis of 1-forms on A. Let $\alpha: \bar{X} \to A$ be the canonical map and let $\omega_i = \alpha^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{i-1} \wedge \mathrm{d}x_{i+1} \wedge \cdots \wedge \mathrm{d}x_n)$ for $i = 1, \ldots, n$. We shall prove first that these are linearly independent $(n-1)$-forms on $\bar{X}$. Suppose the contrary: $\sum_{i=1}^{n} a_i\omega_i = 0$ for $a_i \in \mathbb{C}$. Pick a smooth point p on X.

Suppose X is defined in A near p by an equation $x_n = F(x_1, \ldots, x_{n-1})$, where F is a certain holomorphic function. Then $\omega_i = (-1)^{n-i-1} \frac{\partial F}{\partial x_i} \omega_n$ for $i = 1, \ldots, n-1$. Therefore, $\sum_{i=1}^{n-1} (-1)^{n-i-1} a_i \frac{\partial F}{\partial x_i} + a_n = 0$, which means that there is a non-zero subgroup B of A such that $B + X \subseteq X$, which is a contradiction.

Put $\omega_I = \alpha^*(dx_{i_1} \wedge \cdots \wedge dx_{i_k})$ for each set I of k-distinct integers $1 \leq i_1 < \cdots < i_k \leq n$. Since ω_i are linearly independent, $\{\omega_I\}_I$ gives a linearly independent system of k-forms on $\bar{X}$. Thus, $q_k(\bar{X}) \geq \binom{n}{k}$.

Before we prove the second part of theorem 1, we shall prove the following theorem, due to the first author.

THEOREM 5: *Let A and X be as in theorem* 1. *Let $f : X \to \mathbb{P}^{n-1}$ be the rational map defined by the system $\{\omega_1, \ldots, \omega_n\}$. If X is an algebraic variety of general type, then f is dominant.*

PROOF: Assume the contrary. Let Y be the image variety of f and let q be a smooth point of Y such that $f^{-1}(q)$ is also smooth near some smooth point $p \in f^{-1}(q)$ of X. Our assumption means that $\dim f^{-1}(q) \geq 1$. Consider everything in the universal cover $\mathbb{C}^n$ of A. Let H be the tangent plane of X at p, which we assume is defined by an equation $x_n = 0$. Then, X is defined near p by an equation $x_n = F(x_1, \ldots, x_{n-1})$, where $\{x_1, \ldots, x_n\}$ is a global coordinate system centered at p and $\frac{\partial F}{\partial x_i}(0) = 0$ for $i = 1, \ldots, n-1$. $f^{-1}(q)$ is defined near p by the equations $\frac{\partial F}{\partial x_i} = 0$ for $i = 1, \ldots, n-1$. Y is contained near q in a smooth divisor D of $\mathbb{P}^{n-1}$ (near q). After a suitable linear transformation of $x_1, \ldots, x_{n-1}$, the equation of D can be written as $\frac{\partial F}{\partial x_1} = G\left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_{n-1}}\right)$, where G is a holomorphic function of degree ≥ 2. By the rule of derivation of products, we have on $f^{-1}(q)$ $\frac{\partial}{\partial x_1}\left(\frac{\partial F}{\partial x_i}\right) = \frac{\partial G}{\partial x_i} = 0$ for $i = 1, \ldots, n-1$. Thus, $f^{-1}(q)$ is invariant under translations in the direction of x_1 and hence contains a translation of an abelian subvariety of A generated by the line $x_2 = \cdots = x_n = 0$. Let B be the maximal abelian subvariety of A such that $p + B$ is contained in X. We have proved that $B \neq 0$. Since there are only countably many abelian subvarieties, B does not depend on p. Thus, $B+X \subseteq X$, a contradiction. Q.E.D.

PROOF OF THEOREM 1 CONTINUED: Suppose $p_g(\bar{X}) = n$. Let p be a smooth point of X and let $x_1, \ldots, x_n$ be as in the proof of theorem 5. Let ω be an arbitrary k-form on $\bar{X}$. Write near p $\omega = \sum_{n \notin I} g_I(x_1, \ldots, x_{n-1})\omega_I$. Put $I^c = \{1, \ldots, n-1\} - I$. Then $\omega \wedge \omega_{I^c} = \epsilon(I, I^c) g_I \omega_n$, where ϵ is the sign of permutations. Therefore, we have

$$g_I = g_I(0) + \sum_{i=1}^{n-1} a_{Ii} \frac{\partial F}{\partial x_i}$$

for some $a_{Ii} \in \mathbb{C}$. Let J be a subset of $\{1, \ldots, n-1\}$ such that Card $J = n-k-2$. Since

$$\omega \wedge \omega_J \wedge \mathrm{d}x_n = \sum_{\{i\} \cup I = J^c} \epsilon(I, J, i) g_I \frac{\partial F}{\partial x_i} \omega_n,$$

we have

$$\sum_{\{i\} \cup I = J^c} \epsilon(I, J, i) \left(g_I(0) + \sum_{j=1}^{n-1} a_{Ij} \frac{\partial F}{\partial x_j} \right) \frac{\partial F}{\partial x_i} = \sum_{i=1}^{n-1} b_{Ji} \frac{\partial F}{\partial x_i}$$

for some $b_{Ji} \in \mathbb{C}$. Since $\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_{n-1}}$ are algebraically independent, we can compare the coefficients and we get (1) $a_{Ij} = 0$ for $j \in J$, (2) $a_{Ii} = 0$ for $I \cup \{i\} = J^c$, and (3) for each $K = \{i_1, \ldots, i_{k-1}\}$ such that $i_1 < \cdots < i_{k-1}$ and $K \cup \{i\} \cup \{j\} = J^c$,
$\epsilon(K \cup \{i\}, J, j) a_{K \cup \{i\}, i} + \epsilon(K \cup \{j\}, J, i) a_{K \cup \{j\}, j} = 0$, that is,
$\epsilon(K, i) a_{K \cup \{i\}, i} = \epsilon(K, j) a_{K \cup \{j\}, j}$. Put $a_K = \epsilon(K, i) a_{K \cup \{i\}, i}$.
Then, $\omega = \sum_{n \notin I} q_I(0) \omega_I + \sum_K a_K \omega_K \wedge \mathrm{d}x_n$. Q.E.D.

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