THE CLASSIFICATION OF SURFACES AND OTHER VARIETIES

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Vanishing Theorems and Pathologies in Characteristic $p > 0$

Let X be a projective complex manifold, and let $\mathcal{L}$ be an invertible sheaf on X. For $\mathcal{L}$ ample the Kodaira Vanishing Theorem says that

$$H^i(X, \mathcal{L}^{-1}) = 0, \quad \text{for} \quad i < \dim X.$$

Recall that the Kodaira-dimension $\kappa(\mathcal{L})$ of an invertible sheaf $\mathcal{L}$ is the maximum of the dimensions $\dim \varphi_m(X)$, where φ_m denotes the rational map given by $H^0(X, \mathcal{L}^m)$, for $m \geq 1$. If $H^0(X, \mathcal{L}^m) = 0$ for all $m > 0$, we write $\kappa(\mathcal{L}) = -\infty$. The invertible sheaf $\mathcal{L}$ is called semi-ample if, for some $m \geq 1$, the global sections generate $\mathcal{L}^m$. Mumford's Vanishing Theorem in [M3] says:

Theorem 1. *If $\mathcal{L}$ is semi-ample, and $\kappa(\mathcal{L}) \geq 2$, then*

$$H^i(X, \mathcal{L}^{-1}) = 0 \quad \textit{for} \quad i = 0 \quad \textit{or} \quad 1.$$

C. P. Ramanujam [78] weakened the assumption "$\mathcal{L}$ semi-ample" in Theorem 1 to "$\mathcal{L}$ numerically effective", i.e. $(\deg \mathcal{L}|_C) \geq 0$, for all curves C in X. D. Mumford later obtained Ramanujam's Vanishing Theorem on surfaces as a corollary of Bogomolov's Theorem [9] on the stability of vector bundles (Appendix to [80]).

Using analytic methods, as Mumford did in the first of the two proofs of Theorem 1 given in [M3], H. Grauert and O. Riemenschneider proved in [34] that the vanishing in Theorem 1 holds true for $i = 0, 1, \ldots, \kappa(\mathcal{L}) - 1$.

The second proof of Theorem 1 in [M3] uses two methods which reappear in the proof of several generalizations of the Kodaira Vanishing Theorem: the one-connectedness of divisors (see [78], [10] and [3], IV § 8) and the construction of cyclic coverings. The latter was used by Ramanujam in [78] and it played an essential role in the proof of the vanishing theorem for numerically effective invertible sheaves by Y. Kawamata [46] and the author [93]. The latter also allows a small normal crossing divisor as "correction term". For surfaces a similar results has been shown before by Y. Miyaoka [67].

Theorem 2. *Assume that for an effective normal crossing divisor $D = \sum_{i=1}^{r} \nu_i D_i$ and for some $N > 0$ the sheaf $\mathcal{L}^N(-D)$ is numerically effective. Then*

$$H^i(X, \mathcal{L}^{-1}([\tfrac{D}{N}])) = 0, \quad \textit{for} \quad i < \kappa(\mathcal{L}(-[\tfrac{D}{N}])).$$

Here $[\frac{D}{N}]$ denotes the integral part of the $\mathbb{Q}$-divisor $\frac{D}{N}$.

By Serre Duality, Theorem 2 is equivalent to the vanishing of $H^i(X, \omega_X \otimes \mathcal{L}(-[\frac{D}{N}]))$, for $i > \dim X - \kappa(\mathcal{L}(-[\frac{D}{N}]))$.

April 1996, updated May 2003.

For $D = 0$ and $\kappa(\mathcal{L}) = \dim(X)$ one obtains for all i the surjectivity of the "adjunction map"

$$H^i(X, \mathcal{L} \otimes \omega_X \otimes \mathcal{O}_X(B)) \longrightarrow H^i(B, \mathcal{L} \otimes \omega_B), \tag{1}$$

where B is any effective divisor. J. Kollár has shown in [51] that the latter remains true under much weaker assumptions:

Theorem 3. *If $\mathcal{L}$ is a semi-ample invertible sheaf, B an effective divisor and if*

$$H^0(X, \mathcal{L}^M \otimes \mathcal{O}_X(-B)) \neq 0,$$

for some $M > 0$, then the adjunction map (1) is surjective, for all i.

Again, one can formulate and prove a slight generalization of Theorem 3, assuming just that $\mathcal{L}^N(-D)$ is semi-ample for some small normal crossing divisor D (see for example [29] and [30]).

In the Vanishing Theorem 2 one may allow the effective divisors D to have singularities worse than normal crossings. To this aim one considers a birational morphism $\tau : X' \to X$ with X' non-singular and $D' = \tau^* D$ a normal crossing divisor. Theorem 2 allows to show that $R^i\tau_*(\omega_{X'} \otimes \mathcal{O}_{X'}(-[\frac{D'}{N}])) = 0$, and that the sheaf $\tau_*(\omega_{X'/X} \otimes \mathcal{O}_{X'}(-[\frac{D'}{N}]))$, called the algebraic multiplier ideal sheaf, is independent of τ. One obtains (see [28]) for

$$i > \dim X - \kappa(\mathcal{L}^N(-D)) \geq \dim X - \kappa(\tau^*\mathcal{L}(-[\frac{D'}{N}]))$$

the vanishing of

$$H^i(X, \mathcal{L} \otimes \tau_*(\omega_{X'} \otimes \mathcal{O}_{X'}(-[\tfrac{D'}{N}]))). \tag{2}$$

The name "multiplier ideal" goes back to A. Nadel, who defined in [75] general multiplier ideals in the analytic context, and extended (2) to this case. In [21] J.-P. Demailly studies in a similar way how far the positivity condition can be violated along subvarieties, without effect on the vanishing of cohomology groups. Nadel's multiplier ideals, or their algebraic analogue, turned out to be a powerful tool in higher dimensional algebraic geometry, in particular for Y.-T. Siu's proof of the invariance of the plurigenera (see [87] and [88]).

There is much more to be said on vanishing theorems over a field of characteristic zero, for example: Kollár's easy proof of the Kawamata-Fujita Positivity Theorem and the Generic Vanishing Theorem of M. Green and R. Lazarsfeld. Some of those aspects are discussed in [30] and more references are given there.

Whereas Mumford's second proof of Theorem 1 is in the framework of algebraic geometry, for a long time the only known proofs of Kodaira's Vanishing Theorem and the generalizations given above were based on analytic methods. In [29] for example it is shown that Theorems 2 and 3 are both consequences of the E_1-degeneration of the Hodge to de Rham spectral sequence for logarithmic differential forms, a result shown by P. Deligne [19], using analytic methods.

G. Faltings gives in [31] the first algebraic proof of the E_1-degeneration and thereby of the vanishing theorems. K. Kato [44] in the projective case, and J.-M. Fontaine and W. Messing [32] in the proper case, obtained the E_1-degeneration for manifolds X defined over a perfect field of characteristic $p > \dim X$, provided X lifts to the ring of Witt vectors $W(k)$. Finally P. Deligne and L. Illusie [20] gave an elementary proof of this degeneration under the weaker condition that X lifts to the second Witt vectors $W_2(k)$ and that $p \geq \dim X$. The degeneration of the Hodge to de Rham spectral sequence in characteristic p implies the degeneration in characteristic zero. As explained in [30] one obtains algebraic proofs of Theorems 2 and

3. For Kodaira's original statement, one finds in [20] a short and elegant proof, due to M. Raynaud, which works in characteristic $p \geq \dim(X)$ whenever X lifts to $W_2(k)$. It is not known whether similar lifting properties imply the analogues of the Grauert-Riemenschneider Vanishing Theorem and of Theorem 2 and 3 in characteristic $p \neq 0$.

The examples of "pathologies" in characteristic $p > 0$, given by Mumford in [M1], [M2] and [M3], i.e. examples showing that neither the Vanishing Theorem 1, nor the closedness of global differential forms (and hence the E_1-degeneration), nor the symmetry of the Hodge numbers hold true in general, were complemented by counterexamples of Raynaud [79] to Kodaira's Vanishing Theorem and extended in [91]. There the reader also finds a proof, due to L. Szpiro [90], of analogues of Theorem 1 for surfaces in characteristic p. Examples of surfaces of general type in characteristic $p > 0$ with non-trivial vector fields were given by P. Russel, H. Kurke and W. E. Lang (see [62]). A list of further pathological examples can be found in [41], § 15.

Fujita's Conjecture

As mentioned in the first section, D. Mumford proved Ramanujam's Vanishing Theorem on surfaces as a corollary of Bogomolov's Theorem [9] on stability of vector bundles (published by Reid as an appendix to [80]). By a similar method I. Reider [84] found numerical conditions for an ample sheaf $\mathcal{L}$ on a surface S, which imply that $\omega_S \otimes \mathcal{L}$ is spanned by global sections, and some slightly stronger conditions for the very ampleness of $\omega_S \otimes \mathcal{L}$.

Reider's Theorem gives for surfaces an affirmative answer to Fujita's conjecture, asserting that for an ample invertible sheaf $\mathcal{L}$ on a smooth projective variety X of dimension n the sheaf $\omega_X \otimes \mathcal{L}^{n+1}$ is generated by global sections and that $\omega_X \otimes \mathcal{L}^{n+2}$ is very ample.

In the higher dimensional case, the first numerical criterion for very ampleness was obtained by Demailly [22]. His result implies that $\omega_X^2 \otimes \mathcal{L}^m$ is very ample for $m \geq 12n^n$. The bound was lowered by Siu and by Demailly himself. Their methods of proof are of analytic nature. Using tools, developed in the framework of "Mori's Program", Kollár [57] gave an explicit lower bound for numbers ν which implies the very ampleness of $(\omega_X \otimes \mathcal{L}^{n+2})^\nu$. Using algebro-geometric methods, close to those used in diophantine approximation, L. Ein, R. Lazarsfeld and M. Nakamaye [24] reproved the results of Demailly (with a slightly larger bound for m).

The first part of Fujita's conjecture, the base point freeness of $\omega_X \otimes \mathcal{L}^{n+1}$, has been verified for threefolds by Ein and Lazarsfeld [23]. Their proof, based on the Vanishing Theorem 2 and on multiplier ideals, gives an alternative approach towards Reider's result (see also [30]). Kawamata [49] proves the base point freeness of $\omega_X \otimes \mathcal{L}^5$ on four dimensional manifolds.

Without any restriction on the dimension the best numerical criterion for base point freeness is due to U. Angehrn and Y.-T. Siu [2]. They show, using analytic multiplier ideals and the Vanishing Theorem of Nadel, that $\omega_X \otimes \mathcal{L}^m$ is globally generated for $m \geq \frac{1}{2}(n^2 + n + 2)$ (slightly improved by G. Heier in [35]). Kollár and independently Ein and Lazarsfeld gave an algebraic proof of the same criterion. Lazarsfeld's forthcoming book [64] provides an excellent presentation of vanishing theorems, multiplier ideals and of the results mentioned in this section.

Classification Theory of Surfaces

The Enriques-Kodaira classification of complex surfaces has been extended to algebraic surfaces defined over an algebraically closed field k of characteristic $p \geq 0$, by D. Mumford [M5], and by E. Bombieri and D. Mumford in [M6] and [M7]. A survey on those results and methods can be found in [11].

The pathological behavior of surfaces in characteristic $p > 0$, mentioned above, has to be taken in account, and the classification table gains some additional lines in characteristic 2 and 3.

Theorem 4. *Let S be a projective surface, without exceptional curves of the first kind, defined over an algebraically closed field k of any characteristic.*

(1) *If $\kappa(S) = \kappa(\omega_S) = 2$, then $H^0(S, \omega_S^2) \neq 0$ and ω_S is semi-ample. In particular, the canonical ring*

$$R(S) = \bigoplus_{\nu \geq 0} H^0(S, \omega_S^\nu)$$

is finitely generated.

(2) *If $\kappa(S) = 1$, then either $H^0(S, \omega_S^4) \neq 0$ or $H^0(S, \omega_S^6) \neq 0$. Moreover ω_S is semi-ample and, for $n \gg 0$, ω_S^n defines an elliptic or quasi-elliptic fibration. The latter only occurs if* char $k = 2$ *or 3.*

(3) *If $\kappa(S) = 0$, then either $\omega_S^6 \cong \mathcal{O}_S$ or $\omega_S^4 \cong \mathcal{O}_S$. Moreover S belongs to one of the following classes ($q = \dim H^0(S, \Omega_S^1)$, $p_g = \dim H^0(S, \omega_S)$ and b_i are the Betti-numbers):*

b_2	b_1	$\chi(\mathcal{O}_S)$	q	p_g	*type of surface*
22	0	2	0	1	*K-3*
10	0	1	0	0	*Enriques (classical)*
10	0	1	1	1	*Enriques (non-classical) (only in char. 2)*
6	4	0	2	1	*Abelian*
2	2	0	1	0	*bi-elliptic (classical)*
2	2	0	2	1	*bi-elliptic (non-classical) (only in char. 2 or 3)*

(4) *If $\kappa(S) = -\infty$, then S is ruled. In particular, "adjunction terminates", i.e. for all invertible sheaves $\mathcal{L}$ there exists some $\nu_0 > 0$ such that $H^0(S, \mathcal{L} \otimes \omega_S^\nu) = 0$, for $\nu \geq \nu_0$.*

Some of the arguments used in [M5], [M6] and [M7] supply new proofs for the known results in characteristic 0.

Due to lack of space and knowledge I am not able even to sketch the recent developments in the theory of algebraic surfaces. [11], [77] [16] and [41] are excellent surveys, with a lot of references. A pretty complete picture of the theory of compact complex surfaces, except of the use of Mori's methods and Reider's results (discussed below) can be found in the book of Barth, Peters and Van de Ven [3] (may be there will be a second extended edition?).

Special surfaces in characteristic $p > 0$ were constructed and studied by W. E. Lang (see for example [63]).

The semi ampleness of ω_S and the finite generation of the ring $R(S)$ in Theorem 4, a) had been shown by Mumford already in an appendix to O. Zariski's paper [100]. The question, which power of ω_S defines a birational morphism or is generated by global sections was studied by E. Bombieri [10], in characteristic 0, and by T. Ekedahl [25], in positive characteristic.

Building up on Mumford's Geometric Invariant Theory [74], D. Gieseker constructed in [33] quasi-projective moduli schemes for surfaces of general type, i.e. for surfaces S with $\kappa(S) = 2$. Properties of those moduli schemes have been studied by F. Catanese (see for example [14], [15] and the survey [16]).

Attempts to generalize the classification of surfaces, as stated in Theorem 4, to higher dimensional complex manifolds were made along two lines. The first one, started and programmed by S. Iitaka and developed by K. Ueno, E. Viehweg, T. Fujita, Y. Kawamata and J. Kollár, is explained in S. Mori's survey [71]. It builds up on:

- The Kodaira dimension $\kappa(X) = \kappa(\omega_X) \in \{-\infty, 0, 1, \ldots, \dim X\}$.
- For $1 \leq \kappa(X) \leq \dim X$ on the Iitaka map $\varphi_m : X \to \mathbb{P}^{p_m - 1}$, defined by $H^0(X, \omega_X^m)$ for the multiples m of some sufficiently large number m_0. We write $p_m = \dim H^0(X, \omega_X^m)$.
- For $\kappa(X) \leq 0$ on the Albanese morphism $\alpha_X : X \to A(X)$, where $A(X)$ is an abelian variety of dimension $q(X) = \dim H^0(X, \Omega_X^1)$.

The main results obtained in the framework of Iitaka's program can be summarized in the following three theorems (we assume from now on that the characteristic is zero).

Theorem 5. *Let X be a non-singular complex projective variety of dimension n and let $1 \leq \kappa(X) \leq n - 1$. Then, for $\nu \gg 0$, the non-singular fibres $F_y = \varphi_\nu^{-1}(y)$ of the Iitaka map*

$$\varphi_\nu : X \longrightarrow \varphi_\nu(X) \subset \mathbb{P}^{p_\nu - 1}$$

have Kodaira dimension $\kappa(F_y) = 0$.

One hopes, that the statement holds true for all y in a non empty Zariski open subset of $\varphi_\nu(X)$. In Theorem 5 one uses the result of S. Iitaka [36], saying that for all y outside of a countable union of proper closed subvarieties of $\varphi_\nu(X)$ one has $\kappa(F_y) = 0$. To obtain 5, as stated one also has to use the deformation invariance of the plurigenera (Siu, [87]).

Theorem 5 reduces the study of higher dimensional varieties to the cases: $\kappa(X) = \dim(X)$, $\kappa(X) = 0$ and $\kappa(X) = -\infty$, and to families of lower dimensional varieties F with $\kappa(F) = 0$. The structure of manifolds with Kodaira dimension zero is partly described in the following theorem, which presents results of Kawamata [45], Kollár [51], [56] and Viehweg [94]. Again, the reader finds the history and complete references in [71].

Theorem 6. *Let X be a non-singular projective variety of dimension n.*

1. *If for some $\nu > n$ one has $p_\nu = 1$ then $q \leq n$ and the Albanese morphism $\alpha_X : X \to A(X)$ is surjective with connected fibres. In particular, this holds true whenever $\kappa(X) = 0$.*
2. *The following conditions are equivalent:*
   - (a) *X is birational to an abelian variety.*
   - (b) *$q = n$ and $p_\nu = 1$, for some $\nu > 2$.*
   - (c) *$q = n$ and $\kappa(X) = 0$.*
3. *If $q \geq n - 2$ and if $\kappa(X) = 0$ then, up to birational equivalence, α_X is an étale fibre bundle whose fibre F is a curve or surface with $\kappa(F) = 0$.*

In dimension 3 or 4 one has (see [94] and [71]):

Theorem 7. *Let X be a non-singular projective variety,* $\dim(X) \leq 4$ *and $\kappa(X) = -\infty$. Let $f : X \to Y$ be the Stein factorization of the Albanese map, Y' a desingularization of Y and X' a desingularization of $X \times_Y Y'$. Then $\kappa(Y') \geq 0$, $q(Y') = q(X)$ and the induced morphism $X' \to Y'$ has a general fibre F with $\kappa(F) = -\infty$.*

The Theorems 5, 6 and 7, applied to threefolds, give a partial generalization of Theorem 4. However they say nothing about the difficult cases "$\kappa(X) = 3$", "$\kappa(X) = q(X) = 0$" and "$\kappa(X) = -\infty, q(X) = 0$".

The proofs of Theorems 6 and 7 require the study of positivity properties of the sheaves $f_*\omega^\nu_{X/Y}$ for surjective morphisms $f : X \to Y$ of manifolds. To this aim, one has to know the structure of the general fibre F and one has to study families of $(\dim X - \dim Y)$-dimensional varieties. For example, one possible proof of 6, (3) and of 7 uses results on moduli of curves and surfaces, obtained in [74] and [33]. Needless to underline that this inductive structure of the "Iitaka Program" makes it difficult to generalize Theorem 7 to higher dimensional varieties or to extend Theorem 6, (3) to the case $q(X) < n - 2$. Recently the "Iitaka program" has been made more precise and extended by F. Campana [13] and S.S.Y. Lu [65].

The positivity properties of $f_*\omega^\nu_{X/Y}$ mentioned above have been verified only if $\kappa(F) = \dim(F)$ (Kollár) or if F has a nice "minimal model" (Kawamata). Before turning our attention to the minimal model problem, the second main line of higher dimensional classification theory, let us point out that the positivity properties of the sheaves $f_*\omega^\nu_{X/Y}$ can be used to construct quasi-projective moduli schemes for canonically polarized manifolds and for certain polarized manifolds of arbitrary dimension ([95], [53], [43] and [89]).

Mori's Minimal Model Program

The most powerful tools in the classification theory of threefolds came with Mori's construction of minimal models in characteristic zero.

Let us first recall the surface case. In Theorem 4 we assumed that for a surface S' with $\kappa(S') \geq 0$ there exists a surface S birational to S', with ω_S semi-ample. In order to construct such S one contracts exceptional curves E on S', i.e. curves with $\deg(\omega_{S'}|_E) = -1$. Repeating this finitely many times, one obtains a *minimal model* S of S', i.e. a surface S without exceptional curves or, equivalently, with ω_S numerically effective.

To prove Theorem 4, a) in the Appendix to [100], Mumford goes one step further. Contracting all curves C with $\deg(\omega_S|_C) = 0$, he constructs for a surface S of general type a *canonical model*, i.e. a normal surface $\tilde{S}$ with rational double points as singularities and with $\omega_{\tilde{S}}$ ample.

Unfortunately things look more complicated in the higher dimensional case (see [92] or [71]). If $\dim(X') \geq 3$ and $\kappa(X') \geq 0$, one cannot expect the existence of some manifold X birational to X', with ω_X numerically effective.

Building up on the methods he developed for his proof of the Hartshorne conjecture on the characterization of the projective space [69], Mori studied in [70] the cone of the effective curves on a threefold X' and he contracted "extremal" rational curves C with $\deg(\omega_{X'}|_C) < 0$. Inevitably such a contraction process leads to singular varieties and to proceed by induction one has to allow certain singularities.

A suitable class of singularities had been introduced by M. Reid, studying manifolds X' whose canonical ring is finitely generated. He introduced in [81] the notion of a canonical singularities. A normal variety X with such singularities is Cohen-Macaulay, as shown by Shepherd-Barron, Elkik and Flenner, and for some $r > 0$ the reflexive hull $\omega_X^{[r]}$ of $\omega_X^{\otimes r}$ is invertible. Terminal singularities are special canonical singularities.

A projective variety X is called a minimal model if it belongs to the category $\mathfrak{C}$ of $\mathbb{Q}$-Gorenstein normal varieties, with at most terminal singularities and if for some $r > 0$ the sheaf $\omega_X^{[r]}$ is invertible and numerically effective. Mori's minimal model program proposes for non uniruled manifolds X' the construction of a minimal model X in the category $\mathfrak{C}$. Unfortunately this can not be done just repeating contractions. It might happen that one has to blow up certain bad loci in order to stay in $\mathfrak{C}$. Some parts of this program have been verified in all dimensions, but for four and higher dimensional varieties it is far from being completed.

Building up on the work of Y. Kawamata, X. Benveniste, M. Reid, V. Shokurov and J. Kollár, Mori was able to finish the minimal model program in the three dimensional case [72]. The reader finds an outline of the minimal model program, of the singularities involved, of the implications and of the history of the subject in the survey articles [82], [52], [50], [98], [54], in the Lecture Notes [18] and in the contributions [73], [47] and [55] to the ICM 1990. Let us just indicate some of the tremendous applications of Mori's result to the classification of threefolds.

Benveniste and Kawamata (for threefolds), and Kawamata (in general), have shown that for a minimal model X with $\kappa(X) = \dim X$, the sheaf $\omega_X^{[r]}$ is semi-ample. This implies that the canonical ring $R(X)$ is finitely generated and, by [81], that $\mathrm{Proj}(R(X))$ has canonical singularities. One obtains:

Theorem 8. *If X' is a threefold of general type, then $R(X')$ is finitely generated and X' has a canonical model $\tilde{X}$, i.e. a model $\tilde{X}$ with canonical singularities and with $\omega_{\tilde{X}}^{[r]}$ ample invertible, for some $r > 0$.*

For threefolds of Kodaira dimension $-\infty$, Miyaoka [68] (using Mori's result) gives the following analogue of Theorem 4 (3) for threefolds:

Theorem 9. *Let X' be a projective threefold*

(1) *X' has a minimal model X if and only if $\kappa(X') \geq 0$.*
(2) *The following three conditions are equivalent:*
   (a) *$\kappa(X') = -\infty$.*
   (b) *X' is uniruled.*
   (c) *Adjunction terminates.*

Building up on earlier work of Miyaoka, Kawamata proved in [48] the "abundance conjecture" for threefolds:

Theorem 10. *If X is a minimal threefold, then for some $r > 0$ the sheaf $\omega_X^{[r]}$ is invertible and generated by its global sections.*

Studying the blowing up and down (flips) in the construction of minimal models more closely, Kollár and Mori [60] established the invariance of the plurigenera p_ν under small deformations of threefolds, as mentioned, a result obtained in between by Siu [87] in general.

There has been a lot of progress in our understanding of 3-folds during the last years (see M. Reid's survey [83], and the references given there).

The methods developed for the minimal model program turned out to be powerful tools in the higher dimensional algebraic geometry. For example, they allow a more precise study of manifolds of Kodaira dimension $-\infty$ (see [59] and [12], the survey [76] or the book [58], and the references given there).

They were also used to define and to study "stable surfaces" in [61]. In [53] and [1] it was shown that smoothable stable surfaces of general type have a projective moduli scheme, compactifying the Gieseker moduli space. K. Karu [43] reproved their result, by showing that the existence of minimal models in dimension $n+1$ allows to define stable n-folds of general type, and to construct the corresponding projective moduli scheme.

The Lüroth Problem

Around 1971 three independent papers gave examples of unirational manifolds which are not rational. It was known that all cubic threefolds, and for some of the quartic threefolds are unirational. In [17] C. Clemens and P. Griffiths showed that the intermediate Jacobian of a rational threefold must be the product of Jacobians of curves, whereas the Jacobian of a cubic hypersurface in $\mathbb{P}^4$ does not have this property, hence it can not be rational. Studying the group $\mathrm{Bir}(X)$ of birational morphisms, V. A. Iskovskih and Yu. I. Manin showed in [40] that all quartic hypersurfaces in $\mathbb{P}^4$ are not rational.

M. Artin and D. Mumford [M8] construct in all dimensions and all characteristics $\neq 2$ unirational conic bundles X over a surface with 2-torsion in $H^3(X, \mathbb{Z})$ or in $H^3_{\text{ét}}(X, \mathbb{Z}_\ell)$. Since those cohomology groups are shown to be birational invariants for threefolds in any characteristic and for higher dimensional manifolds in characteristic zero, X can not be rational in those cases. Their examples can not even be stably rational, i.e. for all $m \geq 0$ the manifolds $X \times \mathbb{P}^m$ are non rational.

In [5] the authors construct non-rational complex threefolds, which are stably rational. The appendix to the second edition of Manin's book [66] gives a survey of results on rationality questions, on cubic threefolds and more references.

Rational Equivalence of Zero-Cycles

Let $CH^n(X)_0$ denote the Chow group of zero-cycles of degree zero on a complex n-dimensional manifold X. One says that $CH^n(X)_0$ is finite dimensional if the natural map

$$\begin{array}{rcl} S^m X(\mathbb{C}) \times S^m X(\mathbb{C}) & \rightarrow & CH^n(X)_0 \\ (a, b) & \mapsto & a - b \end{array}$$

is surjective for some $m > 0$.

Mumford discovered in [M4] that for projective surfaces the finite dimensionality of $CH^2(X)_0$ implies that $p_g = \dim H^0(X, \omega_X) = 0$.

Theorem 11 (Mumford for $p = n = 2$, generalized by Roitman [85])**.** *Let X be a complex projective manifold of dimension n. If $h^{p,0} = \dim H^0(X, \Omega^p_X) > 0$ for some $p \geq 2$, then $CH^n(X)_0$ is not finite dimensional.*

In [85] Roitman showed as well that $CH^n(X)_0$ is finite dimensional if and only if $CH^n(X)_0$ is representable, i.e. if the Albanese morphism induces an isomorphism

$$\alpha_X : CH^n(X)_0 \rightarrow A(X).$$

Mumford's proof of Theorem 11 uses in an essential way the existence of a trace map for the sheaf of differential forms under finite morphisms. Unfortunately such a trace does not exist for pluricanonical forms and Mumford's arguments do not say anything about the finite dimensionality of $CH^n(X)_0$ for surfaces of general type. In fact S. Bloch proposed in [7]:

Conjecture 12. If X is a complex projective surface with $p_g = 0$, then $CH^2(X)_0$ is finite dimensional.

This conjecture was verified for surfaces X with $\kappa(X) \leq 1$ by S. Bloch, A. Kas and D. Lieberman [6]. The first examples of surfaces of general type for which Bloch's conjecture holds true were given by H. Inose and M. Mizukami [37]. A larger class of such surfaces was obtained by C. Voisin in [96].

In spite of all progresses made in the classification theory of surfaces, the conjecture 12 remains unsolved, perhaps one of the most challenging open problems for surfaces.

In [7] Bloch uses the decomposition of the diagonal, coming from the condition "$CH^n(X)_0$ finite dimensional" to reprove Mumford's Theorem 11. This method was taken up and extended in [8]. Variants of Bloch's method have been considered by several authors (see for example [42], [26] and [27]). Bloch's lecture notes [7] and the survey articles [42] and [97] explain more about motivic aspects of Mumford's Theorem and Bloch's Conjecture and about the history of the subject.

The Fake Projective Plane

D. Mumford constructs in [M9] a complex projective surface X with an ample canonical class ω_X, whose Betti-numbers $b_0, \ldots, b_4$ and Chern-numbers c_2 and c_1^2 coincide with those of $\mathbb{P}^2$. In particular one has $\chi(\mathcal{O}_X) = 1$ and $H^0(X, \omega_X) = 0$. The construction method, the p-adic uniformization, introduced by A. Kurihara and G. A. Mustafin, and the computation of the numerical invariants have been considered by several authors in the higher dimensional case (see for example [86] and [39]).

Little is known about this surface usually called "Mumford's fake $\mathbb{P}^2$." S.-T. Yau's results [99] imply that X is the quotient of the unit ball D_2 by a discrete cocompact subgroup $\Gamma \subset SU(2,1)$, but nothing is known about Γ. The surface X is not among the ball-quotients constructed in [4] as Galois cover of $\mathbb{P}^2$, ramified along configurations of lines (loc.cit. p. 145).

In [38], the author shows that X covers an elliptic surface and he studies its bad fibres. In [39] one finds other examples of "fake $\mathbb{P}^2$'s", non-isomorphic to Mumford's surface.

Geometric properties of X itself have not been studied and, for example, Bloch's conjecture on the zero-cycles has not been verified for the fake $\mathbb{P}^2$.

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